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ASYMPTOTIC CONFIGURATION OF STATIONARY INTERFACIAL PATTERNS FOR REACTION DIFFUSION SYSTEMS

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Abstract

Asymptotic configuration of interfacial patterns of reaction diffusion systems is considered when the interfacial thickness tends to zero. Under several hypotheses derived by formal asymptotic analysis, it is shown that there are no smooth limiting configuration of interface for generic domains. This partially explains the fact that interfacial patterns become fine and complicated as in the micro-phase separation of block copolymer in this limit.

1. Introduction

Morphology of *final* patterns in phase transition are usually simple ones: only one phase dominates the whole domain (non-conserved) or it is decomposed into simple subdomains (conserved) after coarsening process. This is due to the tendency to minimize the area of interface. However, if there is a microscopic constraint to the system, the final pattern becomes much richer and has in general a variety of morphologies from lamellar to labyrinthine patterns. Block copolymer is one of such materials where two monomers (say, A and B) are connected at some point (constraint), and this is responsible for the formation of very fine and complicated structures depending on the ratio of composite monomers in the process of micro-phase separation [3][2][13]. Locally each monomer moves in a random way and tends to segregate each other (bistability), however connectivity does not allow them to form a large domain consisting of only one monomer (nonlocality). Ohta and Kawasaki [9] proposed the following model system to describe such a phenomenon.

$$(1.1)\qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u,v) & \\ & \text{in } \Omega, \\ 0 = D\Delta v + u & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

where u is the *order parameter* indicating A-rich or B-rich phase, v represents the nonlocal effect due to connectivity, $\varepsilon(\ll 1)$ corresponds to the interfacial thickness and $D(\gg 1)$ is proportional to the square of the polymerization index (namely, the length of block copolymer) which is usually quite large, and $f(u,v)$ is a cubic nonlinearity (typically of the form $u - u^3 - v$). It is anticipated that many other phenomena could be described by similar models to (1.1), since the basic mechanism creating a variety of patterns is due to the competition between local dynamics and nonlocal effect. In fact similar patterns are observed in liquid crystal, magnetic thin film, and so on. The arguments in this note is valid to slightly more general system:

$$(1.2) \qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u,v) & \\ & \text{in } \Omega, \\ \delta v_t = D\Delta v + g(u,v) & \\ & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

where δ is a nonnegative constant. Although the precise assumptions for (f,g) are delegated to [7], they are qualitatively the same as (1.1). A naive approach to find nontrivial patterns of (1.2) is to consider the limiting case either $\varepsilon \downarrow 0$ or $D \uparrow \infty$. For the latter case it is known (see [5][4]) that the resulting equations become a scalar equation with a constraint of integral type and it is unlikely to have a stable complicated pattern for such a system, since there are no stable multi-layered solutions even in 1D case [6]. On the other hand we know very little about the former case in higher space dimensions, since it has been regarded to be extremely difficult to find the first approximate stationary solutions in the limit of $\varepsilon \downarrow 0$. Especiallly we are interested in the behavior of the asymptotic configuration of the interface Γ^ε (see (2.1)). The aim of this note is to answer (at least partially) the following question.

Does (1.2) *has an ε-family of stationary layered solutions with smooth interface Γ^ε up to $\varepsilon = 0$?*

The answer is obviously affirmative, since we know planar and spherical layered solutions (see [12][10]). However those domains have very *special* geometries, i.e., rectangles and spheres and it is not a priori clear that such smooth interfaces persist up to $\varepsilon = 0$ for *generic* domains. It turns out that the answer is negative for generic ones under several hypotheses derived by the formal asymptotic analysis. Here we only consider the case where Γ^ε is a simple closed curve inside of Ω. $(U^\varepsilon, V^\varepsilon)$ is called an *ε-family of matched asymptotic solutions* to (1.2), if it has an matched asymptotic expansion mentioned in Section 2.

MAIN THEOREM

(a) (Disk Symmetry) Suppose that (1.2) *with $n = 2$ has an ε-family of matched asymptotic solutions with simple closed smooth interfaces Γ^ε (see* (2.1)*) up to $\varepsilon = 0$ and that the matching condition $(MC)_1$ holds (see Lemmas* 3.4.1 *and* 3.4.2*), then Γ^0 must be a circle.*

(b) (Non-existence) Moreover under Hypothesis 4.1 *in Section* 4, *(a) implies that*

the reduced problem (2.26) has no solutions for generic domains Ω, and hence there does not exist associated ε-family of matched asymptotic solutions.

REMARK 1.1 *The hypothesis in Main Theorem (a) can be weakened, in fact, it suffices to assume the existence of principal orders of outer and inner expansions. The extension to $\{n \geq 3\}$-case can be done without difficulty. More precisely, see the forthcoming paper [8].*

The above non-existence result is not a dissapoiting result and, in fact, it suggests an important thing about the behavior of the interface as $\varepsilon \downarrow 0$. Namely, if some stationary pattern of (1.2) exists up to $\varepsilon = 0$, but does not have a smooth limiting interface, then the configuration of the interface must become fine and complicated as $\varepsilon \downarrow 0$. In order to understand the morphology of the complicated patterns, it seems necessary to apply an appropriate rescaling to blow up the degenerate situation since there are no well-defined asymptotic limit of interfaces in the original framework. The study in this direction is now currently being done and will be reported elsewhere. The outline of this note is as follows. In Section 2 we shall display the expansion of ε-family of matched asymptotic solutions. In section 3 we shall derive $(v_0)_y = constant$ along the interface, which is a byproduct of the analysis of the formal matched asymptotic expansions of the eigenvalue problem of Allen-Cahn operator associated with the original system (1.2). Based on these observations we shall prove the Main Theorem in Section 4. The key ingredient is the Serrin's result [11] (and its generalizations) for the over-determined Poisson equation.

2.Matched asymptotic expansion of singularly perturbed stationary solutions

In this section, we summarize the hypothesis for the stationary solutions which have interior transition layers, especially, a precise definition of matched asymptotic expansion of them is presented. We assume that there exist an ε-family of smooth stationay solutions $(U^\varepsilon(x), V^\varepsilon(x))$ to (1.2) with interior transition layers such that the interface Γ^ε:

$$\Gamma^\varepsilon \equiv \{x \in \Omega | U^\varepsilon(x) = \frac{1}{2}(h_+(v^*) + h_-(v^*))\} \tag{2.1}$$

is a smooth simple closed curve in $\mathbf{R}^2$ and have a definite limit Γ_0 with same properties as $\varepsilon \downarrow 0$, where $u_\pm$ are two stable branches of $f(u, v) = 0$ (see [7]). Let Ω_0^+ be the region surrounded by Γ_0 and $\Omega_0^- \equiv \Omega \backslash \bar{\Omega}_0^+$. In some tubular neighbourhood $T(\Gamma_0)$ of Γ_0, local coordinate system (s, y) is defined and for $x \in T(\Gamma_0)$

$$x = \Gamma_0(s(x)) + y(x)\nu(s(x)) \tag{2.2}$$

holds, where $s(x)$ is the parameter measuring the arclength along Γ_0 to the point on Γ_0 closest to x, $\nu(s)$ is outward unit normal vector at s, and $y(x)$ is signed distance from x to Γ_0 that is positive if $x \in \Omega_0^-$. We use the notation $u(s,y)$ for the representation of $u(x)$ by the local coordinate system. Using this local coordinate system, Γ^ε can be expanded into

$$\Gamma^\varepsilon = \Gamma_0(s) + \gamma(s,\varepsilon)\nu(s), \quad \gamma(s,\varepsilon) = \sum_{k=1}^{m} \gamma_k(s)\varepsilon^k + \varepsilon^m \hat{\gamma}_{m+1}(s,\varepsilon). \tag{2.3}$$

$(U^\varepsilon(x), V^\varepsilon(x))$ is called an *ε-family of matched asymptotic solutions* when it has the following expansion (2.4) (matched asymptotic expansion (MAE) procedure). Roughly speaking, $(U^\varepsilon(x), V^\varepsilon(x))$ is expanded separately in two regions $\Omega_\varepsilon^\pm$ and they are matched smoothly at Γ^ε. More precisely we have

$$\begin{cases} U_\pm^\varepsilon(x) = \sum_{k=0}^{m} u_k^\pm(x)\varepsilon^k + \Phi_m^\pm(x,\varepsilon) + \varepsilon^m R^\pm(x,\varepsilon) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega_\varepsilon^\pm \\ V_\pm^\varepsilon(x) = \sum_{k=0}^{m} v_k^\pm(x)\varepsilon^k + \varepsilon^2\Psi_m^\pm(x,\varepsilon) + \varepsilon^m S^\pm(x,\varepsilon) \end{cases} \tag{2.4}$$

where

$$\Phi_m^\pm(x,\varepsilon) = \begin{cases} \omega\left(\frac{y(x)-\gamma(s,\varepsilon)}{d}\right) \sum_{n=0}^{m} \phi_n^\pm\left(s(x), \frac{y(x)-\gamma(s,\varepsilon)}{\varepsilon}\right)\varepsilon^k, \\ \qquad\qquad |y(x)-\gamma(s,\varepsilon)| \le d, \\ 0, \qquad\qquad |y(x)-\gamma(s,\varepsilon)| > d, \end{cases} \tag{2.5}$$

$$\Psi_m^\pm(x,\varepsilon) = \begin{cases} \omega\left(\frac{y(x)-\gamma(s,\varepsilon)}{d}\right) \sum_{k=0}^{m-2} \psi_k^\pm\left(s(x), \frac{y(x)-\gamma(s,\varepsilon)}{\varepsilon}\right)\varepsilon^k, \\ \qquad\qquad |y(x)-\gamma(s,\varepsilon)| \le d, \\ 0, \qquad\qquad |y(x)-\gamma(s,\varepsilon)| > d, \end{cases} \tag{2.6}$$

$\omega(\tau) \in C^\infty(\mathbf{R})$ is a cut off function such that

$$\omega(\tau) = 1 \text{ for } |\tau| \le \frac{1}{2}, \omega(\tau) = 0 \text{ for } |\tau| \ge 1, 0 \le \omega(\tau) \le 1, \tag{2.7}$$

$d > 0$ is some small constant, and $R(x,\varepsilon)$ and $S(x,\varepsilon)$ are remainders. $\phi_k^\pm$ and $\psi_k^\pm$ are functions of s and ξ, and ξ is stretched variable $\xi \equiv (y - \gamma(s,\varepsilon))/\varepsilon$. The coefficients $u_k^\pm$, $v_k^\pm$, $\phi_k^\pm$, and $\psi_k^\pm$ satisfy some equations and relations. We can obtain these equations by making outer and inner expansions and equating the same powers of ε^k. The resulting relations are, what we call, *matching conditions* between inner and outer expansions and *C^1-matching conditions* between

$(U_+^\varepsilon, V_+^\varepsilon)$ and $(U_-^\varepsilon, V_-^\varepsilon)$. Let $\beta_\varepsilon(s) = v^* + \sum_{k=1}^m \beta_k(s)\varepsilon^k + \varepsilon^m \hat{\beta}_{m+1}(s,\varepsilon)$ be the value of $V^\varepsilon(x)$ on Γ^ε, Ω_ε^+ be the region surrounded by Γ^ε, and $\Omega_\varepsilon^- \equiv \Omega \backslash \bar{\Omega}_\varepsilon^+$.

We display the equations and the relations up to order $O(\varepsilon)$. Higher order terms are given by the same procedures.

$O(\varepsilon^0)$:

$$\text{(2.8)} \qquad \begin{cases} u_0^\pm = h_\pm(v_0^\pm) \\ D\Delta v_0^\pm + g(h_\pm(v_0^\pm), v_0^\pm) = 0 \end{cases} \qquad x \in \Omega_0^\pm$$

$$\text{(2.9)} \qquad v_0^\pm(s,0) = v^*, \quad \frac{\partial v_0^-}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$\text{(2.10)} \qquad (v_0^+)_y(s,0) = (v_0^-)_y(s,0)$$

$$\text{(2.11)} \qquad \begin{aligned} &\phi_0^\pm(s,0) = \tfrac{1}{2}(h_+(v^*) + h_-(v^*)) - u_0^\pm(s,0), \quad \phi_0^\pm(s,\mp\infty) = 0 \\ &\psi_0^\pm(s,\mp\infty) = 0 = \dot{\psi}_0^\pm(s,\mp\infty) \end{aligned}$$

$$\text{(2.12)} \qquad \dot{\phi}_0^+(s,0) = \dot{\phi}_0^-(s,0)$$

$$\text{(2.13)} \qquad \begin{cases} \ddot{\phi}_0^\pm + f(h_\pm(v^*) + \phi_0^\pm, v^*) = 0 \\ D\ddot{\psi}_0^\pm = g(h_\pm(v^*), v^*) - g(h_\pm(v^*) + \phi_0^\pm, v^*) \end{cases} \qquad \xi \in I^\pm, \quad 0 \le s \le \ell$$

$O(\varepsilon^1)$:

$$\text{(2.14)} \qquad \begin{cases} f_u^{0\pm} u_1^\pm + f_v^{0\pm} v_1^\pm = 0 \\ D\Delta v_1^\pm + g_u^{0\pm} u_1^\pm + g_v^{0\pm} v_1^\pm = 0 \end{cases} \qquad x \in \Omega_0^\pm$$

$$\text{(2.15)} \qquad v_1^\pm(s,0) = \beta_1(s) - (v_0^\pm)_y(s,0)\gamma_1(s), \quad \frac{\partial v_1^-}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$\text{(2.16)} \qquad \begin{aligned} &(v_0^+)_{yy}(s,0)\gamma_1(s) + (v_1^+)_y(s,0) + \dot{\psi}_0^+(s,0) = \\ &\qquad (v_0^-)_{yy}(s,0)\gamma_1(s) + (v_1^-)_y(s,0) + \dot{\psi}_0^-(s,0) \end{aligned}$$

$$\text{(2.17)} \qquad \begin{cases} \ddot{\phi}_1^\pm + \tilde{f}_u^{0\pm}\phi_1^\pm = -D_1\phi_0^\pm - \tilde{F}^{1\pm} \\ D\ddot{\psi}_1^\pm = -DD_1\phi_0^\pm - E^{1\pm}(s,\xi) - \tilde{G}^{1\pm}, \end{cases} \qquad \xi \in I^\pm, \quad 0 \le s \le \ell$$

$$\text{(2.18)}\quad \begin{aligned} &\phi_1^{\pm}(s,0) = -(u_0^{\pm})_y(s,0)\gamma_1(s) - u_1^{\pm}(s,0), \quad \phi_1^{\pm}(s,\mp\infty) = 0, \\ &\psi_1^{\pm}(s,\mp\infty) = 0 = \dot{\psi}_1^{\pm}(s,\mp\infty) \end{aligned}$$

$$\text{(2.19)}\quad (u_0^{+})_y(s,0) + \dot{\phi}_1^{+}(s,0) = (u_0^{-})_y(s,0) + \dot{\phi}_1^{-}(s,0),$$

where $\cdot = \frac{\partial}{\partial\xi}$, $I^+ \equiv (-\infty,0)$, $I^- \equiv (0,\infty)$, ℓ is the total arclength of Γ_0, $f_u^{0\pm} \equiv \frac{\partial}{\partial u} f(u_0^{\pm}, v_0^{\pm})$, $\tilde{f}_u^{0\pm} \equiv \frac{\partial}{\partial u} f(h_{\pm}(v^*) + \phi_0^{\pm}, v^*)$, and D_k $(k = 0,1,\cdots m)$ are the coefficients of the expansion of Laplacian Δ in the local coordinate system (s,ξ);

$$\text{(2.20)}\quad \Delta_x \equiv \frac{1}{\varepsilon^2}\sum_{k=0}^{m} \varepsilon^k D_k.$$

Here D_k is at most second order differential operator in s and ξ. For example,

$$\text{(2.21)}\quad \begin{aligned} &D_0 \equiv \frac{\partial^2}{\partial\xi^2}, \qquad D_1 \equiv -\kappa(s)\frac{\partial}{\partial\xi}, \qquad D_2 \equiv \Lambda_0 - \kappa(s)^2(\xi + \gamma_1(s))\frac{\partial}{\partial\xi}, \\ &\Lambda_0 \equiv \frac{\partial^2}{\partial s^2} - 2\gamma_1'\frac{\partial^2}{\partial\xi\partial s} - \gamma_1''\frac{\partial}{\partial\xi} + (\gamma_1')^2\frac{\partial^2}{\partial\xi^2}, \end{aligned}$$

where $' = \frac{\partial}{\partial s}$ and $\kappa(s)$ is curvature of $\Gamma_0(s)$. $\tilde{F}^{k\pm}$, $\tilde{G}^{k\pm}$, and $E^{k\pm}$ are defined by

$$\text{(2.22)}\quad \begin{aligned} \tilde{F}^{k\pm}(s,\xi) \equiv \frac{1}{k!}\frac{d^k}{d\varepsilon^k} f\Bigg(&\sum_{i=0}^{m} u_i^{\pm}(s,\varepsilon\xi + \gamma(s,\varepsilon))\varepsilon^i + \sum_{i=0}^{m}\phi_i^{\pm}(s,\xi)\varepsilon^i, \\ &\sum_{i=0}^{m} v_i^{\pm}(s,\varepsilon\xi + \gamma(s,\varepsilon))\varepsilon^i + \varepsilon^2\sum_{i=0}^{m-2}\psi_i^{\pm}(s,\xi)\varepsilon^i\Bigg)\Bigg|_{\varepsilon=0} \end{aligned}$$

$$\text{(2.23)}\quad \begin{aligned} \tilde{G}^{k\pm}(s,\xi) \equiv \frac{1}{k!}\frac{d^k}{d\varepsilon^k} g\Bigg(&\sum_{i=0}^{m} u_i^{\pm}(s,\varepsilon\xi + \gamma(s,\varepsilon))\varepsilon^i + \sum_{i=0}^{m}\phi_i^{\pm}(s,\xi)\varepsilon^i, \\ &\sum_{i=0}^{m} v_i^{\pm}(s,\varepsilon\xi + \gamma(s,\varepsilon))\varepsilon^i + \varepsilon^2\sum_{i=0}^{m-2}\psi_i^{\pm}(s,\xi)\varepsilon^i\Bigg)\Bigg|_{\varepsilon=0} \end{aligned}$$

$$\text{(2.24)}\quad E^{k\pm}(s,\xi) \equiv \frac{1}{k!}\frac{d^k}{d\varepsilon^k}\left[D\Delta\left(\sum_{i=0}^{m} v_i^{\pm}\varepsilon^i\right)(s,\varepsilon\xi + \gamma(s,\varepsilon))\right]\Bigg|_{\varepsilon=0}$$

Here we present the precise form of $\tilde{F}^{1\pm}$ for later use in Section 3;

$$\text{(2.25)}\quad \begin{aligned} \tilde{F}^{1\pm} \equiv &\{\tilde{f}_u^{0\pm}(u_0^{\pm})_y(s,0) + \tilde{f}_v^{0\pm}(v_0^{\pm})_y(s,0)\}(\gamma_1(s) + \xi) \\ &- \tilde{f}_u^{0\pm}u_1^{\pm}(s,0) - \tilde{f}_v^{0\pm}v_1^{\pm}(s,0). \end{aligned}$$

We call the second equation of (2.8) with (2.9) and (2.10) the *reduced problem*, that is

$$(2.26)\qquad \begin{cases} D\Delta v_0^{\pm} + g(h_{\pm}(v_0^{\pm}), v_0^{\pm}) = 0, & \text{in } \Omega_0^{\pm} \\ v_0^{\pm}(s,0) = v^*, \quad \dfrac{\partial v_0^-}{\partial n} = 0 \text{ on } \partial\Omega, & \\ (v_0^+)_y(s,0) = (v_0^-)_y(s,0) & \end{cases}$$

3. Asymptotic formula for critical eigenvalues of Allen-Cahn operator associated with Activator-Inhibitor systems

In this section, we consider the following eigenvalue problem

$$(3.1)\qquad \begin{cases} L^{\varepsilon}w = \lambda w & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

where

$$(3.2)\qquad L^{\varepsilon} \equiv \varepsilon^2\Delta + f_u^{\varepsilon},$$

$f_u^{\varepsilon} = \frac{\partial}{\partial u} f(U^{\varepsilon}, V^{\varepsilon})$, and $(U^{\varepsilon}, V^{\varepsilon})$ is a stationary solution of (1.2) having the expansion as in Section 2. The information on the spectral behavior of (3.1) is basic for the study of the stability of $(U^{\varepsilon}, V^{\varepsilon})$. Especially the behavior of critical eigenvalues (i.e., those which tend to zero as $\varepsilon \downarrow 0$) play the key role to investigate the spectrum of the linearized problem for the full system. However in this short note, we only focus on the asymptotic expansions of eigen-pairs of (3.1) and their matching conditions, which gives us an important result Proposition 3.4 to prove the Main theorem.

Our approach is the matched asymptotic expansion method. We divide (3.1) into two problems as follows;

$$(3.3)_-\qquad \begin{cases} \varepsilon^2\Delta w^- + f_u^{\varepsilon}w^- = \lambda^{\varepsilon}w^- & \text{in } \Omega_{\varepsilon}^-, \\ \dfrac{\partial w^-}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad w^- = \Theta^{\varepsilon} & \text{on } \Gamma^{\varepsilon}. \end{cases}$$

$$(3.3)_+\qquad \begin{cases} \varepsilon^2\Delta w^+ + f_u^{\varepsilon}w^+ = \lambda^{\varepsilon}w^+ & \text{in } \Omega_{\varepsilon}^+, \\ w^+ = \Theta^{\varepsilon} \quad \text{on } \Gamma^{\varepsilon}, & \end{cases}$$

where

$$(3.4)\qquad \Theta^{\varepsilon} \equiv \sum_{k=0}^{m} \varepsilon^k\Theta_k(s), \qquad \lambda^{\varepsilon} \equiv \sum_{k=1}^{m} \varepsilon^k\lambda_k.$$

$\Omega_\varepsilon^\pm$ and Γ^ε are defined in Section 2. Θ^ε and λ^ε are determined by C^1-matching condition of w_-^ε and w_+^ε on Γ^ε.

OUTER EXPANSION Let

$$w^\pm = \sum_{k=0}^m \varepsilon^k w_k^\pm(x) \quad \text{and} \quad f_u^{\varepsilon\pm} = \sum_{k=0}^m \varepsilon^k F_u^{k\pm} \tag{3.5}$$

where

$$F_u^{k\pm} \equiv \frac{1}{k!}\frac{d^k}{d\varepsilon^k} f_u\left(\sum_{i=0}^m \varepsilon^i u_i^\pm(x), \sum_{i=0}^m \varepsilon^i v_i^\pm(x)\right)\Bigg|_{\varepsilon=0}. \tag{3.6}$$

Substituting (3.5) and (3.6) into $(3.3)_\pm$, we have

$$\sum_{k=2}^m \varepsilon^k \Delta w_{k-2}^\pm + \sum_{k=0}^m\left(\varepsilon^k \sum_{i+j=k} F_u^{i\pm} w_j^\pm\right) = \sum_{k=1}^m\left(\varepsilon^k \sum_{i+j=k, i\geq 1} \lambda_i w_j^\pm\right).$$

Equating like power of ε^k, we obtain the following equations:

$$k=0: \quad F_u^{0\pm} w_0^\pm = 0,$$

$$k=1: \quad F_u^{0\pm} w_1^\pm + F_u^{1\pm} w_0^\pm = \lambda_1 w_0^\pm,$$

$$2\leq k\leq m: \quad \Delta w_{k-2}^\pm + \sum_{i+j=k} F_u^{i\pm} w_j^\pm = \sum_{i+j=k, i\geq 1}\lambda_i w_j^\pm.$$

By using induction arguments, we see $w_k^\pm \equiv 0 \quad (0\leq k\leq m)$.

INNER EXPANSION We introduce the stretched variable ξ in the neighbourhood of Γ^ε as in Section 2. Substituting

$$w^\pm = \sum_{k=0}^m \varepsilon^k \zeta_k^\pm(s,\xi) \quad \text{and} \quad f_u^{\varepsilon\pm} = \sum_{k=0}^m \tilde{F}_u^{k\pm}\varepsilon^k$$

into $(3.3)_\pm$

$$\sum_{k=0}^m\left(\varepsilon^k \sum_{i+j=k} D_i\zeta_j^\pm\right) + \sum_{k=0}^m\left(\varepsilon^k \sum_{i+j=k}\tilde{F}_u^{i\pm}\zeta_j^\pm\right) = \sum_{k=1}^m\left(\varepsilon^k \sum_{i+j=k, i\geq 1}\lambda_i\zeta_j^\pm\right),$$

where

$$\tilde{F}_u^{k\pm} \equiv \frac{1}{k!}\frac{d^k}{d\varepsilon^k} f_u\left(\sum_{i=0}^m \varepsilon^i u_i^\pm(s,\varepsilon\xi+\gamma(s,\varepsilon)) + \sum_{i=0}^m \varepsilon^i\phi_i^\pm(s,\xi),\right.$$

$$\left.\sum_{i=0}^m \varepsilon^i v_i^\pm(s,\varepsilon\xi+\gamma(s,\varepsilon)) + \sum_{i=2}^m \varepsilon^i\psi_i^\pm(s,\xi)\right)\Bigg|_{\varepsilon=0}.$$

Equating like powers of ε^k, we have the following problems for $\zeta_k^{\pm}$.

$$(3.7)\qquad \begin{cases} \ddot{\zeta}_0^{\pm} + \tilde{F}_u^{0\pm}\zeta_0^{\pm} = 0, \\ \zeta_0^{\pm}(s, \pm\infty) = 0, \qquad \zeta_0^{\pm}(s, 0) = \Theta_0(s), \end{cases} \quad \xi \in I^{\pm},\ 0 \le s \le \ell$$

$$(3.8)\qquad \begin{cases} \ddot{\zeta}_k^{\pm} + \tilde{F}_u^{0}\zeta_k^{\pm} = \sum_{i=1}^{k} R_i^{\pm}\zeta_{k-i}^{\pm}, \\ \zeta_k^{\pm}(s, \pm\infty) = 0, \qquad \zeta_k^{\pm}(s, 0) = \Theta_k(s), \end{cases} \quad \xi \in I^{\pm},\ 0 \le s \le \ell$$

where

$$(3.9)\qquad R_i^{\pm}(s, \xi) \equiv \lambda_i - D_i - \tilde{F}_u^{i\pm}.$$

By the similar argument in Fife [1], we can see that the inhomogeneous terms of (3.8) and their derivatives of any order with respect to s and ξ decay exponentially as $|\xi| \to \infty$. Then noting that $\dot{\phi}_0^{\pm}$ (> 0) are fundamental solutions of (3.7) and (3.8), the solutions of them are given by

$$(3.10)\qquad \zeta_0^{\pm} = \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\Theta_0(s)$$

$$(3.11)\qquad \begin{aligned} \zeta_k^{\pm}(s, \xi) &= \tfrac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\Theta_k(s) + \dot{\phi}_0(\xi)\textstyle\int_0^{\xi}(\dot{\phi}_0(t))^{-2} \\ &\times \textstyle\int_{\mp\infty}^{t}\left(\sum_{i=1}^{k}\dot{\phi}_0(\tau)R_i^{\pm}(s, \tau)\zeta_{k-i}^{\pm}(s, \tau)\right) d\tau dt, \quad (1 \le k \le m). \end{aligned}$$

Finally we define the formal approximation to the solution $W_m^{\pm}(x)$ of $(3.3)_{\pm}$. We take a sufficiently small $\varepsilon_0 > 0$ and let $\kappa_\varepsilon(s)$ be the curvature of Γ^ε. Define $d > 0$ by

$$\max_{0 \le s \le \ell, 0 \le \varepsilon \le \varepsilon_0} |\kappa_\varepsilon(s)| \le \frac{1}{2d}$$

and $W_m^{\pm}(x)$ on $\Omega_\varepsilon^{\pm}$ by

$$(3.12)\qquad W_m^{\pm}(x) = \begin{cases} \omega\left(\frac{y(x) - \gamma(s;\varepsilon)}{d}\right) \sum_{k=0}^{m} \varepsilon^k \zeta_k^{\pm}(s, \dfrac{y(x) - \gamma(s;\varepsilon)}{\varepsilon}), \\ \qquad\qquad |y(x) - \gamma(s;\varepsilon)| \le d, \\ 0, \qquad\qquad |y(x) - \gamma(s;\varepsilon)| > d, \end{cases}$$

LEMMA 3.1. *There exist $K > 0$ independent of ε, such that for all $x \in \Omega^{\pm}$ and sufficiently small ε,*

$$|(\varepsilon^2\Delta + f_u^\varepsilon - \lambda^\varepsilon)[W_m^{\pm}]| \le K\varepsilon^{m+1}$$

holds.

In order to determine Θ^ε and λ^ε, $W_m^{\pm}$ must satisfy the C^1-matching conditions

$$\varepsilon(W_m^+)_y(s,\gamma(s,\varepsilon)) - \varepsilon(W_m^-)_y(s,\gamma(s,\varepsilon)) = O(\varepsilon^{m+1}), \tag{3.13}$$

which can be rewritten as

$$\varepsilon(W_m^+)_y(s,\gamma(s,\varepsilon)) - \varepsilon(W_m^-)_y(s,\gamma(s,\varepsilon)) = \sum_{k=0}^{m} \varepsilon^k \left\{\dot\zeta_k^+(s,0) - \dot\zeta_k^-(s,0)\right\} + O(\varepsilon^{m+1}). \tag{3.14}$$

Noting (2.12), we see that $\dot\zeta_0^+(s,0) - \dot\zeta_0^-(s,0) = 0$ is already satisfied. For (3.14), we see that

LEMMA 3.2. *Each C^1-matching condition*

$$\dot\zeta_k^+(s,0) - \dot\zeta_k^-(s,0) = 0 \qquad (1 \le k \le m) \tag{3.15}$$

is equivalent to the formal solvability condition for (3.8):

$$\left\langle \sum_{i=1}^{k} \hat R_i(s,\xi)\hat\zeta_{k-i}, \dot\phi_0(\xi) \right\rangle_\xi = 0, \tag{$(MC)_k$}$$

where $\hat R_i(s,\xi)$ and $\hat\zeta_i(s,\xi)$ are

$$\hat R_i(s,\xi) = \begin{cases} R_i^+(s,\xi) & \xi \in (-\infty,0) \\ R_i^-(s,\xi) & \xi \in [0,\infty) \end{cases}$$

and

$$\hat\zeta_i(s,\xi) = \begin{cases} \zeta_i^+(s,\xi) & \xi \in (-\infty,0) \\ \zeta_i^-(s,\xi) & \xi \in [0,\infty) \end{cases}$$

respectively, and $\langle\cdot,\cdot\rangle_\xi$ denotes L^2-inner product with respect to ξ.

PROOF. Differentiating (3.11) with respect to ξ at $\xi = 0$, we have

$$\dot\zeta_k^{\pm}(0,\xi) = \frac{\ddot\phi_0(\xi)}{\dot\phi_0(0)}\Theta_k(s) + \frac{1}{\dot\phi_0(0)}\sum_{i=1}^{k}\int_{\mp\infty}^{0} \dot\phi_0(\tau)R_i^{\pm}(s,\tau)\zeta_{k-i}^{\pm}(s,\tau)d\tau dt.$$

Then,

$$0 = \dot{\zeta}_k^+(s,0) - \dot{\zeta}_k^-(s,0) = \tfrac{1}{\dot{\phi}_0(0)} \Big[\int_{-\infty}^0 \sum_{i=1}^k \dot{\phi}_0(\tau) R_i^+(s,\tau)\zeta_{k-i}^+(s,\tau)d\tau dt$$

$$- \int_{+\infty}^0 \sum_{i=1}^k \dot{\phi}_0(\tau) R_i^-(s,\tau)\zeta_{k-i}^-(s,\tau)d\tau dt\Big] = \left\langle \sum_{i=1}^k \hat{R}_i(s,\xi)\hat{\zeta}_{k-i}, \dot{\phi}_0(\xi) \right\rangle_\xi .$$

In the following of this section, we study how Θ^ε and λ^ε are determined. When $k = 1$, we have the following result.

LEMMA 3.3. *λ_1 is determined by $(MC)_1$ and is given by*

$$\lambda_1 = (v_0)_y(s,0) \int_{h_-(\beta^*)}^{h_+(\beta^*)} f_v(u,\beta^*)du \Big/ \int_{-\infty}^{+\infty} (\dot{\phi}_0(t))^2 dt \tag{3.16}$$

PROOF. $\zeta_1^\pm$ satisfies

$$\ddot{\zeta}_1^\pm + \tilde{F}_u^{0\pm}\zeta_1^\pm = \frac{\Theta_0}{\dot{\phi}_0(0)}[\lambda_1\dot{\phi}_0^\pm + H^\pm], \qquad H^\pm \equiv \kappa\ddot{\phi}_0 - \tilde{F}_u^{1\pm}\dot{\phi}_0$$

and $(MC)_1$ is rewritten as

$$\dot{\zeta}_1^+(s,0) - \dot{\zeta}_1^-(s,0) = \frac{\Theta_0(s)}{(\dot{\phi}_0(0))^2}\left[\lambda_1 \int_{-\infty}^{+\infty} (\dot{\phi}_0(t))^2 dt\right.$$

$$\left. + \int_{-\infty}^0 H^+\dot{\phi}_0(t)dt + \int_0^{+\infty} H^-\dot{\phi}_0(t)dt\right] = 0.$$

We see that $r^\pm \equiv \dot{\phi}_1$ satisfy the next equation;

$$\ddot{r}^\pm + \tilde{F}_u^{0\pm} r^\pm = \Omega^\pm$$

$$\Omega^\pm(s,\xi) \equiv H^\pm(s,\xi) - \{(u_0^\pm)_y(s,0)\tilde{f}_u^{0\pm} + (v_0^\pm)_y(s,0)\tilde{f}_v^{0\pm}\}.$$

This can be solved as

$$r^\pm(s,\xi) = \frac{r^\pm(0)}{\dot{\phi}_0(0)}\dot{\phi}_0^\pm(\xi) + \dot{\phi}_0(\xi)\int_0^\xi (\dot{\phi}_0(t))^{-2}\int_{\mp\infty}^t \Omega^\pm(s,\tau)\dot{\phi}_0(\tau)d\tau dt. \tag{3.17}$$

Differentiating (3.17) with respect to ξ at $\xi = 0$, we have

$$\dot{r}^\pm(s,0) = \frac{r^\pm(0)}{\dot{\phi}_0(0)}\ddot{\phi}_0^\pm(0) + \frac{1}{\dot{\phi}_0(0)}\int_{\mp\infty}^0 \Omega^\pm(s,\tau)\dot{\phi}_0(\tau)d\tau. \tag{3.18}$$

Using

$$\int_{\mp\infty}^{0} \tilde{f}_u^{0\pm}\dot{\phi}_0(\tau)d\tau = \ddot{\phi}_0^{\pm}(0), \qquad \int_{\mp\infty}^{0} \tilde{f}_v^{0\pm}\dot{\phi}_0(\tau)d\tau = \int_{h_\pm(\beta_0^*)}^{\alpha} f_v(u,\beta_0^*)du$$

and after some computation, (3.18) can be rewritten as

$$\int_{\mp\infty}^{0} H^{\pm}(s,\tau)\dot{\phi}_0(\tau)d\tau = \ddot{\phi}_1^{\pm}(s,0)\dot{\phi}_0(0) - \ddot{\phi}_0(0)\{\dot{\phi}_1^{\pm}(s,0) + (u_0^{\pm})_y(s,0)\}$$

$$+(v_0^{\pm})_y(s,0)\int_{h_\pm(\beta_0^*)}^{\alpha} f_v(u,\beta_0^*)du,$$

where $\alpha = (h_+(\beta_0^*) + h_-(\beta_0^*))/2$. Moreover, noting that $\ddot{\phi}_1^-(s,0) = \ddot{\phi}_1^+(s,0)$, $v_0^{\pm}$ are C^1-matched on Γ_0 (see (2.10)) (so we omit the superscript $\pm$ of $(v_0)_y$) and (2.19), we have

$$\dot{\zeta}_1^+(s,0)-\dot{\zeta}_1^-(s,0) = \frac{\Theta_0(s)}{(\dot{\phi}_0(0))^2}\left[\lambda_1\int_{-\infty}^{+\infty}(\dot{\phi}_0(t))^2dt + (v_0)_y(s,0)\int_{h_+(\beta_0^*)}^{h_-(\beta_0^*)} f_v(u,\beta_0^*)du\right],$$

which implies (3.16).

Noting that the inhomogeneous terms of the equation ζ_1^+ and ζ_1^- are continuous at $\xi = 0$, once $\lambda_1$is determined, we can regard $\hat{\zeta}_1$ as C^2-solution on $\mathbb{R}$. So we omit ˆ and write ζ_1 as

$$\zeta_1(s,\xi) = \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\Theta_1(s)+\frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\left[\int_0^{\xi}(\dot{\phi}_0(t))^{-2}\int_{-\infty}^{t}\dot{\phi}_0(\tau)(R_1(s,\tau)\dot{\phi}_0(\tau))d\tau dt\right]\Theta_0(s)$$

As a corollary of Lemma 3.3, we have an important consequence.

PROPOSITION 3.4.

$$(v_0)_y \equiv \textit{constant} \quad \textit{on } \Gamma_0$$

Next, for $(MC)_2$ we have

LEMMA 3.5. *$(MC)_2$ is equivalent to the following problem;*

$$(3.19)\qquad \begin{cases} \dfrac{d^2}{ds^2}\Theta_0 + P(s)\Theta_0 = \lambda_2\Theta_0, & s \in (0,\ell), \\[2ex] \Theta_0(0) = \Theta_0(\ell), & \dfrac{d\Theta_0}{ds}(0) = \dfrac{d\Theta_0}{ds}(\ell). \end{cases}$$

where, $P(s)$ is a smooth bounded function of s.

PROOF. First note that $(MC)_2$ is rewritten as

$$(3.20)\qquad 0 = \langle R_1\zeta_1 + R_2\zeta_0, \dot{\phi}_0\rangle_\xi = \langle R_1\zeta_1, \dot{\phi}_0\rangle_\xi + \langle R_2\zeta_0, \dot{\phi}_0\rangle_\xi$$

Substituting the representation of ζ_1 into the first term of (3.20), we have

$$\langle R_1\zeta_1,\dot\phi_0\rangle_\xi = \frac{1}{\dot\phi_0(0)}\Big[\Theta_1\langle R_1\dot\phi_0,\dot\phi_0\rangle_\xi + \Theta_0\int_{-\infty}^{+\infty}\dot\phi_0(\xi)(R_1(s,\xi)\dot\phi_0(\xi))$$

$$\times\int_0^\xi(\dot\phi_0(\xi))^{-2}\int_{-\infty}^t(R_1(s,\tau)\dot\phi_0(\tau))d\tau dt d\xi\Big] = \mathcal{G}(\Theta_0),$$

where

$$\mathcal{G}(\Theta_0) \equiv \frac{\Theta_0}{\dot\phi_0(0)}\int_{-\infty}^{+\infty}\dot\phi_0(\xi)(R_1(s,\xi)\dot\phi_0(\xi))\int_0^\xi(\dot\phi_0(t))^{-2}$$

$$\times\int_{-\infty}^t\dot\phi_0(\xi)(R_1(s,\tau)\dot\phi_0(\tau))d\tau dt d\xi.$$

On the other hand, recalling (2.21), (3.9) and (3.10), the second term of (3.20) is computed as follows

$$\begin{aligned}\langle R_2\zeta_0,\dot\phi_0\rangle_\xi &= \frac{\Theta_0}{\dot\phi_0(0)}\int_{-\infty}^{+\infty}(\lambda_2-\tilde F_u^2)(\dot\phi_0(\xi))^2d\xi - \frac{1}{\dot\phi_0(0)}\int_{-\infty}^{+\infty}\Big[\dot\phi_0(\xi)\partial_{ss}\Theta_0\\
&\quad -2\gamma_1'\ddot\phi_0(\xi)\partial_s\Theta_0 - \gamma_1''\ddot\phi_0(\xi)\Theta_0 + (\gamma_1')^2(\phi_0)_{\xi\xi\xi}(\xi)\Theta_0 - \kappa^2\xi\ddot\phi_0(\xi)\Theta_0\\
&\quad -\kappa^2\gamma_1\ddot\phi_0(\xi)\Theta_0\Big]\dot\phi_0(\xi)d\xi\\
&= \frac{1}{\dot\phi_0(0)}\Big[\lambda_2\Theta_0\int_{-\infty}^{+\infty}(\dot\phi_0(\xi))^2d\xi - \partial_{ss}\Theta_0\int_{-\infty}^{+\infty}(\dot\phi_0(\xi))^2d\xi\\
&\quad -\Theta_0\Big\{\int_{-\infty}^{+\infty}\tilde F_u^2(\dot\phi_0(\xi))^2d\xi - (\gamma_1')^2\int_{-\infty}^{+\infty}(\ddot\phi_0(\xi))^2d\xi\\
&\quad +(\kappa(s))^2\int_{-\infty}^{+\infty}\xi\dot\phi_0(\xi)\ddot\phi_0(\xi)d\xi\Big\}\Big] \equiv \mathcal{F}(\Theta_0).\end{aligned}$$

Hence $(MC)_2$ is equivalent to

$$\mathcal{F}(\Theta_0) + \mathcal{G}(\Theta_0) = 0,$$

and this yields (3.19).

REMARK 3.6. *(3.19) is a Sturm-Liouville eigenvalue problem with periodic boundary condition, and the existence and asymptotic behavior of the eigenvalues and their eigenfunctions are well-studied. We denote those eigenpairs by* $\{(\hat\lambda_2^{(n)},\hat\Theta_0^{(n)})\}_{n=1}^\infty$.

By Proposition 3.5, λ_2 and Θ_0 are determined. Generally λ_k and Θ_{k-2} $(k\ge 3)$ are determined by $(MC)_k$, that is

PROPOSITION 3.7. *For each $n\ge 1$, $(MC)_k$ $(k\ge 3)$ is equivalent to the following problem*

$$\frac{d^2}{ds^2}\Theta_{k-2} + P(s)\Theta_{k-2} - \lambda_2^{(n)}\Theta_{k-2} = \Psi_k(s), \qquad 0\le s\le \ell \tag{3.21}$$

with Periodic boundary condition, where $P(s)$ is the same one appeared in Proposition 3.5 and $\Psi_k(s)$ is a smooth bounded function of s depending on $\lambda_k, \lambda_{k-1}, \cdots, \hat{\lambda}_2^{(n)}, \lambda_1, \Theta_{k-3}, \cdots, \hat{\Theta}_0^{(n)}$, and s. Then λ_k and Θ_{k-2} are determined once $\lambda_{k-1}, \cdots, \lambda_1$ and $\Theta_{k-3}, \cdots, \hat{\Theta}_0^{(n)}$ are known. In fact, λ_k is uniquely determined by the solvability condition

$$\langle \Psi_k, \hat{\Theta}_0^{(n)} \rangle_s = 0, \tag{3.22}$$

where $\langle \cdot, \cdot \rangle_s$ denotes L^2-inner product with respect to s, and then, Θ_{k-2} is also uniquely determined by (3.21) with $\langle \Theta_{k-2}, \hat{\Theta}_0^{(n)} \rangle_s = 0$.

PROOF. We prove by using induction arguments. Assume that λ_j and Θ_{j-2} $(2 \leq j \leq k)$ are known, and we determine λ_{k+1} and Θ_{k-1}. $(MC)_{k+1}$ is rewritten as

$$\begin{aligned} 0 &= \left\langle \sum_{i=1}^{k+1} R_i(s,\xi)\zeta_{k+1-i}, \dot{\phi}_0(\xi) \right\rangle_\xi \\ &= \left\langle R_1\zeta_k, \dot{\phi}_0 \right\rangle_\xi + \left\langle R_2\zeta_{k-1}, \dot{\phi}_0 \right\rangle_\xi + \left\langle \sum_{i=3}^{k+1} R_i\zeta_{k+1-i}, \dot{\phi}_0 \right\rangle_\xi . \end{aligned} \tag{3.23}$$

Then note that the third term of (3.23) does not depend on Θ_k and Θ_{k-1}, and λ_{k+1} is only involved in the third term and the coefficient of λ_{k+1} is given by $\dfrac{\phi_0(\xi)}{\dot{\phi}_0(0)}\Theta_0(s)$. Noting that the coefficients of Θ_k and Θ_{k-1} in the representation of ζ_k are $\dfrac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}$ and $\dfrac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)} \displaystyle\int_0^\xi (\dot{\phi}_0(t))^{-2} \int_{-\infty}^t \dot{\phi}_0(\tau)(R_1(s,\tau)\dot{\phi}_0(\tau))d\tau dt$, respectively, the first term of (3.23) is computed as follows:

$$\begin{aligned} \langle R_1\zeta_k, \dot{\phi}_0 \rangle_\xi &= \frac{1}{\dot{\phi}_0(0)} \Bigg[\Theta_k \langle R_1\dot{\phi}_0, \dot{\phi}_0 \rangle_\xi + \Theta_{k-1} \int_{-\infty}^{+\infty} (\dot{\phi}_0(\xi))^2 \int_0^\xi (\dot{\phi}_0(t))^{-2} \\ &\quad \times \int_{-\infty}^t \dot{\phi}_0(\tau)(R_1(s,\tau)\dot{\phi}_0(\tau))d\tau dt d\xi \Bigg] + \Sigma_{k-1}(s) = \mathcal{G}(\Theta_{k-1}) + \Sigma_{k-1}(s). \end{aligned} \tag{3.24}$$

Here, $\mathcal{G}(\cdot)$ is the same one defined in Proposition 3.5 and $\Sigma_{k-1}(s)$ is a known function depending on λ_j $(1 \leq j \leq k)$, Θ_i $(0 \leq i \leq k-2)$, its derivatives, and so on. On the other hand, noting that ζ_{k-1} is can be represented by

$$\zeta_{k-1}(s,\xi) = \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)}\Theta_{k-1}(s) + Z_{k-1}(s,\xi),$$

($Z_{k-1}(s,\xi)$ is a known function independent of Θ_{k-1} and λ_{k+1}) we have

$$\left\langle R_2\zeta_{k-1}, \dot{\phi}_0 \right\rangle_\xi = \mathcal{F}(\Theta_{k-1}) + \left\langle R_2 Z_{k-1}, \dot{\phi}_0 \right\rangle_\xi . \tag{3.25}$$

Using (3.24) and (3.25), we see that (3.23) is equivalent to (3.21). The remainder of the statement is obvious from Riesz-Schauder theory.

4 Over-determined reduced elliptic problem and the proof of Main Theorem

Formal asymptotic analysis in the previous sections tells us that the normal derivative of the C^1-matched solution v_0 at Γ_0 is constant along the interface (Prop. 3.4). On the other hand, v_0 satisfies the reduced problem (2.26) subject $v_0 = v^*$ on Γ_0. This is apparently an *over-determined* problem and only special Γ_0 and v_0 are allowable. In fact, the first part of the Main Theorem in Section 1 is clear from Theorem 2 of J. Serrin [11]. To show the second part, the following hypothesis is necessary.

HYPOTHESIS 4.1. *The nonlinear elliptic problem*

$$D\Delta v_0 + g(h_-(v_0), v_0) = 0 \qquad \textit{in} \quad \mathbf{R}^2 \backslash \Omega_0^+,$$

$$v_0 = v^* \quad \textit{and} \quad (v_0)_y \equiv \textit{constant} \qquad \textit{on} \quad \Gamma_0 = \partial\Omega_0^+$$

has only axisymmetric solutions.

Under this hypothesis it is clear that the Neumann boundary condition on $\partial\Omega$ of the reduced problem (2.26) cannot be satisfied for generic domains. This completes the proof of the second part of the Main Theorem.

References

[1] P.C.Fife, Semilinear elliptic boundary value problems with small parameters, Arch. Rational Mech. Anal., **52** (1973), 205-232.

[2] H.Hasegawa, H.Tanaka, K.Yamasaki, and T.Hashimoto, Macromolecules, **20**, 1651 (1987).

[3] T.Hashimoto, H.Tanaka, H.Hasegawa, (M.Nagasawa Ed.) *Molecular Conformation and Dynamics of Macromolecules in Condensed Systems*, Elsevier (1988).

[4] D.Hilhorst, Y.Nishiura, and M.Mimura, A free boundary problem arising in some reacting-diffusing system, Proc. Roy. Soc. Edinburgh, **118A** (1991), 355-378.

[5] Y.Nishiura, Global structure of bifurcating solutions of some reaction-diffusion systems, SIAM J. Math. Anal., **13** (1982), 555-593.

[6] Y.Nishiura, Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit, Dynamics reported, Springer-Verlag, in press.

[7] Y.Nishiura and H.Fujii, Stability of singularly perturbed solutions to systems of reaction diffusion equations, SIAM J. Math. Anal., **18** (1987), 1726-1770.

[8] Y.Nishiura and H.Suzuki, in preparation.

[9] T.Ohta and K.Kawasaki, Macromolecules, **19**, 261 (1986).

[10] T.Ohta, M.Mimura, and R.Kobayashi, Higher-dimensional localized pattern in excitable media, Physics **34** D (1989), 115-144.

[11] J.Serrin A symmetry problem in potential theory, Arch. Rational Mech. Anal., **43** (1971), 304-318.

[12] M.Taniguchi and Y.Nishiura, Instability of planar interfaces in reaction diffusion systems, SIAM J. Math. Anal., in press.

[13] E.L.Thomas, D.B.Alward, D.J.Kinning, D.C.Martin, D.L.Handlin, and Jr.,L.J.Fetters, Macromolecules, **19**, 2197 (1986).

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