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Coexistence of Infinitely Many Stable Solutions to Reaction Diffusion Systems in the Singular Limit

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1. Introduction and singular limit slow dynamics

Recognition stems from realization of the separation boundary between two different physical or chemical *states*. In other words we can observe natural phenomena through the emergence and evolution of the *interface* between these states as in solidification, combustion, chemical reaction, and biological patterns. The interface studied here results from the balance between two opposing tendencies: a *diffusive* effect and a (physical or chemical) *separation kinetics* built in the system. The former attempts to smooth out the inhomogeneity as in the heat equation, and the latter drives the system to one or the other pure state such as solid or liquid (see, for instance, Fife [19] for details). Turing's contribution [58] is one of the pioneering works related to the onset of spatial patterns through a cooperative work of diffusion and separation kinetics. Besides the existence of these two tendencies, another key ingredient to produce interesting interfacial patterns is the differences of the strength of the above mixing and unmixing effects among species involved in the system. In fact, reaction diffusion systems for two components u and v, which are the main concern in this paper, can be classified formally as

(i) There is a difference in the diffusion rates of u and v;
(ii) There is a difference in the reaction rates of u and v;
(ii) There are differences in the diffusion and reaction rates of u and v, i.e., a combination of (i) and (ii).

Steady interfacial patterns, which usually originate in the onset of symmetry breaking patterns through Turing's diffusion driven instability, are commonly observed in the first category. A typical example is an activator-inhibitor system describing morphogenetic patterns (see Meinhardt[38] and Murray [40; chapters 14 and 15]). Propagator-controller systems, including a simple skeleton model for the Belousov-Zhabotinsky reaction, lie in the second category (see, for instance, Fife [18], and Keener and Tyson [33]). Layer oscillation ("breather") is one of the characteristic phenomena in the third category (see Nishiura and Mimura [46]).

In this paper we focus on the first category and consider the following system in one-dimensional space:

$$\delta u_t = \varepsilon^2 u_{xx} + f(u, v) \tag{1.1a}$$

$$\text{in } I$$

$$v_t = D v_{xx} + g(u, v) \tag{1.1b}$$

$$u_x = 0 = v_x \qquad \text{on } \partial I, \tag{1.1c}$$

where I is the unit interval $(0, 1)$, δ denotes the ratio of reaction rates of u and v, and ε^2 and D are the diffusion coefficients of u and v, respectively. We assume that $0 < \varepsilon << 1$ but $D = O(1)$, i.e., (1.1) belongs to the first category.

Although $\delta = O(1)$ is a typical situation in this category, the main results hold for the wider regime where $\varepsilon/\delta = o(1)$ as $\varepsilon \downarrow 0$, in particular, one can take $\delta = \varepsilon^\alpha$ with $0 \le \alpha < 1$ (see Remark 2.8). The nullclines for typical f and g are drawn in Figure 1.1; $f = 0$ is of sigmoidal shape; $g = 0$ intersects with $f = 0$ transversally, and $f > 0$, $g > 0$ in lower regions of those nullclines. More precise assumptions will be stated at the end of this section.

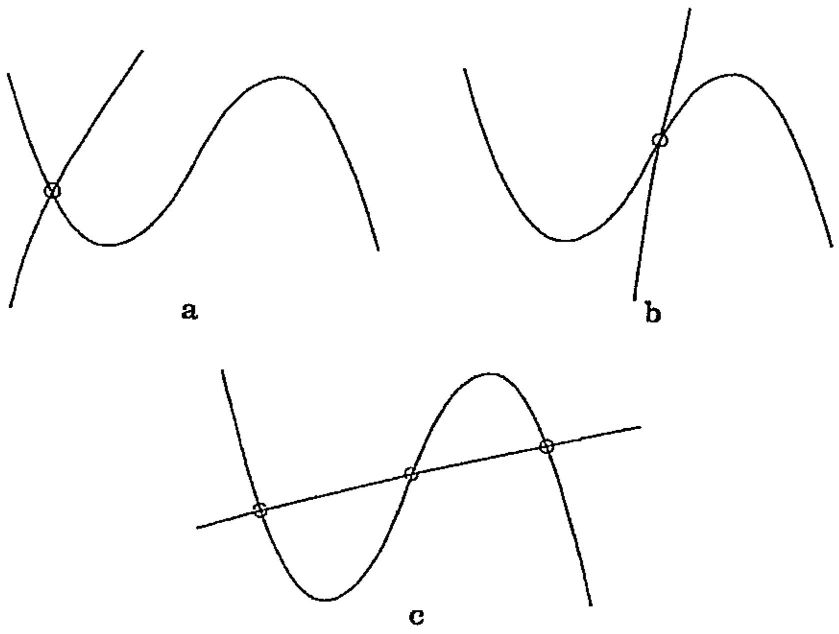

Figure 1.1.

One of the most important features of the nonlinearity is the bistable nature of f. Namely, for a fixed v, the kinetics $u_t = f(u, v)$ drives u toward either the left-end zero or the right-end zero of $f = 0$ depending on the initial data. This separating force causes the emergence of transition layers when the initial data is distributed in spatial direction (see Figure 1.2(a)). It turns out that the width of the resulting layers becomes $O(\varepsilon)$. Then these layers start to propagate slowly ($O(\varepsilon/\delta)$-speed) in some direction, but this wave can be blocked at some stage by the inhibitor v, and settles down to a steady state (see Figure 1.2(b)), since v can diffuse much faster than u and make an environment that controls the motion of u. Note that the ratio of unit time scales of Figures (a) and (b) is 3 : 40, i.e., layers propagate very slowly. Our main concern is the stability of such a layered solution obtained as the final pattern in Figure 1.2(b).

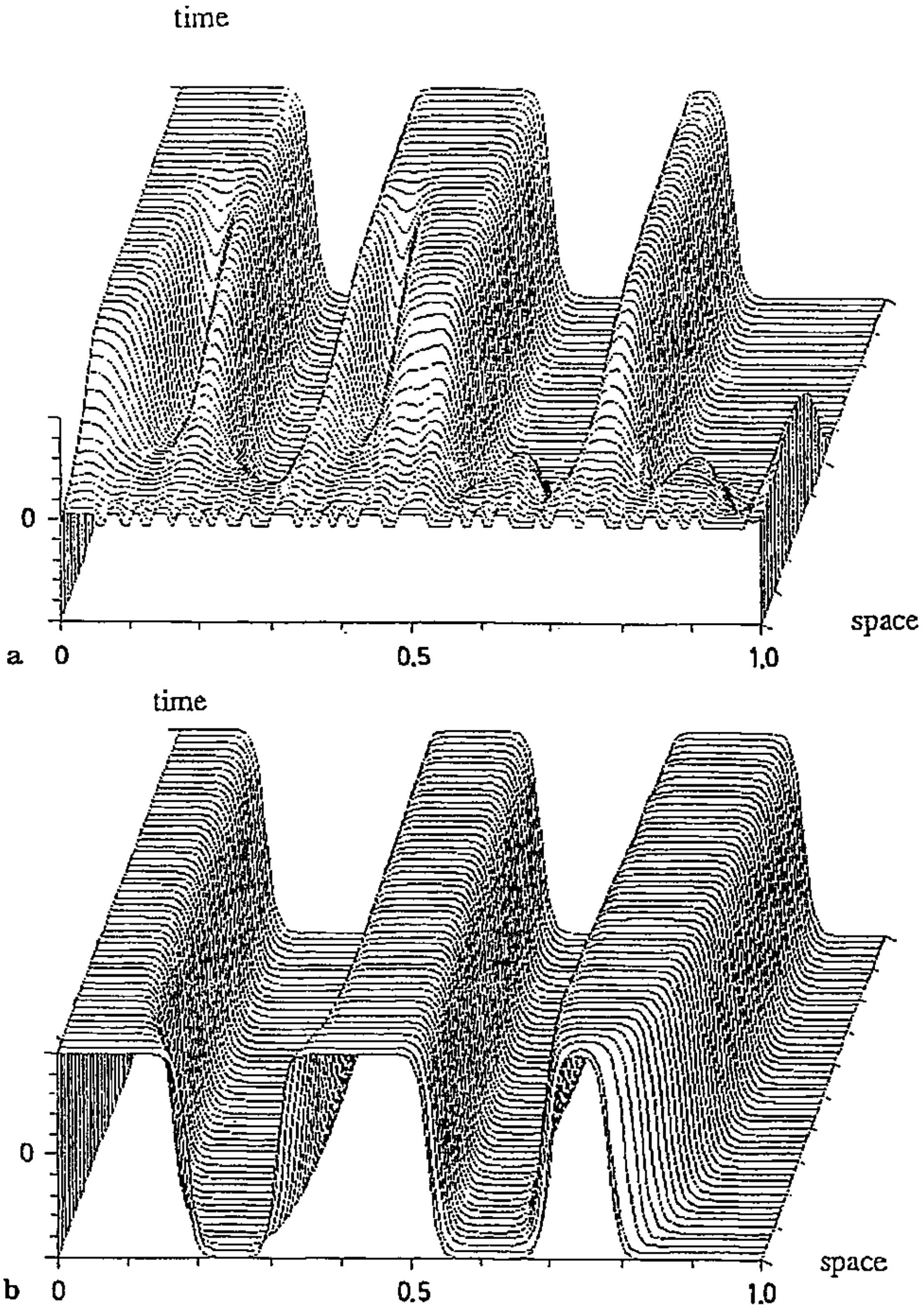

Figure 1.2.

The basic question is " How many such stable layered solutions and what do they look like? " The answer is, which is our final goal, is really remarkable, namely,

Main Theorem. *The number of non-constant stable steady states of* (1.1) *becomes "infinite " in the singular limit* $\varepsilon \downarrow 0$. *In other words, for any large number* N, *one can find an* ε_0 *such that* (1.1) *has at least* N *asymptotically stable steady states for* $0 < \varepsilon \leq \varepsilon_0$.

In fact we will prove in Section 3 that for an arbitrary number n, the *normal n-layered solution* (see Corollary 3.9) as in Figure 1.3 becomes stable for small ε. Throughout the paper we use the word "stable" in the sense of "asymptotically stable" in an appropriate norm.

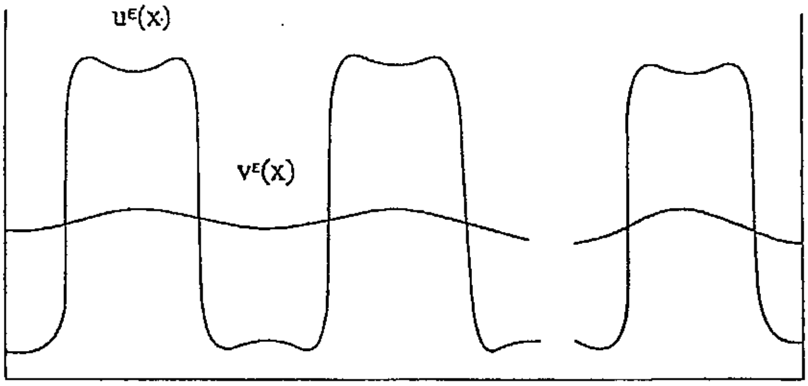

Figure 1.3.

Apparently this makes a sharp contrast with a scalar reaction diffusion equation.

$$\delta u_t = \varepsilon^2 u_{xx} + f(u), \qquad u_x = 0 \text{ on } \partial I, \tag{1.2}$$

where f is typically a cubic-like function of u. It is known (Casten and Holland [10], and Matano [37]) that any non-constant solution of (1.2) is, if it exists, *unstable*. If some constraint (for instance, mass conservation) is imposed on (1.2), then (1.2) may have a unique (up to reflection) non-constant stable solution with one internal transition layer (see Carr, Gurtin, and Slemrod [8]). However, all the remaining multi-layered solutions are unstable. A natural question is "What is the mechanism that causes the above difference between scalar and system?" It is obvious that the second component v somehow controls the behavior of u, but in what manner ?

In order to see the role of the controller v more clearly, we shall derive a *singular limit slow dynamics* from (1.1) in a heuristic way. Hereafter we assume for definitness that (f, g) is of type (b) in Figure 1.1. The dynamics of (1.1) consists of two stages with different time scales; outer dynamics (phase separation process) and then followed by the slow layer dynamics (propagation process). Let $(u_0(x), v_0(x))$ be a smooth and moderate initial data, then the diffusion term $\varepsilon^2 u_{xx}$ could be neglected for a while until u_{xx} becomes sufficiently large. The resulting *outer dynamics* is

$$\begin{aligned} \delta U_t &= f(U, V) \\ V_t &= DV_{xx} + g(U, V) \end{aligned} \tag{1.3a}$$

with

$$V_x = 0 \qquad \text{on } \partial I. \tag{1.3b}$$

If layers move slowly compared with the relaxation time of (1.3) (see (1.11)), we can expect that the solution of (1.3) approaches a steady state in regions away from layers:

$$\begin{aligned} 0 &= f(U, V) \\ 0 &= DV_{xx} + g(U, V) \end{aligned} \tag{1.4}$$

Since f is of bistable type for a fixed V as in Figure 1.1, U is attracted to either $u = h_-(v)$ or $u = h_+(v)$ branch everywhere except in neighbourhoods of finitely many points $\{\varphi_i\}_{i=1}^n$ for generic data. Hence, to the lowest degree of approximation, (1.4) can be rewritten as

$$0 = DV_{xx} + G_\Phi(V) \tag{1.5}$$

with $(1.3)_b$, where Φ denotes a collection of layer positions $\{\varphi_i\}_{i=1}^n$ ($0 < \varphi_1 < \varphi_2 < \cdots < \varphi_n < 1$), and $G_\Phi(V)$ is equal to either $G_-(V) \equiv g(h_-(V), V)$ or $G_+(V) \equiv g(h_+(V), V)$, according to the chosen branch, on each subinterval partitioned by $\{\varphi_i\}_{i=1}^n$. $G_+(V)$ (resp. $G_-(V)$) is defined for $v < \overline{v}$ (resp. $v > \underline{v}$), positive (resp. negative), and strictly decreasing (see Figure 1.4 and Remark 1.1).

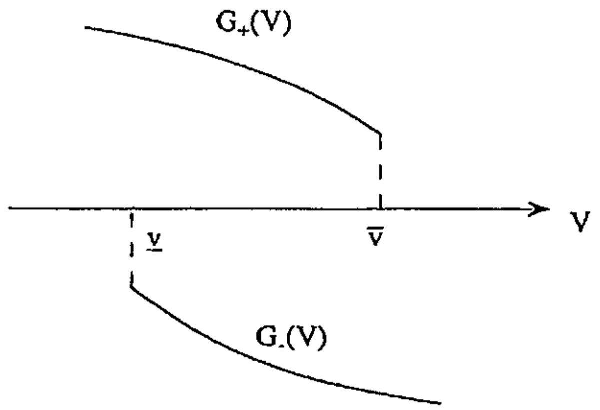

Figure 1.4.

Since G_Φ has discontinuities at layer positions, we say that V is a solution of (1.5) if and only if it satisfies (1.5) in a classical sense in each subinterval as well as (1.3b), and it is matched in C^1-sense at each layer position, i.e., the right and left limits coincide each other up to first derivatives. The associated U-component has jump discontinuities at φ_i's where two stable branches are switched. Note that (1.5), in general, does *not* have a solution for arbitrary partition Φ. In fact, even for mono-layer case ($n = 1$), φ_1 cannot be arbitrary close to boundary points 0 and 1, because $|G_\pm(V)|$ are bounded away from zero (see Theorem 2.1). On the other hand, if (f, g) is of bistable type (Figure 1.1 (c)), $G_\pm$ has a unique zero respectively, and hence φ_1 can be taken arbitrarily.

After the completion of outer dynamics, we move into the next stage: the process of layer propagation. The diffusion term $\varepsilon^2 u_{xx}$ plays an important role. Once (u, v) approaches a solution of (1.4) in outer region, it is supposed to be held rigidly there, since we assume that layers propagate slowly. Therefore we can localize our analysis in the neighbourhood of each layer position to derive the propagation dynamics. We shall introduce the following stretched coordinate and slow time scale to study the dynamics of u inside of thin layers.

Let φ be an arbitrary layer position and define a *stretched* coordinate y and a *slow* time s:

$$y \equiv \frac{x - \varphi}{\varepsilon}, \qquad s \equiv \frac{\varepsilon}{\delta} t. \tag{1.6}$$

The κ-neighbourhood $I_\kappa \equiv (\varphi - \kappa, \varphi + \kappa)$ of $x = \varphi$ is stretched to $\tilde{I}_\kappa \equiv (-\kappa/\varepsilon, \kappa/\varepsilon)$. What we want to know is the time dependence of φ, and it turns out that φ becomes a function of slow time s. In fact, noting the relation,

$$\frac{\partial}{\partial t} = \frac{\varepsilon}{\delta}\frac{\partial}{\partial s} - \frac{1}{\varepsilon}\frac{d\varphi}{dt}\frac{\partial}{\partial y} \qquad \text{and} \qquad \frac{\partial}{\partial x} = \frac{1}{\varepsilon}\frac{\partial}{\partial y},$$

the first equation of (1.1) becomes

$$\varepsilon u_s - \frac{\delta}{\varepsilon}\varphi_t u_y = u_{yy} + f(u, V(\varphi + \varepsilon y)). \tag{1.7a}$$

Here v is replaced by the solution V of (1.5). In view of the second term of the left-hand side of (1.7a), we see that it is natural to regard φ to be a function of s instead of t. Then (1.7b) becomes

$$\varepsilon u_s - \varphi_s u_y = u_{yy} + f(u, V(\varphi + \varepsilon y)). \tag{1.7b}$$

Taking a formal limit of (1.7b) as $\varepsilon \downarrow 0$, we have

$$u_{yy} + \varphi_s u_y + f(u, V(\varphi)) = 0 \qquad \text{on } \mathbf{R}. \tag{1.8a}$$

The stretched interval $\tilde{I}_\kappa$ becomes a whole line $\mathbf{R}$ in this limit and the boundary conditions become

$$u(\pm\infty) = h_\pm(V(\varphi)) \qquad (\text{ resp. } h_\mp(V(\varphi))) \tag{1.8b}$$

if the outer solution to the right of $x = \varphi$ is attracted to the branch $u = h_+(V)$ (resp. $h_-(V)$). Hereafter we focus on the former case (h_+-case). It is well-known (see, for instance, Fife and McLeod [22] and references therein) that (1.8) has a unique solution $u = u(y; V(\varphi))$ provided that

$$\varphi_s = c(V(\varphi)) \tag{1.9}$$

holds, where $c(\cdot)$ is, what is called, the *velocity function* of traveling waves for the bistable nonlinearity f. Typically c is a strictly monotone increasing function of V and has a unique zero at $v = v^*$ (see (A.2)). In fact, when f is given by

$$f(u, V) = u(1 - u)(u - V), \tag{1.10a}$$

the solution and its velocity function with $u(-\infty) = 0$ and $u(\infty) = 1$ are uniquely determined as

$$u(y) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{y}{2\sqrt{2}}\right) \tag{1.10b}$$

$$c(V) = \sqrt{2}(V - \frac{1}{2}). \tag{1.10c}$$

In terms of original time scale t, (1.9) becomes

$$\varphi_t = \frac{\varepsilon}{\delta}c(V(\varphi)), \tag{1.11}$$

which shows that each internal layer moves slowly compared with the relaxation time of the outer dynamics (1.3) so long as $\varepsilon/\delta = o(1)$ as $\varepsilon \downarrow 0$. Note that, when $\delta = O(\varepsilon)$, the motion of φ is not slow for small ε, hence the associated dynamics becomes different from (1.12) below. See the discussion in Section 5.

Summarizing the above discussion, the slow dynamics for n layers, to the lowest degree of approximation, is given by

$$(\varphi_i)_s = (-1)^{i-1} c(V(\varphi_i(s))) \tag{1.12a}$$

$$DV_{xx} + G_{\Phi_n(s)}(V) = 0 \tag{1.12b}$$

with the ordering property

$$0 < \varphi_1(s) < \varphi_2(s) < \cdots < \varphi_n(s) < 1, \tag{1.12c}$$

where $\Phi_n(s) = \{\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s)\}$ denotes the locations of n layers. Here the coefficient $(-1)^{i-1}$ of (1.12a) comes from the assumption that $G_{\Phi_n(s)}(V)$ is equal to $G_-(V)$ on the first subinterval $(0, \varphi_1(s))$. Note that (1.12b) has a *unique* solution for a given $\Phi_n(s)$ because of (1.22).

As far as the number of layers remains unchanged, (1.12) can be regarded as a system of nonlinear ODEs with respect to $(\varphi_1, \varphi_2, \cdots, \varphi_n)$. However (1.12) has several quite different features from usual ODE systems. Firstly the definition domain for $\Phi_n = (\varphi_1, \cdots, \varphi_n)$ is not a priori clear since the vector field $(1.12)_a$ is defined only on Φ_n where (1.12b) is satisfied. In fact, as was remarked earlier, the solution of (1.12) does not always exist for an arbitrary Φ_n with (1.12c). However it turns out in Section 2 that (1.12) is well-defined for any type of nonlinearity in Figure 1.1 at least in a neighbourhood of a critical point Φ_n^* defined later on, which is sufficient for the study of local stability of it. Secondly the number of unknowns may decrease when time evolves. Namely, two layers may collide with each other and *disappear* after that. This reminds us what is called the *coarsening* process in solidification theory. In Figure 1.5 there are four layers initially, however, after a finite time, two layers collide and disappear, then approach two-layered solution. Figure 1.5(a) shows the orbits of (1.12) with $n = 4$, and Figure 1.5(b) shows the u-profile of the solution to the original system (1.1) for the corresponding initial data.

Despite this singularity, the solution is well-defined even at collision points, and can be continuated after that, since the C^1-matching conditions do not break down at the hitting time. Taking into account this reduction of layer number, we see that, when we start with an *N-layered* solution, more natural definition domain for Φ_n is given by

$$\hat{\mathcal{M}}_N = \bigcup_{n=1}^{N} \mathcal{M}_n, \tag{1.13}$$

which is finite dimensional, where $\mathcal{M}_n$ is defined by

$$\mathcal{M}_n = \{\Phi_n | \text{ there exists a solution of (1.12b) for } \Phi_n \}. \tag{1.14}$$

The whole space for (1.12) is apparently given by

$$\hat{\mathcal{M}}_\infty = \bigcup_{n=1}^{\infty} \mathcal{M}_n \tag{1.15}$$

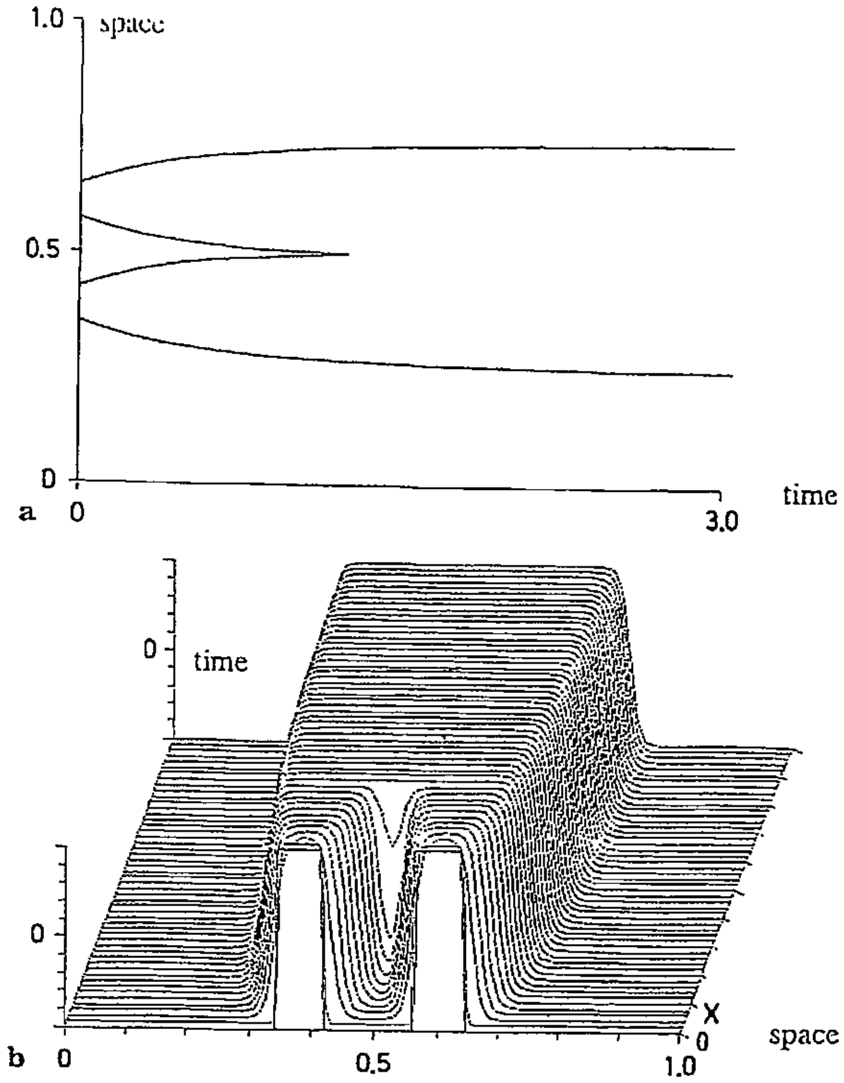

Figure 1.5.

which is an *infinite* dimensional space. Sometimes it is more convenient to consider the pair $(\Phi_n, v(\Phi))$ instead of only Φ_n where $v(\Phi)$ denotes the n-dimensional vector $(V(\varphi_1), V(\varphi_2), \cdots, V(\varphi_n))$, since the motion of each layer is determined by the value of V there. We use the same notation as before for this new definition such as

$$\mathcal{M}_n = \{(\Phi_n, v(\Phi_n)) | \text{ there exists a solution of } (1.12)_b \text{ for } \Phi_n\}. \qquad (1.16)$$

We call $\mathcal{M}_n$ the *slow manifold for n-layered solutions* and $\hat{\mathcal{M}}_\infty$ the slow manifold for (1.1). It may not be appropriate to call $\hat{\mathcal{M}}_N$, $\hat{\mathcal{M}}_\infty$ "manifold", since they are the union of manifolds of different dimension. However we abuse this terminology to call these objects. Note that the velocity of each layer is determined locally by $(1.12)_a$, however $\varphi_i(s)$ have strong linkage with each other through the relation $(1.12)_b$ for the controller V, which is a nonlocal relation of them.

It should be noted that, for a given number of layers, there is a *unique* critical point for (1.12). In fact suppose that $\Phi_n^* = (\varphi_1^*, \cdots, \varphi_n^*)$ is a critical point, then $V(\varphi_i^*) = v^*$ because of $c(v^*) = 0$. By using a phase plane analysis for $(1.12)_b$, which is glued at $V = v^*$, and the assumption $\left(\dfrac{dG_\pm}{dV} < 0\right)$, we can prove without difficulty that there is a unique orbit that rotates around $(v^*, 0)$ in the phase plane $[\frac{n}{2}]$ (resp. $[\frac{n}{2}]$ and half) times depending on n being even (resp. odd). We call this unique critical point Φ_n^* the *normal n-layered solution for the singular limit slow dynamics* (1.12). The normal layered solutions are important, since they seem to form the attractor of (1.12).

Conjecture. *After the coarsening process, any solution of (1.12) approaches one of the normal layered solutions* $\{\Phi_n^*\}_{n=1}^{\infty}$.

When the number of initial layers is less than or equal to 2, this is true (see Nishiura and Suzuki [48]).

Now we are ready to answer the question concerning the role of controller v. First we consider the scalar equation (1.2). Since there are no controllers in this case, the associated slow dynamics consists of *only* (1.12a) with $V(\varphi_i(s))$ being equal to some fixed value ξ. Recalling (1.10b), we easily see that internal layers cannot persist as a steady state, since they steadily move with constant velocity unless $\xi = 0$. Even for $\xi = 0$, they are *neutrally stable*, because arbitrary positions are equilibrium points of (1.12a). Recent progress on slow motions or metastable patterns (see Carr and Pego [9], Fusco and Hale [26], and Alikakos, Bates, and Fusco [2]) which gives us a more accurate approximation than (1.12a), shows that, when $\xi = 0$, layers are not neutrally stable but move with transcendentally small velocities. In any case layer structure cannot persist without a controller V. Then the question is how the controller V stabilizes the layer structure. The key for this lies in the slow manifold $\mathcal{M}_n$ on which interfaces move around. In order to see the role of the controller and slow manifold more clearly, we digress a little bit from (1.12) and consider intuitively how we can stabilize a mono-layered solution (Figure 1.6 (a)) of the scalar equation

$$\delta u_t = \varepsilon^2 u_{xx} + f(u, V) \tag{1.17a}$$

with f being given by (1.10a). Note that $v^* = 1/2$ in this case.

A naive way to stabilize this layer is to control the area $A = \int_I u dx$ so that it approaches neither 0 and 1. To do this, we add the following auxiliary equation to the scalar equation (1.17a) so that the *scalar* controller $V \equiv \xi$ steers u toward an internal layer solution with the assigned area A with $0 < A < 1$.

$$\frac{d\xi}{dt} = \left(\int_I u dx - A\right) - (\xi - \frac{1}{2}), \tag{1.17b}$$

Similar arguments to derive the slow dynamics (1.12) also work for this system (1.17), and the resulting one is given by

$$\varphi_s = c(\xi) \tag{1.18a}$$

$$\frac{3}{2} - \varphi - A - \xi = 0. \tag{1.18b}$$

Since the area of the corresponding u-profile is equal to $1 - \varphi$, (1.18b) is easily obtained by computing the right-hand side of (1.17b). One can regard the scalar relation (1.18b) to be a slow manifold $\mathcal{M}$ in (φ, ξ)-space. Apparently (1.18) is equivalent to the scalar

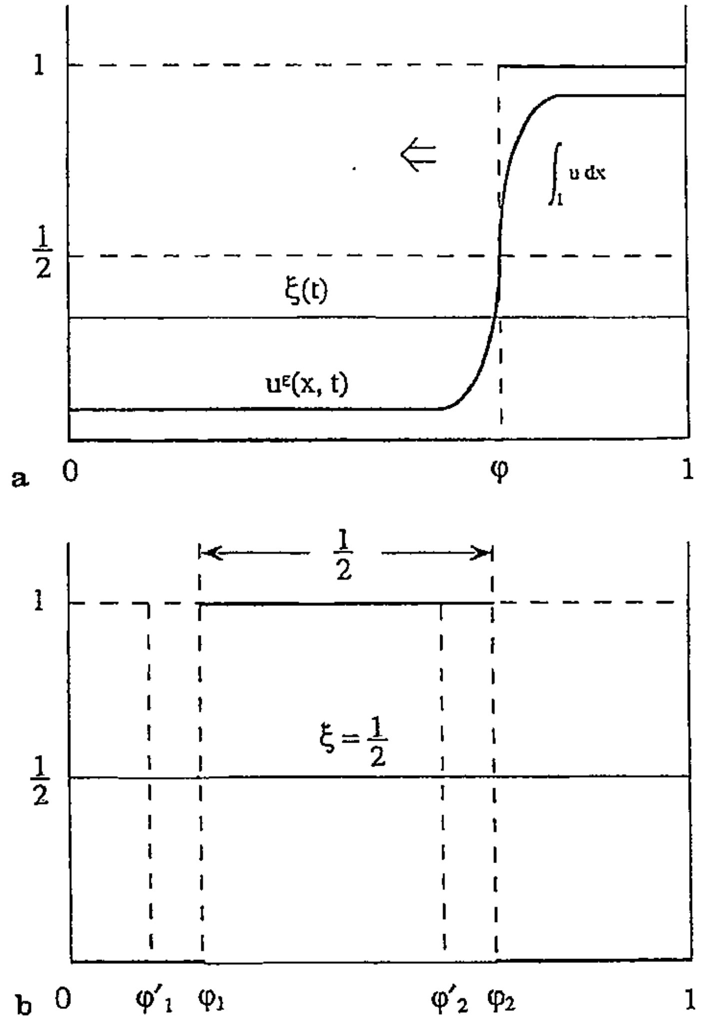

Figure 1.6.

ODE:

$$\varphi_s = c\left(\frac{3}{2} - \varphi - A\right). \tag{1.19}$$

Since the velocity function c is strictly monotone increasing, we see from (1.19) and $v^* = 1/2$ that $\varphi^* \equiv 1 -, A$ is a unique critical point which is *globally asymptotically stable on* $\mathcal{M}$. Note that the area of the corresponding u-profile is equal to A as we expected. Although the above introduction of (1.17b) looks very artificial, it is possible to derive it in a natural way from (1,12). In fact, when D becomes sufficiently large, we can show that V tends to be flat (see [41] and [30]), i.e., a constant function in spacial direction, which is denoted by $V = \xi(s)$. On the other hand $\int_I G_{\Phi(s)}(V)dx = 0$ always holds because of Neumann boundary conditions. We, therefore, have the following relation

$$\int_I G_{\Phi(s)}(\xi)dx = 0 \tag{1.20}$$

in the limit of $D \uparrow \infty$ instead of (1.12b). It is clear that, when $g(u, v) = u-(v-1/2)-A$, which is one of the typical cases, (1.20) coincides with (1.18b).

The above discussion is easily generalized to multi-layered solution; for instance, if there are two layeres as in Figure 1.6 (b), then the associated slow dynamics is given by

$$\begin{aligned}(\varphi_1)_s &= c(\xi)\\ (\varphi_2)_s &= -\, c(\xi)\\ 0 &= (\varphi_2 - \varphi_1) - \left(\xi - \frac{1}{2}\right) - A.\end{aligned} \tag{1.21}$$

Apparently $(\varphi_1, \varphi_2; \xi) = \left(1 - \frac{A}{2}, 1 + \frac{A}{2}, \frac{1}{2}\right)$ is a unique equilibrium point of (1.21), which corresponds to double-layered solution. However, this is *not* stable, since any translate of it with keeping $\varphi_2 - \varphi_1 = A$ and $\xi = 1/2$ is again an equilibrium point. Namely there exists a continuum of steady states for (1.21). This suggests that *scalar* controller ξ is *not* sufficient to stabilize *two* layers simultaneously. One may guess that the controller should have n degrees of freedom in order to control n layers. This is true, in fact, if we add another appropriate scalar variable η to (1.21) which controlls the sum of φ_1 and φ_2 (the difference is already controlled by ξ), then the resulting system has a unique asymptotically stable double-layered solution. In view of the slow dynamics (1.12), the controller V is a function of x (i.e., it has *infinite* degrees of freedom), hence V has a potentiality to control arbitrary many layers. In fact, we will see in Sections 2 that all critical points $\{\Phi_n^*\}$ of (1.12) are stable at least locally (although the discussions in Section 2 is restricted to the double-layer case, the generalization to n-layer case is straightforward by using the results in Section 3.).

Summarizing the above discussions, we can say that the slow manifold $\mathcal{M}_n$ forms a field for Φ where the unique equilibrium Φ_n^* sits at the bottom of the local basin.

Thus the singular limit system (1.12) admits the *coexistence of infinitely many stable solutions simultaneously*. However, when ε becomes positive, this is no longer true, in fact, the number of the steady states of (1.1) becomes finite for a fixed $\varepsilon > 0$, and so is the number of stable ones, although it goes to infinity as $\varepsilon \downarrow 0$. This reduction of number is caused by the diffusion effect $\varepsilon^2 u_{xx}$ which reduces the precision of discrimination of different layers. To see more clearly, we consider the scalar reaction diffusion equation of bistable type on **R**

$$\delta u_t = \varepsilon^2 u_{xx} + f(u),$$

where f is a cubic-like function such as (1.10a) with V being fixed to be a constant. Note that this simplification is plausible in the following discussion, since the controller v is close to a constant in a small neighborhood of layer position. Set an initial data $u_0(x)$ which has several layers within $O(\varepsilon)$-distance like Figure 1.7.

As time proceeds, these layers merge into a *mono*-layered travelling front (see, for example, Fife and McLeod [22]). On the other hand, if the mutual distance of layers are larger than ε, say $O(\varepsilon|\log\varepsilon|)$, these, in general, stay apart. This suggests that the resolution of layers for the original system (1.1) is proportional to the size of ε.

The arguments so far have been mainly focused on the singular limit system (1.12) and the dynamics on its slow manifold.

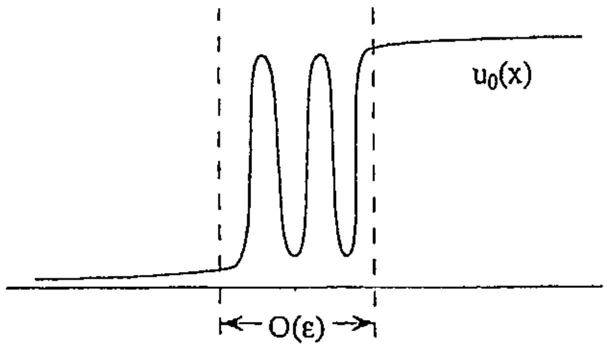

Figure 1.7.

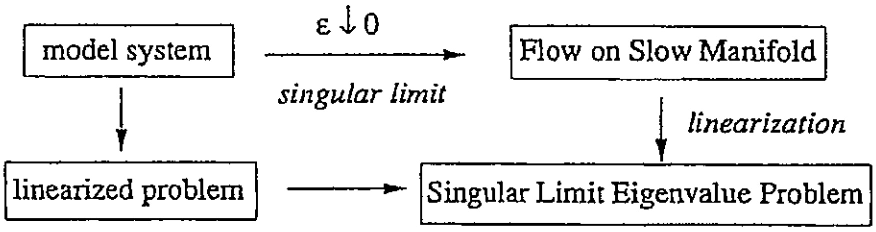

Figure 1.8.

However, it is, of course, not obvious that how the dynamics of (1.12) is related to that of the original system (1.1) for small ε. As far as stability is concerned, the Singular Limit Eigenvalue Problem (*SLEP*) method originated in Nishiura and Fujii [44] gives us a satisfactory answer, in fact, we have a commutative diagram as in Figure 1.8. Namely, a formal linearization of the singular limit slow dynamics at Φ_n^* gives us a rigorous result about the stability of normal n-layered solution to (1.1) for small ε. This not only supports the validity of the limiting slow dynamics concerning local flows near equilibrium solutions, but also is quite useful in practical sense, i.e., *formal* stability analysis of the limiting dynamics gives you a *correct* answer for the original system (1.1). We shall illustrate this more precisely in Section 2. It turns out that perturbation in outer region decays quickly, however perturbation at layer positions, essentially related to *shifting* the location of layers, behaves more slowly and delicately, and hence needs more precise analysis. Loosely speaking, the singular limit slow dynamics essentially describe the locus of each fully developped layers after outer part settles down. Accordingly, the linearized spectra at a layered solution to (1.1) is divided into two parts: *noncritical* and *critical* eigenvalues, where critical ones tend to zero as $\varepsilon \downarrow 0$ and the stability is determined by the behavior of finitely many critical eigenvalues, the number of which is proportional to that of layers. The SLEP system corresponds to the limiting eigenvalue problem for those dangerous perturbations concentrated at layer positions. Moreover we will show in Section 2 that the *formal* linearization of the limiting slow dynamics coincides exactly with the SLEP system.

The SLEP method is very close, in sprit, to the Lyapunov-Schmidt method in bifurcation theory in the sense that it enables us to reduce the linearized problem of (1.1), which is infinite dimensional, to the finite dimensional problem the size of which is proportional to the number of layers. In other words, the entire problem is contracted to the one on layers. In higher space dimension $\mathbb{R}^n$, the problem can be similarly reduced to the one on the interface, which is a $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$, however it is no more a finite dimensional problem. In fact it becomes a PDE problem associated with infinitely many number of critical eigenfunctions. We shall discuss more about this in Section 5.

Apparently the linearized eigenvalue problem of (1.1) at a layered solution (see (3.12)) degenerates in a singular way and its coefficients have discontinuities at layer positions as $\varepsilon \downarrow 0$. Moreover the associated eigenfunctions with critical eigenvalues, which control the stability properties and bifurcation, do *not* remain in a usual function space, say L^2, when $\varepsilon \downarrow 0$. Technically this is the main obstacle to overcome. The basic idea of the SLEP method lies in that those dangerous critical eigenfunctions can be characterized as *distributions* by means of appropriate ε-scaling in the limit of $\varepsilon \downarrow 0$. Especially, in one-dimensional case, they become a combination of Dirac's point mass distribution on layer positions. The linearized eigenvalue problem is well-defined up to $\varepsilon = 0$ by this characterization. Also the idea of the SLEP method is free from the forms of nonlinearities, boundary conditions, and the space dimension (see Section 5).

There are several different approaches for the stability problems of large amplitude, especially, singularly perturbed solutions: One is the stability index developped by Alexander, Gardner, and Jones (see [1] and [27]). Making use of a topological approach, they presented a beautiful framework of counting the number of critical eigenvalues for a general class of systems in one-dimensional space, however their method seems inadequate to keep track of the asymptotic behavior of the critical eigenvalues. It should be noted that their approach is closely related to ours when the parameters of the system belong to the regime of singular perturbation setting (see Suzuki, Nishiura, and Ikeda [56]). Another nice work was done by Hale and Sakamoto [28] who showed existence and stability simultaneously for the inhomogeneous scalar equation, i.e., $f = f(u, x)$ in (1.2). Then, Sakamoto [55] extended this to the system case based on the results of Nishiura and Fujii [45].

Now we state the assumptions for f and g (Figure 1.9).

(A.0) f and g are smooth functions of u and v defined on some open set $\mathcal{O}$ in $\mathbf{R}^2$.

(A.1a) The nullcline of f is sigmoidal and consists of three smooth curves $u = h_-(v)$, $h_0(v)$ and $h_+(v)$ defined on the intervals I_-, I_0, and I_+, respectively. Let $\min I_- = \underline{v}$ and $\max I_+ = \overline{v}$, then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $v \in I^* \equiv (\underline{v}, \overline{v})$ and $h_+(v)$ (resp. $h_-(v)$) coincides with $h_0(v)$ at only one point $v = \overline{v}$ (resp. $\underline{v}$) respectively.

(A.1b) The nullcline of g intersects with that of f at one or three points transversally as in Fig.1.9. The critical point on $u = h_-(v)$ (resp. $h_+(v)$ or $h_0(v)$), if exists, is denoted by $P = (u_-, v_-) = (h_-(v_-), v_-)$ (resp. $Q = (u_+, v_+) = (h_+(v_+), v_+)$ or $R = (u_0, v_0) = (h_0(v_0), v_0)$).

(A.2) $J(v)$ has an isolated zero at $v = v^* \in I^*$ such that $dJ/dv < 0$ at $v = v^*$, where $J(v) = \displaystyle\int_{h_-(v)}^{h_+(v)} f(s, v)ds$. Moreover we assume that $v_- < v^* < v_+$.

(A.3) $f_u < 0$ on $\mathcal{H}_+ \cup \mathcal{H}_-$, where $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) denotes the part of the curve $u = h_-(v)$ (resp. $h_+(v)$) defined by $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) $= \{(u, v) | u = h_-(v)$ (resp. $h_+(v)$) for $v_- \leq v < v^* (v^* < v \leq v_+)\}$, respectively. Note that v_- (resp. v_+) is replaced by $\underline{v}$ (resp. $\overline{v}$) when there are no critical points on the branch $u = h_-(v)$ (resp. $h_+(v)$). See thick solid part of $f = 0$ in Figure 1.9.

(A.4a) $$g|_{\mathcal{H}_-} < 0 < g|_{\mathcal{H}_+}$$

(A.4b) $$\det\left(\frac{\partial(f,g)}{\partial(u,v)}\right)\Big|_{\mathcal{H}_+\cup\mathcal{H}_-} > 0.$$

(A.5) $$g_v|_{\mathcal{H}_+\cup\mathcal{H}_-} \leq 0.$$

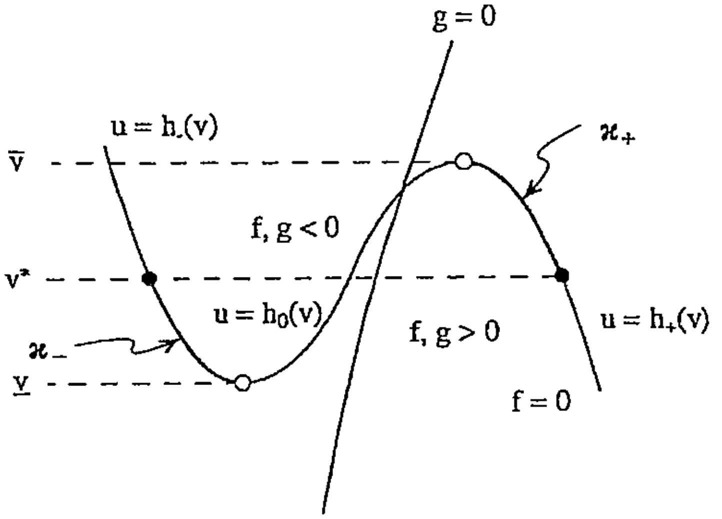

Figure 1.9.

Remark 1.1. *Let $G_\pm(v) \equiv g(h_\pm(v), v)$ for $v \in I_\pm$. Then, the assumption (A.4)(b) is equivalent to*

$$\frac{d}{dv}G_\pm(v)\Big|_{\mathcal{H}_\pm} < 0, \text{ respectively,} \tag{1.22}$$

since it follows from $f(h_\pm(v), v) = 0$ and (A.3) that

$$\frac{d}{dv}G_\pm(v)\Big|_{\mathcal{H}_\pm} = \frac{f_u g_v - f_v g_u}{f_u}\Big|_{\mathcal{H}_\pm}.$$

Remark 1.2. *It holds that $f_u = 0$ at $(h_+(\overline{v}), \overline{v})$ and $(h_-(\underline{v}), \underline{v})$.*

The outline of this paper is as follows. In Section 2, we study the slow dynamics from a geometrical point of view and show that formal linearized eigenvalue problem of (1.12) at an equilibrium point is exactly the same as the SLEP system in Section 3 derived from the original linearized problem for (1.1). In Section 3, we prove the stability of multi-layered solutions for the original model system (1.1), and show that how the SLEP method is used to reduce the whole problem to a finite dimensional one. It turns out that the resulting SLEP system is equivalent to solving the eigenvalue problem of a tri-diagonal symmetric matrix. The signs of eigenvalues of this matrix determine the stability, and lead us to the Main Theorem. In Section 4, we explain through the intermediary of the shadow system (4.1) why the stability of layered solutions makes a sharp contrast between the system (1.1) and the single equation (1.2). In Section 5, we briefly discuss about the case of different scaling for reaction and diffusion rates (the third category mentioned at the beginning of this section), the higher dimensional problems, and several related topics for which the SLEP method is useful.

We use the following notation throughout the paper:

$C^p(\bar{I})$ = the space of p-times continuous differentiable functions on $\bar{I}$ with usual supremum norm.

$C^p_\varepsilon(\bar{I})$ = the space of p-times continuous differentiable functions on $\bar{I}$ with the norm

$$\| u \|_{C^p_\varepsilon} = \sum_{k=0}^{p} \max \left| \left(\varepsilon \frac{d}{dx} \right)^k u(x) \right|,$$

$H^p(I)$ = the usual Sobolev space of order $p(\geq 0)$ in $L^2(I)$-framework,

$H^p_N(I)$ = the space of closure of $\{\cos(n\pi x/|I|)\}_{n=0}^{\infty}$ in $H^p(I)$,

$H^p_0(I)$ = the space of closure of $\{\sin(n\pi x/|I|)\}_{n=1}^{\infty}$ in $H^p(I)$,

$H^{-1}(I)$ = the dual space of $H^1_N(I)$,

$< \cdot, \cdot >$ = the inner product in $L^2(I)$-space,

$C^k_{c.u.}(I)$-topology = the compact uniform convergence in C^k-sense in I, namely, the uniform convergence on any compact subset of I in C^k-sense.

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2. Intuitive Approach to the Stability of Multi-layered Solutions

— Slow Manifold and Formal Linearization —

2.1. Slow Manifold for Mono-layered Solution

As we observed in Section 1, the singular limit slow dynamics is finite-dimensional when we *fix* a number of layers, and the dimension of it is exactly equal to the number of layers. In this section we shall construct the slow manifolds for single and double layered solutions, respectively, and study the flow on each manifold. Let us begin with the mono-layer case (see (1.12)):

$$(\varphi)_s = c(V(\varphi(s))) \tag{2.1a}$$

$$DV_{xx} + G_{\Phi(s)}(V) = 0 \quad \text{with} \quad V_x = 0 \quad \text{on } \partial I \tag{2.1b}$$

where

$$G_{\Phi(s)}(V) = \begin{cases} G_-(V) & \text{on } I_- = (0, \varphi(s)) \\ G_+(V) & \text{on } I_+ = (\varphi(s), 1). \end{cases} \tag{2.1c}$$

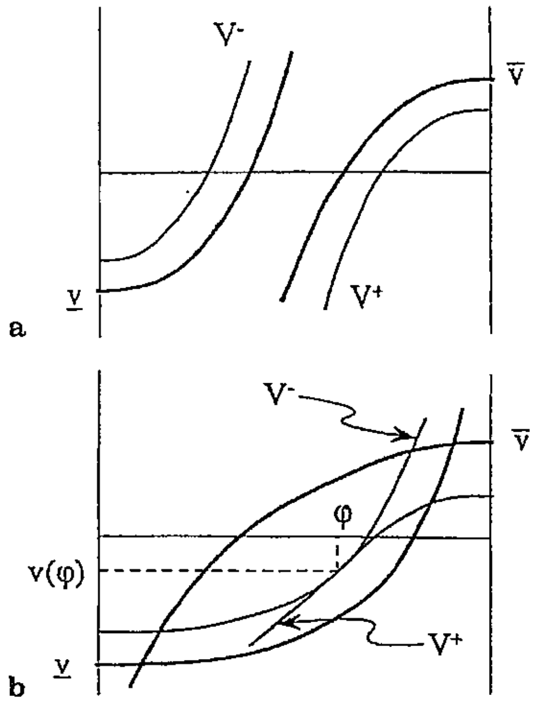

Figure 2.1.

For definiteness we assume that $G_{\pm}(v)$ take the form as in Figure 1.4 which is equivalent to assume that (f, g) is of Turing type (see Figure 1.1 (b)). More precisely, (A.4) and Remark 1.1 imply that

(G.1) $G_{\pm}(v)$ are smooth and satisfy $\dfrac{dG_{\pm}}{dv} < 0$ for $\underline{v} \le v < \overline{v}$ and $\underline{v} < v \le \overline{v}$, respectively, i.e., strictly monotone decreasing.

(G.2) $G_{+}(\overline{v}) > 0$ and $G_{-}(\underline{v}) < 0$.

First we explain the geometrical meaning of C^1-matched solution of (2.1b). We denote a solution of (2.1b) on $I^{\pm}$ by $V^{\pm}$, respectively. It is clear from (G.2) that V^- (resp. V^+) is strictly convex (resp. concave) on I^- (resp. I^+) (see Figure 2.1(a)). Also we see from (G.1) and the boundary conditions that V is order-preserving on I^- in the sense that $V_1^-(0) > V_2^-(0)$ implies $V_1^-(x) > V_2^-(x)$ on I^-, and similar property also holds for V^+ on I^+. In view of Figure 2.1(b), it is apparent that, for a given φ, (2.1b) has a C^1-matched solution at $x = \varphi$ if and only if V^+ and V^- are tangent with each other at $x = \varphi$. We denote this solution and its value at $x = \varphi$ by V_{φ} and $v(\varphi)$, respectively, which are uniquely determined by φ. The slow manifold $\mathcal{M}_1 = \mathcal{M}_1(D)$ for (2.1) is defined by

$$\mathcal{M}_1(D) = \{(\varphi, v(\varphi)) \mid \text{there is a } C^1\text{-matched solution to (2.1b) with } 0 < \varphi < 1\}. \tag{2.2}$$

In what follows we shall find an analytical expression for $\mathcal{M}_1$ (see (2.7)), which is more convenient to study the flow on it.

It is clear from (G.2) and the order preserving property that $(2.1)_b$ cannot have C^1-matched solutions when D is small (see Figure 2.1(a)), in fact, there is a unique threshold value $D = \underline{D}_1$ such that there are no C^1-matched solutions for $D < \underline{D}_1$. Here $\underline{D}_1$ is defined by

$\underline{D}_1$: When $D = \underline{D}_1$, $(2.1)_b$ has two solutions $V^-(x)$ with $V^-(0) = \underline{v}$ and $V^+(x)$ with $V^+(1) = \overline{v}$ which are tangent with each other. (2.3)

Remark 2.1. *When the nonlinearity (f, g) is of bistable type like Figure 1.1(c), we have no limitation of D for the existence of C^1-matched solutions. In fact, since both G_- and G_+ have zero points, we can always construct C^1-matched solutions of $(2.1)_b$ for any $D(> 0)$ and φ $(0 < \varphi < 1)$.*

To be more precise, it is convenient to introduce the mappings $d^-(\varphi, v)$ and $d^+(1-\varphi, v)$ defined as follows: For a given $\varphi(0 < \varphi < 1)$, consider the boundary value problem

$$DV_{xx} + G_-(V) = 0 \quad \text{on} \quad (0, \varphi) \tag{2.4a}$$

$$V_x(0) = 0, \qquad V(\varphi) = v. \tag{2.4b}$$

In view of the assumptions (G.1) and (G.2), it is easy to verify by contradiction that the solution of (2.4), if it exists, is unique. We denote by $d^-(\varphi, v)$ the x-derivative of this solution at $x = \varphi$. Similarly $d^+(1 - \varphi, v)$ is defined as the derivative at $x = 1 - \varphi$ of the solution to

$$DV_{xx} + G_+(V) = 0 \quad \text{on} \quad (0, 1 - \varphi) \tag{2.5a}$$

$$V_x(0) = 0, \qquad V(1 - \varphi) = v. \tag{2.5b}$$

Then the C^1-matching condition at $x = \varphi$ can be represented by

$$\mathcal{F}(\varphi, v) = 0, \tag{2.6a}$$

where

$$\mathcal{F}(\varphi, v) \equiv d^-(\varphi, v) + d^+(1 - \varphi, v). \tag{2.6b}$$

Using this expression, we can give an analytical definition for nthe one-dimensional slow manifold in (φ, v)-space:

$$\mathcal{M}_1(D) \equiv \{(\varphi, v) \mid \mathcal{F}(\varphi, v) = 0, 0 < \varphi < 1\}. \tag{2.7}$$

Since

$$d\mathcal{F} = \left(\frac{\partial d^-}{\partial v} + \frac{\partial d^+}{\partial v}\right) dv + \left(\frac{\partial d^-}{\partial \varphi} + \frac{\partial d^+}{\partial \varphi}\right) d\varphi,$$

and from (G.1) and (G.2), we have

$$\frac{\partial d^-}{\partial v} > 0, \ \frac{\partial d^+}{\partial v} > 0, \ \frac{\partial d^-}{\partial \varphi} > 0, \ \frac{\partial d^+}{\partial \varphi} > 0$$

at any solution of (2.6). Let (φ_0, v_0) be any solution of (2.6a), then by using the implicit function theorem, there is a unique smooth curve $v = \bar{v}(\varphi)$ with $v_0 = v(\varphi_0)$ locally near (φ_0, v_0) and $\dfrac{d\bar{v}}{d\varphi} < 0$ holds. □

Using this result and continuous dependency on parameters of solutions, we can prove the following.

Theorem 2.2. *There exists a unique* $\underline{D}_1 > 0$ *such that*

(1) $\mathcal{M}_1(D) = \phi$ *(empty set) for* $D < \underline{D}_1$.

(2) For $D > \underline{D}_1$, *there exist continuous functions* $\underline{\varphi}(D)$ *and* $\overline{\varphi}(D)$ *of* $D(0 < \underline{\varphi}(D) < \overline{\varphi}(D) < 1)$ *such that* $\mathcal{M}_1(D)$ *is a smooth one-dimensional manifold defined by*

$$\mathcal{M}_1(D) \equiv \{(\varphi, v) | v = \bar{v}(\varphi) \text{ for } \underline{\varphi}(D) < \varphi < \overline{\varphi}(D)\},$$

i.e., a graph on $(\underline{\varphi}(D), \overline{\varphi}(D))$. *Moreover* $\dfrac{d\bar{v}}{d\varphi} < 0$ *holds. Finally, suppose a unique equilibrium point* $\Phi_1^* = (\varphi^*, v^*)$ *is contained in* $\mathcal{M}_1(D)$, *then it is globally asymptotically stable on* $\mathcal{M}_1(D)$.

Proof. (1) is clear from the definition of $\underline{D}_1$ (see (2.3)). As for (2), it follows from the previous discussions that $\mathcal{M}_1(D)$ is locally expressed as a smooth strictly decreasing curve $v = \bar{v}(\varphi)$. In view of Figure 2.1 and $\dfrac{d\bar{v}}{d\varphi} < 0$, we see that $\mathcal{M}_1(D)$ is a graph on the interval $(\underline{\varphi}(D), \overline{\varphi}(D))$ where $\overline{\varphi}(D)$ is determined in such a way that V^--solution with $V^-(0) = \underline{v}$ is matched with a V^+-solution at $x = \overline{\varphi}(D)$ (Figure 2.1 (b)), and similaly for $\underline{\varphi}(D)$. Global asymptotic stability of (φ^*, v^*) comes from the fact that $\mathcal{M}_1(D)$ is one-dimensional and (φ^*, v^*) is a unique equilibrium point with $\dfrac{d\bar{v}}{d\varphi} < 0$ everywhere. □

Remark 2.3. *(a)* $\mathcal{M}_1(D)$, *in general, does not contain the equilibrium point* (φ^*, v^*). *In fact, the matching value of* V^- *and* V^+ *is not, in general, equal to* v^* *at* $D = \underline{D}_1$ *(see* (2.3) *) where* $\underline{\varphi}(D)$ *and* $\overline{\varphi}(D)$ *coincide each other, then it holds that* $(\varphi^*, v^*) \notin \mathcal{M}_1(D)$ *when* D $(\geq \underline{D}_1)$ *is close to* $\underline{D}_1$.

(b) On the other hand, suppose that the nonlinearity (f, g) *has an odd symmetry with repect to the middle intersecting point* R *of* $f = 0$ *and* $g = 0$, *the equilibrium point* (φ^*, v^*) $(\varphi^* = \dfrac{1}{2})$ *always lies in* $\mathcal{M}_1(D)$ *for* $D > \underline{D}_1$.

2.2. Local Slow Manifold for Double-layered Solution

We shall construct a local slow manifold around the equilibrium point for the double-layered case and study the flow on it. In this subsection we fix D to be an appropriate value so that (2.8) has a unique steady state $\Phi_2^* = (\varphi_1^*, \varphi_2^*)$. In view of (1.12), we see that the slow dynamics for two layers can be written as follows:

$$(\varphi_1)_s = c(V(\varphi_1(s))) \tag{2.8a}$$

$$(\varphi_2)_s = -c(V(\varphi_2(s))) \tag{2.8b}$$

$$0 = DV_{xx} + G_{\Phi(s)}(V) \tag{2.8c}$$

$$V_x(0) = V_x(1) = 0 \tag{2.8d}$$

where

$$G_{\Phi(s)}(V) \equiv G_-(V)\,[H(\varphi_1(s)-x)+H(x-\varphi_2(s))] + G_+(V)H(x-\varphi_1(s))H(\varphi_2(s)-x),$$

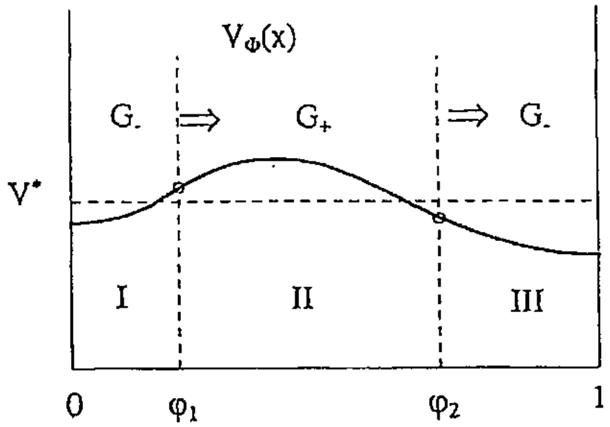

Figure 2.2.

$\Phi(s) = (\varphi_1(s), \varphi_2(s))$, and $H(\xi)$ is the Heaviside function i.e., $H(\xi) = 1$ (resp. 0) for $\xi > 0$ (resp. < 0). Namely $G_\Phi = G_-$ in region I, III, and $G_\Phi = G_+$ in region II (see Figure 2.2). For a given $\Phi = (\varphi_1, \varphi_2)$, we denote the solution of (2.8c) and (2.8d) by $V_\Phi(x)$.

Note that the dynamics (2.8) is valid for $0 < \varphi_1 < \varphi_2 < 1$ and that V_Φ is matched in C^1-sense at layer positions $x = \varphi_i(s)$ $(i = 1, 2)$. Let $(V_\Phi, \varphi_1, \varphi_2) = (V_2, \varphi_1^*, \varphi_2^*)$ be the normal 2-layered solution, i.e., the unique critical point of (2.8). Note that $\varphi_2^* = 1 - \varphi_1^*$ holds because of symmetry. V_2 must satisfy $V_2^*(\varphi_1^*) = v^* = V_2^*(\varphi_2^*)$. What we have to do is to construct the solutions (2.8c), (2.8d) for any (φ_1, φ_2) near $(\varphi_1^*, \varphi_2^*)$ and study the flow on it. Let $(v_1, d_-(\varphi_1, v_1))$ denote the value of V and its derivative at $x = \varphi_1$ by solving (2.8c) in region I. Note that the derivative $d_-(\varphi_1, v_1)$ is uniquely determined as function of (φ_1, v_1) because of the Neumann boundary condition at $x = 0$ and the monotonicity of $G_-(V)$. Using this mapping d_-, we have, for a given v_2, $(v_2, -d_-(1-\varphi_2, v_2))$ at $x = \varphi_2$. The minus sign in front of d_- is necessary, since we solve (2.8c) from right to left in region III. Finally let $(S_+(v, d, x), d_+(v, d, x))$ denote the solution map of $DV_{xx} + G_+(V) = 0$ for a given initial data $(V, V_x) = (v, d)$ at $x = 0$. Because of symmetry, it is more convenient to make a C^1-matching at $x = \dfrac{\varphi_2 - \varphi_1}{2}$, the middle point of region II:

$$\mathcal{F}_1(\varphi_1, \varphi_2, v_1, v_2) = 0 \tag{2.9a}$$

$$\mathcal{F}_2(\varphi_1, \varphi_2, v_1, v_2) = 0 \tag{2.9b}$$

where

$$\mathcal{F}_1(\varphi_1, \varphi_2, v_1, v_2) \equiv S_+\left(v_1, d_-(\varphi_1, v_1), \frac{\varphi_2 - \varphi_1}{2}\right) - S_+\left(v_2, d_-(1-\varphi_2, v_2), \frac{\varphi_2 - \varphi_1}{2}\right) \tag{2.9c}$$

$$\mathcal{F}_2(\varphi_1, \varphi_2, v_1, v_2) \equiv d_+\left(v_1, d_-(\varphi_1, v_1), \frac{\varphi_2 - \varphi_1}{2}\right)$$
$$+d_+\left(v_2, d_-(1-\varphi_2, v_2), \frac{\varphi_2 - \varphi_1}{2}\right). \tag{2.9d}$$

At 2-layer solution, it holds that

$$\begin{aligned} \mathcal{F}_1(\varphi_1^*, \varphi_2^*, v^*, v^*) &= 0 \\ \mathcal{F}_2(\varphi_1^*, \varphi_2^*, v^*, v^*) &= 0. \end{aligned} \tag{2.10}$$

We show that (2.9) can be solved uniquely with respect to (v_1, v_2) in a neighbourhood of $(\varphi_1^*, \varphi_2^*)$. The mappings $d_\pm$ and S_+ are smooth and the next lemma holds in a neighbourhood of $(v, d, x) = (v^*, d_-(\varphi_1^*, v^*), \varphi_1^*)$ (in fact, it holds in larger region).

Lemma 2.4.

(i) $$\frac{\partial S_+}{\partial v} > 0, \ \frac{\partial S_+}{\partial d} > 0$$

(ii) $$\frac{\partial d_+}{\partial v} > 0, \ \frac{\partial d_+}{\partial d} > 0, \ \frac{\partial d_+}{\partial x} < 0$$

(iii) $$\frac{\partial d_-}{\partial v} > 0, \ \frac{\partial d_-}{\partial x} > 0$$

Proof. These are the direct consequences of the fact that $G_\pm(V)$ are strictly monotone decreasing and take definite signs, respectively. □

Using Lemma 2.4, we have, at $(v, d, x) = (v^*, d_-(v^*, \varphi_1^*), \varphi_1^*)$,

$$\begin{aligned} \frac{\partial \mathcal{F}_1}{\partial v_1} &= \frac{\partial S_+}{\partial v_1} + \frac{\partial S_+}{\partial d}\frac{\partial d_-}{\partial v_1} > 0 \\ \frac{\partial \mathcal{F}_1}{\partial v_2} &= -\frac{\partial S_+}{\partial v_2} - \frac{\partial S_+}{\partial d}\frac{\partial d_-}{\partial v_2} < 0 \\ \frac{\partial \mathcal{F}_2}{\partial v_1} &= \frac{\partial d_+}{\partial v_1} + \frac{\partial d_+}{\partial d}\frac{\partial d_-}{\partial v_1} > 0 \\ \frac{\partial \mathcal{F}_2}{\partial v_2} &= \frac{\partial d_+}{\partial v_2} + \frac{\partial d_+}{\partial d}\frac{\partial d_-}{\partial v_2} > 0. \end{aligned} \tag{2.11}$$

This implies $\det \dfrac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(v_1, v_2)} > 0$, and hence, by the implicit function theorem, we obtain the following.

Proposition 2.5. *For arbitrary $\Phi = (\varphi_1, \varphi_2)$ near $\Phi_2^* = (\varphi_1^*, \varphi_2^*)$, there exist a unique solution $V = V_\Phi(x)$ of $(2.8)_c$ and $(2.8)_d$ which converges to normal 2-layer solution in C^1-sense when (φ_1, φ_2) tends to $(\varphi_1^*, \varphi_2^*)$. The values of $V_\Phi(x)$ at two layer positions, denoted by $v_1(\varphi_1, \varphi_2)$, $v_2(\varphi_1, \varphi_2)$, are uniquely determined and smooth functions of (φ_1, φ_2) in a neighbourhood of $(\varphi_1^*, \varphi_2^*)$.*

Using Proposition 2.5, (2.8a) and (2.8b), the vector field on a local slow manifold is given by

$$\begin{aligned} \frac{d}{ds}\varphi_1 &= c(v_1(\varphi_1, \varphi_2)) \\ \frac{d}{ds}\varphi_2 &= -c(v_2(\varphi_1, \varphi_2)). \end{aligned} \tag{2.12}$$

Hence *the original dynamical system* (2.8) *is contracted to two-dimensional nonlinear ODE system* (2.12).

Now it is straightforward to check the stability of critical point $\Phi_2^* = (\varphi_1^*, \varphi_2^*)$ by computing the linearized matrix of (2.12) at Φ_2^*. Namely it suffices to find the signs of the real parts of eigenvalues of

$$J_{\Phi_2^*} \equiv \left.\begin{pmatrix} \dfrac{\partial c}{\partial v_1}\dfrac{\partial v_1}{\partial \varphi_1} & \dfrac{\partial c}{\partial v_1}\dfrac{\partial v_1}{\partial \varphi_2} \\ -\dfrac{\partial c}{\partial v_2}\dfrac{\partial v_2}{\partial \varphi_1} & -\dfrac{\partial c}{\partial v_2}\dfrac{\partial v_2}{\partial \varphi_2} \end{pmatrix}\right|_{\Phi_2^*} \tag{2.13}$$

We shall show $\det J_{\Phi_2^*} > 0$ and $\mathrm{tr} J_{\Phi_2^*} < 0$. From the differential relations,

$$\frac{\partial \mathcal{F}_1}{\partial v_1}dv_1 + \frac{\partial \mathcal{F}_1}{\partial v_2}dv_2 + \frac{\partial \mathcal{F}_1}{\partial \varphi_1}d\varphi_1 + \frac{\partial \mathcal{F}_1}{\partial \varphi_2}d\varphi_2 = 0$$

$$\frac{\partial \mathcal{F}_2}{\partial v_1}dv_1 + \frac{\partial \mathcal{F}_2}{\partial v_2}dv_2 + \frac{\partial \mathcal{F}_2}{\partial \varphi_1}d\varphi_1 + \frac{\partial \mathcal{F}_2}{\partial \varphi_2}d\varphi_2 = 0$$

we have

$$\begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \frac{1}{\dfrac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(v_1, v_2)}} \begin{pmatrix} \dfrac{\partial \mathcal{F}_2}{\partial v_2} & -\dfrac{\partial \mathcal{F}_1}{\partial v_2} \\ -\dfrac{\partial \mathcal{F}_2}{\partial v_1} & \dfrac{\partial \mathcal{F}_1}{\partial v_1} \end{pmatrix} \begin{pmatrix} -\dfrac{\partial \mathcal{F}_1}{\partial \varphi_1} & -\dfrac{\partial \mathcal{F}_1}{\partial \varphi_2} \\ -\dfrac{\partial \mathcal{F}_2}{\partial \varphi_1} & -\dfrac{\partial \mathcal{F}_2}{\partial \varphi_2} \end{pmatrix} \begin{pmatrix} d\varphi_1 \\ d\varphi_2 \end{pmatrix}. \tag{2.14}$$

It is obvious that (i, j) -component of the matrix on the right-hand side of (2.14) is equal to $\dfrac{\partial v_i}{\partial \varphi_j}$, i.e.,

$$\left\{\frac{\partial v_i}{\partial \varphi_j}\right\}_{i,j=1,2} \equiv \left\{\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(v_1, v_2)}\right\}^{-1} \left\{-\frac{\partial \mathcal{F}_i}{\partial \varphi_j}\right\}_{1\le i,j\le 2} \tag{2.15}$$

By similar computation to (2.11), we easily see that

$$\frac{\partial \mathcal{F}_1}{\partial \varphi_1} > 0, \ \frac{\partial \mathcal{F}_1}{\partial \varphi_2} > 0, \ \frac{\partial \mathcal{F}_2}{\partial \varphi_1} > 0, \ \text{and} \ \frac{\partial \mathcal{F}_2}{\partial \varphi_2} < 0,$$

which implies $\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(v_1,v_2)} < 0$, and hence, $\det\left\{\frac{\partial v_i}{\partial \varphi_j}\right\}_{i,j=1,2} < 0$. Noting that $\frac{\partial c}{\partial v_1} = \frac{\partial c}{\partial v_2} > 0$ at Φ_2^*, we obtain

$$\det J_{\Phi_2^*} = -\left(\left.\frac{\partial c}{\partial v_1}\right|_{\Phi_2^*}\right)^2 \det\left\{\frac{\partial v_i}{\partial \varphi_j}\right\}_{i,j=1,2} > 0.$$

On the other hand, we have

$$\mathrm{tr} J_{\Phi_2^*} = \left.\frac{\partial c}{\partial v_1}\right|_{\Phi_2^*} \left.\left\{\frac{\partial v_1}{\partial \varphi_1} - \frac{\partial v_2}{\partial \varphi_2}\right\}\right|_{\Phi_2^*}.$$

In view of (2.11), (2.14), (2.15), and (2.16), we easily see that $\mathrm{tr} J_{\Phi_2^*} < 0$. Thus we conclude that

Proposition 2.6. *The critical point* $(\varphi_1, \varphi_2) = (\varphi_1^*, \varphi_2^*)$ *corresponding to normal 2-layer solution is asymptotically stable equilibrium of* (2.8) *on the local slow manifold constructed in Proposition 2.5.*

In the rest of this subsection we consider the stability of Φ_2^* from more intuitive point of view. Essentially there are two types of perturbation; symmetric and anti-symmetric ones as in Figure 2.3.

The first case (Figure 2.3(a)) can be reduced to the mono-layer case. Namely, it suffices to see only the half part because of symmetry, and hence it was already studied in Section 2.1. The second case (Figure 2.3(*b*)), the anti-symmetric perturbation, where the directions of the shifts of two layer positions are the same, is the main concern here. Recalling that the velocity function $c(V)$ is monotone decreasing and $c(V(\varphi_i^*)) = 0$ $(i = 1, 2)$, we see that, in order to stabilize this perturbation, the v-value at left (resp. right) layer must be up (resp. down) so that they recover their original positions. We will see that this really occurs as in Figure 2.3(b). Arrows in Figure 2.3 indicate the directions in which the perturbed layers move. Since the distance between φ_1 and φ_2 does not change for the anti-symmetric perturbation, it holds from $\varphi_2^* = 1 - \varphi_1^*$ that

$$\varphi_2 - \varphi_1 = 1 - 2\varphi_1^*. \tag{2.17}$$

In order to know how v_1 and v_2 behave on the slow manifold, we consider the differentials of (2.9):

$$\begin{aligned} d\mathcal{F}_1 &= \frac{\partial \mathcal{F}_1}{\partial v_1}dv_1 + \frac{\partial \mathcal{F}_1}{\partial v_2}dv_2 + \frac{\partial \mathcal{F}_1}{\partial \varphi_1}d\varphi_1 + \frac{\partial \mathcal{F}_1}{\partial \varphi_2}d\varphi_2 = 0 \\ d\mathcal{F}_2 &= \frac{\partial \mathcal{F}_2}{\partial v_1}dv_1 + \frac{\partial \mathcal{F}_2}{\partial v_2}dv_2 + \frac{\partial \mathcal{F}_2}{\partial \varphi_1}d\varphi_1 + \frac{\partial \mathcal{F}_2}{\partial \varphi_2}d\varphi_2 = 0 \end{aligned} \tag{2.18}$$

We compute the directional derivative of v_1 and v_2 along the line (2.17). Recalling the form (2.9), we see that

$$\frac{\partial \mathcal{F}_1}{\partial \varphi_1} = \frac{\partial \mathcal{F}_1}{\partial \varphi_2} \quad \text{and} \quad \frac{\partial \mathcal{F}_2}{\partial \varphi_1} + \frac{\partial \mathcal{F}_2}{\partial \varphi_2} = 0 \tag{2.19}$$

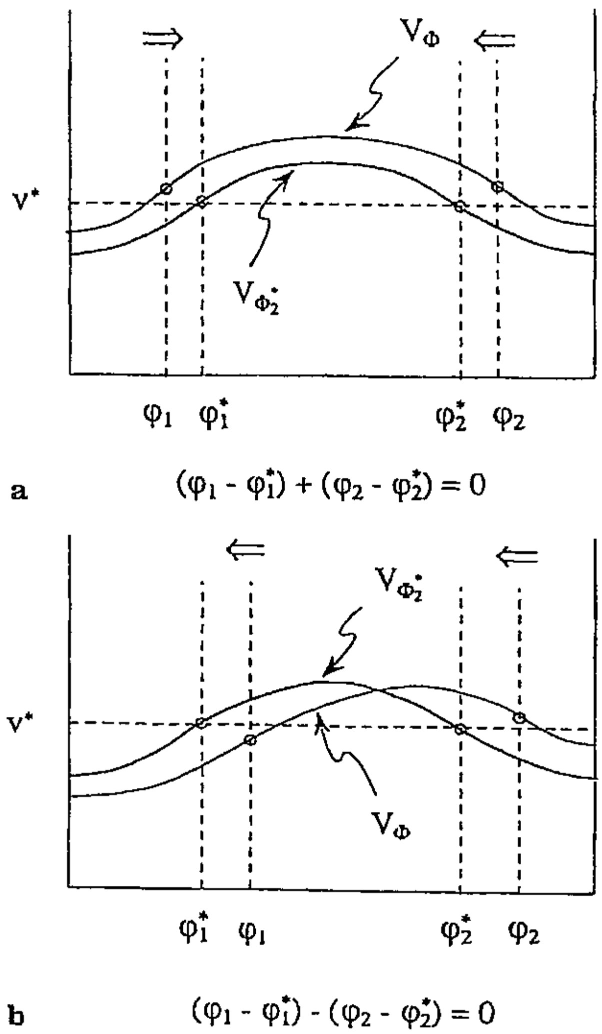

Figure 2.3.

hold at $(\varphi_1, \varphi_2, v_1, v_2) = (\varphi_1^*, \varphi_2^*, v^*, v^*)$. Also, by a simple computation, we see from Lemma 2.4 that $\frac{\partial \mathcal{F}_1}{\partial v_1} > 0$, $\frac{\partial \mathcal{F}_1}{\partial v_2} < 0$, $\frac{\partial \mathcal{F}_1}{\partial \varphi_1} > 0$, $\frac{\partial \mathcal{F}_2}{\partial v_1} > 0$, $\frac{\partial \mathcal{F}_2}{\partial v_2} > 0$, holds. This combined with (2.18) and (2.19) implies that

$$D^+ v_1 < 0 \text{ and } D^+ v_2 > 0 \text{ at } (\varphi_1^*, \varphi_2^*) \tag{2.20}$$

where D^+ denotes the directional derivative along the line (2.17), i.e., (1,1)-direction. This is what we expected before.

2.3. Formal Linearization

In this subsection we shall derive a formal linearized equation for the singular limit equations (2.8). The reason for this is that the resulting linearized eigenvalue problem is exactly the same form as the SLEP system which will be derived from the original system in a rigorous manner in Section 3. This is important since formal analysis gives us a correct answer for the original system. As was mentioned in Section 1, the dynamics has two steps; first, the outer dynamics, and then slow layer dynamics. So is the linearized problem. We will show that the linearized problem for outer dynamics is quite stable, i.e., all the spectrum have strictly negative parts, on the other hand, the one for the layer dynamics, which corresponds to the SLEP system, is subtle and needs more computation.

A. Outer Part

The dynamics in this region is given by

$$\begin{aligned} \delta U_t &= f(U,V) \\ V_t &= \frac{1}{\sigma} V_{xx} + g(U,V), \end{aligned} \tag{2.21}$$

where we replace D by $1/\sigma$ for later convenience. Let (U_2^*, V_2^*) be the normal 2-layer solution associated with Φ_2^*, i.e., V_2^* is the solution of (2.8c) and (2.8d) with $\Phi = \Phi_2^*$, and U_2^* is the associated u -component through the relation $u = h_\pm(v)$. In Section 3 we will write this solution with superscript σ as in Corollary 3.9, however we omit it here for simplicity. We set $U = U_2^* + e^{\lambda t}W$, $V = V_2^* + e^{\lambda t}Z$ where (W, Z) is a perturbation in outer region which become zero at layer positions. Substituting this into (2.21), the resulting linearized problem becomes

$$\begin{aligned} \delta\lambda W &= f_u^* W + f_v^* Z \\ \lambda Z &= \frac{1}{\sigma} Z_{xx} + g_u^* W + g_v^* Z, \end{aligned} \tag{2.22}$$

subject to $Z_x = 0$ at $x = 0, 1$, where $f_u^* \equiv f_u(U_2^*, V_2^*)$ and so on. Since $f_u^* < 0$ (see (A.3)) we can solve the first equation of (2.22) as $W = -f_v^* Z/(f_u^* - \delta\lambda)$ which is well-defined as far as λ varies in

$$C_\mu = \{\lambda \mid Re\delta\lambda > -\mu, \quad 0 > -\mu > \inf_{x \neq \varphi_1^*, \varphi_2^*} f_u(U_2^*(x), V_2^*(x))\}.$$

Clearly this restriction does not affect our stability analysis. Substituting this into the second one of (2.22), we obtain

$$\lambda Z = \frac{1}{\sigma} Z_{xx} + \left(-\frac{f_v^* g_u^*}{f_u^* - \delta\lambda} + g_v^* \right) Z. \tag{2.23}$$

It is known that this problem exactly coincides with the eigenvalue problem for the *noncritical* eigenvalues (see Section 2 of [44]). Typically, $f_v^* < 0$, $g_u^* > 0$, and $g_v^* < 0$ (for instance, $f(u, v) = u(1-u)(u-a) - v$ and $g(u, v) = \gamma u - \sigma v$ with $\gamma > 0$, $\sigma > 0$), hence the coefficient of Z on the right-hand side of (2.23) is strictly negative. Making a bilinear form, we easily see that all the spectrum of (2.23) have strictly negative real parts. Thus there are no dangerous eigenvalues coming from the outer part.

B. Layer Part

Using the *slow* time $s = \varepsilon t/\delta$, the layer dynamics was given by (2.8). $(V, \varphi_1, \varphi_2) = (V_2^*(x), \varphi_1^*, \varphi_2^*)$ is the normal 2-layer solution. Recall that $V_2^*(\varphi_1^*) = V_2^*(\varphi_2^*) = v^*$ and

$c(v^*) = 0$. We derive the linearized equations at $(V_2^*, \varphi_1^*, \varphi_2^*)$ by setting

$$\begin{aligned} V &= V_2^*(x) + e^{\tau s} h(x) \\ \varphi_1 &= \varphi_1^* - e^{\tau s}\psi_1 \\ \varphi_2 &= \varphi_2^* - e^{\tau s}\psi_2 \end{aligned} \tag{2.24}$$

Here we employ $\tau \equiv \lambda\delta/\varepsilon$ as the eigenvalue parameter, since we adopt the slow time s instead of t. Note that we put the minus signin front of $e^{\lambda s}\psi_1$ in order for ψ_1 and ψ_2 to have the same sign for anti-symmetric perturbation (see Figure 2.3(b)). Substituting (2.24) into (2.8), and linearizing it in a formal way, we have

$$-\tau\psi_1 = \frac{dc}{dV}(v^*)\left\{-\frac{dV_2^*}{dx}(\varphi_1^*)\psi_1 + h(\varphi_1^*)\right\} \tag{2.25a}$$

$$\tau\psi_2 = -\frac{dc}{dV}(v^*)\left\{\frac{dV_2^*}{dx}(\varphi_2^*)\psi_2 + h(\varphi_2^*)\right\}, \tag{2.25b}$$

$$\begin{aligned} 0 = \frac{1}{\sigma}h_{xx} &+ \frac{dG_-}{dV}(V_2^*)\{H(\varphi_1^* - x) + H(x - \varphi_2^*)\}h \\ &+ \frac{dG_+}{dV}(V_2^*)\{H(x - \varphi_1^*)H(\varphi_2^* - x)\}h \\ &+ G_-(V_2^*)(-\delta_{\varphi_1^*}\psi_1 - \delta_{\varphi_2^*}\psi_2) \\ &+ G_+(V_2^*)(\delta_{\varphi_1^*}\psi_1)H(\varphi_2^* - x) \\ &+ G_+(V_2^*)H(x - \varphi_1^*)\delta_{\varphi_2^*}\psi_2 \end{aligned} \tag{2.25c}$$

where $\delta_{\varphi_i^*}$ is the Dirac's δ -function with support at $x = \varphi_i^*$. Using

$$H(\varphi_2^* - x)\delta_{\varphi_1^*}\psi_1 = \delta_{\varphi_1^*}\psi_1, \qquad H(x - \varphi_1^*)\delta_{\varphi_2^*}\psi_2 = \delta_{\varphi_2^*}\psi_2,$$

and

$$\{G_+(V_2^*) - G_-(V_2^*)\}\delta_{\varphi_i^*}\psi_i = [g]\delta_{\varphi_i^*}\psi_i \qquad (i = 1, 2),$$

where $[g] = g(h_+(v^*), v^*) - g(h_-(v^*), v^*)$, i.e., the jump of the value of g from left- to right-branch at $v = v^*$, (2.25c) becomes

$$T^{*,\sigma}h = [g]\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right), \tag{2.26}$$

where

$$T^{*,\sigma} \equiv -\frac{1}{\sigma}\frac{d^2}{dx^2} - \frac{dG_-}{dV}(V_2^*)\{H(\varphi_1^* - x) + H(x - \varphi_2^*)\}$$

$$-\frac{dG_+}{dV}(V_2^*)H(x-\varphi_1^*)H(\varphi_2^* - x).$$

Since $G_\pm(s)$ is strictly monotone decreasing (see (G.1)), i.e., $\dfrac{dG_\pm}{dV} < 0$, $T^{*,\sigma}$ has a well-defined inverse $K^{*,\sigma,0} : H^{-1}(I) \to H_N^1(I)$. Hence it follows from (2.26) that

$$h = [g]K^{*,\sigma,0}\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right). \tag{2.27}$$

This leads to the expression:

$$h(\varphi_1^*) = [g]\langle K^{*,\sigma,0}\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right), \delta_{\varphi_1^*}\rangle$$

$$h(\varphi_2^*) = [g]\langle K^{*,\sigma,0}\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right), \delta_{\varphi_2^*}\rangle$$

Substituting these into (2.25a), (2.25b), we have

$$\begin{aligned} -\tau\psi_1 &= \frac{dc}{dV}(v^*)\left\{-\frac{dV_2^*}{dx}(\varphi_1^*)\psi_1 + [g]\langle K^{*,\sigma,0}\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right), \delta_{\varphi_1^*}\rangle\right\} \\ \tau\psi_2 &= \frac{dc}{dV}(v^*)\left\{\frac{dV_2^*}{dx}(\varphi_2^*)\psi_2 + [g]\langle K^{*,\sigma,0}\left(\delta_{\varphi_1^*}\psi_1 + \delta_{\varphi_2^*}\psi_2\right), \delta_{\varphi_2^*}\rangle\right\} \end{aligned} \tag{2.28}$$

In order to compare (2.28) with the SLEP system (3.75) in Section 3, we need the following equalities. For the definitions of notation, see Remark 3.7, Lemmas 3.15 and 3.22 in Section 3.

Lemma 2.7.

$$\frac{dc}{dV}(v^*)\frac{dV_2^*}{dx}(\varphi_1^*) = -\frac{dc}{dV}(v^*)\frac{dV_2^*}{dx}(\varphi_2^*) = \sigma\hat{\zeta}^{*,\sigma} \tag{i}$$

$$\begin{aligned} -\frac{dc}{dV}(v^*)[g] &= \frac{1}{\left\|\dfrac{d}{dy}\tilde{u}^*\right\|_{L^2}^2}\frac{dJ}{dv}(v^*)[g] \\ &= -\frac{1}{\left\|\gamma^*\dfrac{d}{dy}\tilde{u}^*\right\|_{L^2}^2}\left(-\gamma^*\frac{dJ}{dv}(v^*)\right)(\gamma^*[g]) \\ &= -\frac{c_1^*c_2^*}{(c^*)^2}. \end{aligned} \tag{ii}$$

Proof. See Appendix A. □

Using Lemma 2.7, we can rewrite (2.28) in an equivalent form:

$$\left[\frac{c_1^* c_2^*}{(c^*)^2}\begin{pmatrix} \langle K^{*,\sigma,0}\delta_{\varphi_1^*}, \delta_{\varphi_1^*}\rangle & \langle K^{*,\sigma,0}\delta_{\varphi_2^*}, \delta_{\varphi_1^*}\rangle \\ \langle K^{*,\sigma,0}\delta_{\varphi_1^*}, \delta_{\varphi_2^*}\rangle & \langle K^{*,\sigma,0}\delta_{\varphi_2^*}, \delta_{\varphi_2^*}\rangle \end{pmatrix} + (\tau - \sigma\xi^{*,\sigma})I\right]\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = 0. \quad (2.29)$$

This is exactly the *same* as the SLEP system (see (3.75)) for 2- layered case. According to Theorem 3.24, (2.29) has two negative real eigenvalues, which implies the stability of normal 2-layer solutions. In a similar way, but tedious, we can derive the same result for the n-layered case by linearizing the singular limit slow equations (1.12). We leave the details to the reader.

Remark 2.8. *In view of the above computation, the form* (2.29) *does not depend on the behavior of δ as far as it belongs to the regime $\varepsilon/\delta = o(1)$ as $\varepsilon \downarrow 0$, although the asymptotic form of the resulting critical eigenvalues depends on δ (i.e., it behaves like $O(\varepsilon/\delta)$). It should be noted that, when $\delta = o(1)$ as $\varepsilon \downarrow 0$, u reacts much faster than v and hence the system belongs to the third category according to the classification in Section 1. This suggests that more careful classification is needed depending on the asymptotic regime of δ and ε. In fact, as we saw, as far as stability properties of steady states are concerned, there are no drastic change in the regime of $\varepsilon/\delta = o(1)$ as $\varepsilon \downarrow 0$, however when δ becomes comparable to ε, the system behaves in a quite different manner (see Section 5).*

Remark 2.9. *Although, for 2-layer case, we treat only the local dynamics near Φ_2^*, the global picture has been clarified recently for the bistable case (see Figure 1.1(c)) by [48]: (φ_1, φ_2)-space is divided into three regions by two separation orbits and Φ_2^* lies in one of the domains, say Ω_2^*. If the initial data belongs to Ω_2^*, the orbit tends to Φ_2^* as $t \uparrow \infty$. If not, one of the layers hit the boundary and disappear at certain finite time, then it approaches a mono-layered solution.*

3. The SLEP Method for the Stability of Normal N-layered Solutions

In the previous sections we derived the singular limit slow dynamics (1.12) which approximates the dynamics of the original system (1.1) after layers are fully developed, and proved that all the equilibrium points $\{\Phi_n^*\}_{n=1}$ are locally asymptotically stable. However we do not know in a rigorous sense that how the total dynamics of (1.12) is close to that of (1.1). We shall show in this section by means of the SLEP method that, as far as local stability is concerned, the formal linearized stability analysis for (1.12), which was done in Section 2, becomes a true criterion for the original system (1.1). In fact (1.1) has an ε-family of the normal n-layered solutions (Corollary 3.9) which tends to Φ_n^* as $\varepsilon \downarrow 0$, and they all become stable for sufficiently small ε. This immediately leads us to Main Theorem and the commutative diagram (Figure 1.8) in Section 1. Note that linearized stability implies nonlinear stability for the semilinear parabolic system (1.1) (see, for instance, Henry [29]). This section is a detailed version of the paper Nishiura and Fujii [45].

In what follows, we use the notation $1/\sigma$ instead of D for the diffusion coefficient of v, and assume that $\delta = 1$, since, for general δ, a similar result can be obtained by obvious modification (see Remark 2.8). The model system and its stationary problem become

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(u, v) \\ v_t = \dfrac{1}{\sigma} v_{xx} + g(u, v) \end{cases} \tag{3.1}$$

and

$$\begin{cases} 0 = \varepsilon^2 u_{xx} + f(u, v) \\ 0 = \dfrac{1}{\sigma} v_{xx} + g(u, v), \end{cases} \tag{3.2}$$

respectively. The boundary conditions are always Neumann ones unless otherwise stated.

The core of the SLEP method lies in the asymptotic characterizations of critical eigenvalues and the associated eigenfunctions which are valid up to $\varepsilon = 0$. To do this, the study of the spectral behavior of the singular Sturm-Liouville operator $L^{\varepsilon,\sigma}$ (see (3.12b)) is basic. We start this section with the definition of the normal n-layered solution (Figure 1.3).

3.1. Normal N-layered Solution

In this subsection we briefly mention about the construction of, what we call, the normal n-layered solution to (3.2) which converge to the equilibrium point Φ_n^* of (1.12) as $\varepsilon \downarrow 0$. This solution consists of two parts; the *outer* and *inner* parts. The outer one is determined by the formal limiting system of (3.2) (see (3.3)), and the inner one is essentially, after stretching, the stationary front of (1.8) with $V(\varphi) = v^*$ which compensates the jump discontinuity of u from h_--branch to h_+-branch. Here we simply collect necessary results for later discussions. For the detailed proofs of them, see Fife [16], Mimura, Tabata, and Hosono [39], Ito [32], and Nishiura and Fujii [44; appendix].

First we construct a mono-layered solution, which we call the *basic pattern*, then apply the folding up principle (Proposition 3.8) to it to obtain the normal n-layered solutions. The solution of the following reduced problem becomes the first approximation to the basic pattern in outer region.

$$f(u, v) = 0, \tag{3.3a}$$

$$(u, v) \in L^2(I) \times \{H^2(I) \cap H_N^1(I)\},$$

$$\frac{1}{\sigma} v_{xx} + g(u, v) = 0. \tag{3.3b}$$

Since we are interested in the solutions of (3.3) which are the limit of those of (3.2) as $\varepsilon \downarrow 0$, we take

$$u = h^*(v) \equiv \begin{cases} h_-(v) & \text{for } v \le v^*, \\ h_+(v) & \text{for } v \ge v^* \end{cases} \tag{3.4}$$

as a special solution of (3.3a), i.e., u has a jump from h_--branch to h_+-branch at $v = v^*$. The reason why we take this special value $v = v^*$ comes from the fact that the velocity of the inner front must be zero in order to have a stationary solution (see (1.8), (1.9), and (3.10)). Substituting this into (3.3b), we obtain the reduced scalar equation for v:

$$\frac{1}{\sigma}v_{xx} + G^*(v) = 0, \qquad v \in H^2(I) \cap H^1_N(I), \tag{3.5}$$

where $G^*(v) \equiv g(h^*(v), v)$. Since $G^*(v)$ has a jump discontinuity at $v = v^*$ (see Figure 1.4), the solution of (3.5) has to be matched in C^1-sense at this switching value.

Lemma 3.1. *There exists a uniquely determined positive constant σ_1^* such that monotone increasing (resp. decreasing) C^1-matched solution $V_+^{*,\sigma}(x)$ (resp. $V_-^{*,\sigma}(x)$) of (3.5) exists uniquely for $0 < \sigma \le \sigma_1^*$. Moreover, we have $\lim_{\sigma\downarrow 0} V^{*,\sigma}(x) = v^*$ in C^1-sense.*

For definiteness, we only consider the monotone increasing case and write simply $V^{*,\sigma}(x)$ instead of $V_+^{*,\sigma}(x)$. In view of (3.4), the first approximate solution takes the following form:

$$(U^{*,\sigma}(x), V^{*,\sigma}(x)) \qquad \text{for } 0 \le \sigma \le \sigma_1^*, \tag{3.6}$$

where $U^{*,\sigma}(x) \equiv h^*(V^{*,\sigma}(x))$. We call (3.6) the *reduced solution* for the basic pattern.

Corollary 3.2. *The matching point $x_1^*(\sigma)$ is well-defined by*

$$V^{*,\sigma}(x_1^*(\sigma)) = v^* \tag{3.7}$$

due to the monotonicity of $V^{*,\sigma}(x)$. Then $x_1^*(\sigma)$ becomes a continuous function for $0 \le \sigma \le \sigma_1^*$.

Applying the singular perturbation techniques to (3.6), we have the following existence result for the basic pattern ([16], [39], [32], [44]).

Theorem 3.3 (Existence Theorem for the Basic Pattern). *For any σ_0 with $0 < \sigma_0 < \sigma_1^*$, there is an $\varepsilon_0 > 0$ such that (3.2) has an (ε, σ)-family of solutions $D^1(\varepsilon, \sigma) = (u^1(x; \varepsilon, \sigma), v^1(x; \varepsilon, \sigma)) \in C_\varepsilon^2(\bar{I}) \times C^2(\bar{I})$ for $(\varepsilon, \sigma) \in Q^1 = \{(\varepsilon, \sigma) | 0 < \varepsilon < \varepsilon_0, 0 \le \sigma \le \sigma_0\}$. $D^1(\varepsilon, \sigma)$ are uniformly bounded in $C_\varepsilon^2(\bar{I}) \times C^2(\bar{I})$, and satisfy*

$$\lim_{\varepsilon\downarrow 0} u^1(x; \varepsilon, \sigma) = U^{*,\sigma}(x) \text{ uniformly on } \bar{I}\backslash I_\kappa \text{ for any } \kappa > 0 \tag{3.8a}$$

and

$$\lim_{\varepsilon\downarrow 0} v^1(x; \varepsilon, \sigma) = V^{*,\sigma}(x) \text{ uniformly on } \bar{I} \tag{3.8b}$$

where $I_\kappa = (x_1^*(\sigma) - \kappa, x_1^*(\sigma) + \kappa)$. Moreover, $D^1(\varepsilon, \sigma)$ depends continuously on $(\varepsilon, \sigma) \in Q^1$ in $C_\varepsilon^2 \times C^2$-topology, and continuously on $(\varepsilon, \sigma) \in \overline{Q}^1$ in $L^2 \times C^1$-topology.

Using a streched variable y defined by

$$y \equiv \frac{x - x_1^*(\sigma)}{\varepsilon}, \qquad y \in \tilde{I} \equiv \left(-\frac{x_1^*(\sigma)}{\varepsilon}, \frac{1 - x_1^*(\sigma)}{\varepsilon}\right)$$

the inner part of $D_1(\varepsilon,\sigma)$ *behaves as*

$$\lim_{\varepsilon\downarrow 0}(\tilde{u}^{\varepsilon,\sigma}(y),\tilde{v}^{\varepsilon,\sigma}(y)) = (\tilde{u}^*(y), v^*) \ \text{in} \ C^2_{c.u.}(\mathbf{R})\text{-sense}, \tag{3.9}$$

where $(\tilde{u}^{\varepsilon,\sigma},\tilde{v}^{\varepsilon,\sigma})$ *are the stretched solutions defined by*

$$\tilde{u}^{\varepsilon,\sigma}(y) \equiv u^1(x_1^*+\varepsilon y;\varepsilon,\sigma), \qquad \tilde{v}^{\varepsilon,\sigma}(y) \equiv v^1(x_1^*+\varepsilon y;\varepsilon,\sigma).$$

$\tilde{u}^*(y)$ *is a translate of the unique monotone increasing solution of*

$$\begin{cases} \dfrac{d^2}{dy^2}\tilde{u} + f(\tilde{u}, v^*) = 0 \\ \tilde{u}(\pm\infty) = h_\pm(v^*) \\ \tilde{u}(0) = h_0(v^*), \end{cases} \tag{3.10}$$

and v^* *is the unique zero of* $J(v)$ *(see* (A.2)*). Here we use the convention of Remark 3.5 for the stretched functions. Note that the limiting function of* (3.9) *does not depend on* σ.

Remark 3.4. *The convergence result for* $\tilde{v}^{\varepsilon,\sigma}$ *(see* (3.9)*) comes from the following fact: Let* $\varphi^\varepsilon(x) \in C^0(\bar{I})$ *for* $0<\varepsilon<\varepsilon_0$, *and* $\|\varphi^\varepsilon\|_{C^2(I)}$ *be uniformly bounded with respect to* ε. *Suppose that* $\lim_{\varepsilon\downarrow 0}\varphi^\varepsilon(x^*) = \alpha$ *holds at some point* $x^*\in I$, *where* α *is a constant. Then the stretched function* $\tilde{\varphi}^\varepsilon(y)$ *at* $x=x^*$, *i.e.,* $\tilde{\varphi}^\varepsilon(y)\equiv\varphi^\varepsilon(x^*+\varepsilon y)$ *satisfies*

$$\lim_{\varepsilon\downarrow 0}\tilde{\varphi}^\varepsilon(y) = \alpha \quad \text{in} \quad C^2_{c.u.}(\mathbf{R})\text{-sense}.$$

Remark 3.5. *For a given stretched function* $\tilde{\varphi}(y)(y\in\tilde{I})$, *it is convenient to extend the definition domain from* $\tilde{I}$ *to the whole line* $\mathbf{R}$ *in a smooth way so that* $\varphi\equiv 0$ *for large* $|y|$. *We use the same notation for the extended one.*

Corollary 3.6. *Let* $F(u,v)$ *be a smooth function of* u *and* v. *Then, the composite function* $F(\tilde{u}^{\varepsilon,\sigma},\tilde{v}^{\varepsilon,\sigma})$ *converges to* $F(\tilde{u}^*, v^*)$ *in* $C^2_{c.u.}(\mathbf{R})$*-sense.*

Remark 3.7. *Differentiating* (3.10) *by* y, *we see that* $\dfrac{d}{dy}\tilde{u}^*(>0)$ *is a constant multiple of the principal eigenfunction of the following eigenvalue problem associated with the principal eigenvalue* $\zeta=0$:

$$\hat{L}^*\tilde{\phi} \equiv \frac{d^2}{dy^2}\tilde{\phi} + f_u(\tilde{u}^*, v^*)\tilde{\phi} = \zeta\tilde{\phi} \qquad \text{on } \mathbf{R}, \qquad \tilde{\phi}\in L^2(\mathbf{R}). \tag{3.11}$$

We denote by $\hat{\phi}_0^*$ *and* $\tilde{\phi}_L^*$ *the* L^2*-normalized principal eigenfunction and the positive* L^1*-normalized principal eigenfunction, respectively, i.e.,* $\|\hat{\phi}_0^*\|_{L^2(\mathbf{R})}=1$, $\int_{\mathbf{R}}\hat{\phi}_L^*dx = 1$. *The interrelation among these quantities is given by*

$$\hat{\phi}_0^* = \frac{\gamma^*}{c^*}\frac{d}{dy}\tilde{u}^*$$

$$\hat{\phi}_L^* = \gamma^* \frac{d}{dy}\bar{u}^*$$

$$\hat{\phi}_L^* = c^* \hat{\phi}_0^*,$$

where

$$c^* \equiv \|\hat{\phi}_L^*\|_{L^2(\mathbf{R})}$$

$$\gamma^* \equiv 1/(h_+(v^*) - h_-(v^*)).$$

Multi-layered patterns called the normal n-layered solutions are easily constructed by applying the next proposition to the basic pattern.

Proposition 3.8 (Folding Up Principle). *Suppose $W(x;\underline{d})$ is a solution of* (3.2) *at $\underline{d} = (\bar{\varepsilon}^2, \bar{\sigma}^{-1})$, then $R^n(W)(x)$ is a solution of* (3.2) *at $\underline{d}/n^2 = (\varepsilon^2, \sigma^{-1})$ for $n = 1, 2, \cdots$, where $(\varepsilon, \sigma) \equiv (\bar{\varepsilon}/n, n^2\bar{\sigma})$. Here*

$$R^n(W)(x) = \begin{cases} W(n(x - i/n);\underline{d}) & i = \text{even}, \\ W(n(1/n - (x - i)/n);\underline{d}) & i = \text{odd}, \end{cases}$$

for $i/n \le x \le (i+1)/n$ $(i = 0, 1, 2, \cdots, n-1)$.

Intuitively speaking, $R^n(W)$ can be obtained by flipping W n-times and normalizing the length of interval to one.

Corollary 3.9 (Existence of the Normal n-layered solution). *Let $W = D^1(\bar{\varepsilon}, \bar{\sigma})$ $((\bar{\varepsilon}, \bar{\sigma}) \in Q^1)$ in Proposition 3.8, then $D^n(\varepsilon, \sigma) = (u^n(x;\varepsilon,\sigma), v^n(x;\varepsilon,\sigma))$ defined by $R^n(D^1(\bar{\varepsilon}, \bar{\sigma}))$ becomes a solution of* (3.2) *with n interior transition layers for $(\varepsilon, \sigma) = (\bar{\varepsilon}/n, n^2\bar{\sigma}) \in Q^n \equiv \{(\varepsilon, \sigma) | 0 < \varepsilon < \varepsilon_0/n, 0 \le n^2\sigma_0\}$ (see Figure 1.3). We write the reduced solution of $D^n(\varepsilon, \sigma)$ (i.e., the L^2-limit of $D^n(\varepsilon, \sigma)$ as $\varepsilon \downarrow 0$) as $(U_n^{*,\sigma}(x), V_n^{*,\sigma}(x))$. $D^n(\varepsilon, \sigma)$ is called the normal n-layered solution or simply n-layer solution.*

3.2. Asymptotic Behaviors of Critical Eigenvalues and Eigenfunctions of $L^{\varepsilon,\sigma}$

We shall study the stability properties of the normal n-layered solutions $D^n(\varepsilon, \sigma)$ in the following three subsections. Since our model is a semilinear parabolic system, the stability is determined by the spectrum of the following linearized operator at $D^n(\varepsilon, \sigma)$

$$\mathcal{L}^{\varepsilon,\sigma}\begin{pmatrix} w \\ z \end{pmatrix} \equiv \begin{pmatrix} L^{\varepsilon,\sigma} & f_v^{\varepsilon,\sigma} \\ g_u^{\varepsilon,\sigma} & M^{\varepsilon,\sigma} \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \lambda \begin{pmatrix} w \\ z \end{pmatrix}, \tag{3.12a}$$

$$(w, z)^t \in \{H^2(I) \cap H_N^1(I)\}^2$$

where

$$L^{\varepsilon,\sigma} \equiv \varepsilon^2 \frac{d^2}{dx^2} + f_u^{\varepsilon,\sigma} \tag{3.12b}$$

$$M^{\varepsilon,\sigma} \equiv \frac{1}{\sigma}\frac{d^2}{dx^2} + g_v^{\varepsilon,\sigma}, \tag{3.12c}$$

and all partial derivatives are evaluated at $D^n(\varepsilon, \sigma) = (u^n(x; \varepsilon, \sigma), v^n(x; \varepsilon, \sigma))$, i.e., $f_u^{\varepsilon,\sigma} \equiv f_u(u^n(x; \varepsilon, \sigma), v^n(x; \varepsilon; \sigma))$ and so on.

A naive way to find the spectral behavior of (3.12) as $\varepsilon \downarrow 0$ is to put $\varepsilon = 0$ in (3.12), however it turns out that the resulting system is exactly the same as (2.22) in Section 2. Namely just the formal limit of (3.12) tells us only the behavior of noncritical eigenvalues which govern the behavior of solutions in outer region, and does not inherit any information from layer part. As we saw in Section 2, the most subtle part of spectral analysis comes from the layer part. The reason for this is that it is related to neutral (zero) eigenvalue of *translation invariance*. More precisely, rewriting (3.12) by using a stretched coordinate $y = (x - x_i^*)/\varepsilon$ at any layer position x_i^* and taking a formal limit of $\varepsilon \downarrow 0$, we see from Theorem 3.3 that (3.12) restricted to the i-th subinterval $I_i \equiv ((i-1)/n, i/n)$ becomes

$$\begin{aligned} &\frac{d^2}{dy^2}\tilde{w} + f_u(\tilde{u}^*, v^*)\tilde{w} + f_v(\tilde{u}^*, v^*)\tilde{z} = \lambda\tilde{w} \\ & && y \in \mathbf{R} \\ &\frac{d^2}{dy^2}\tilde{z} = 0. \end{aligned} \tag{3.13}$$

It follows from Remark 3.7 that (3.13) has zero eigenvalue with $(\tilde{w}, \tilde{z}) = (\frac{d\tilde{u}^*}{dy}, 0)$. This observation suggests that (3.12) has an eigenvalue which approaches zero as $\varepsilon \downarrow 0$ associated with each layer. In fact the number of these *critical* eigenvalues (i.e., those which tend to zero as $\varepsilon \downarrow 0$) of (3.12) will be proved to be exactly equal to the number of layers ($= n$) and their precise behavior will be clarified by solving the SLEP matrix which eventually leads us to the main result (Theorem 3.25).

In view of (3.13), we see that each zero eigenvalue coincides with that of the limiting (stretched) Sturm-Liouville operator, hence $L^{\varepsilon,\sigma}$ itself is expected to have n eigenvalues going to zero when $\varepsilon \downarrow 0$. In fact, we shall show in Lemma 3.10 that there are *n positive* critical eigenvalues of $L^{\varepsilon,\sigma}$, and the rest of the spectrum is strictly bounded away from zero. The asymptotic behavior of these critical eigenvalues will be investigated in Lemma 3.15. Note that we use the word "critical" both for eigenvalues of $L^{\varepsilon,\sigma}$ and the full system $\mathcal{L}^{\varepsilon,\sigma}$.

The supports of eigenfunctions associated with the critical eigenvalues of $L^{\varepsilon,\sigma}$ are concentrated on layer positions when $\varepsilon \downarrow 0$, in fact, after an appropriate scaling, each eigenfunction tends to a combination of Dirac's point mass distribution on layer positions (Lemma 3.22). This seems very singular, however, we will see that it is equivalent to say that the stretched eigenfunction at each layer position approaches a constant multiple of the principal eigenfunction of the limiting Sturm-Liouville problem (3.11).

The next lemma tells us the number of critical eigenvalues of the Sturm-Liouville problem:

$$\begin{aligned} &L^{\varepsilon,\sigma}\phi = \zeta\phi \qquad \text{on } I \\ &\phi \in H^2(I) \cap H_N(I), \qquad \|\phi\|_{L^2(I)} = 1. \end{aligned} \tag{3.14}$$

We denote the complete orthonormal set of (3.14) by $\{\zeta_i^{\varepsilon,\sigma}, \phi_i^{\varepsilon,\sigma}\}$. Throughout this subsection we omit the superscript σ like L^ε, $\zeta_i^\varepsilon, \phi_i^\varepsilon$ in the proofs of lemmas.

Lemma 3.10. *The first n eigenvalues of $L^{\varepsilon,\sigma}$ tend to zero as $\varepsilon \downarrow 0$, and the rest of the eigenvalues of $L^{\varepsilon,\sigma}$ is strictly negative up to $\varepsilon = 0$.*

Proof. We first consider the behavior of the principal eigenvalue of (3.14). Since a normal n-layered solution is obtained by folding up a mono-layered solution n-times, the principal eigenfunction of L^ε is also obtained by folding that of L^ε restricted to the first subinterval $I_1 \equiv (0, 1/n)$

$$\begin{aligned} &L^\varepsilon \phi_{01}^\varepsilon = \zeta_0^\varepsilon \phi_{01}^\varepsilon \qquad \text{on } I_1 \\ &\phi_{01}^\varepsilon \in H^2(I_1) \cap H_N(I_1), \quad \| \phi_{01}^\varepsilon \|_{L^2(I)} = \frac{1}{n}, \end{aligned} \tag{3.15}$$

where ζ_0^ε is the principal eigenvalue of (3.14) and ϕ_{01}^ε denotes the restriction of the associated principal eigenfunction to I_1. Using a stretched coordinate $y = (x - x_1^*)/\varepsilon$, (3.15) becomes the following with a new normalization :

$$\begin{aligned} &\frac{d^2}{dy^2}\tilde{\phi}_{01}^\varepsilon + \tilde{f}_u^\varepsilon \tilde{\phi}_{01}^\varepsilon = \zeta_0^\varepsilon \tilde{\phi}_{01}^\varepsilon, \qquad y \in \tilde{I}_1, \\ &\tilde{\phi}_{01}^\varepsilon \in H^2(\tilde{I}_1) \cap H_N(\tilde{I}_1), \quad \| \tilde{\phi}_{01}^\varepsilon \|_{L^2(\tilde{I}_1)} = 1 \end{aligned} \tag{3.16}$$

where $\tilde{I}_1 \equiv (-\ell/\varepsilon, r/\varepsilon)$ with $\ell = x_1^*$, $r = 1/n - x_1^*$, and $\tilde{\phi}_{01}^\varepsilon (> 0)$ is the normalized stretched principal eigenfunction. In view of Corollary 3.6, we see that $\lim_{\varepsilon \downarrow 0} \tilde{f}_u^\varepsilon = f_u(\tilde{u}^*, v^*)$ in $C^2_{c.u.}(\mathbf{R})$-sense. Recalling the behavior in outer region (see Theorem 3.3), we can find a finite interval I_0 and positive constants μ and γ such that

$$-\gamma < \tilde{f}_u^\varepsilon < -\mu < 0 \qquad y \in \tilde{I}_1 \backslash I_0 \tag{3.17}$$

holds, where I_0, μ, γ are independent of $(\varepsilon, \sigma) \in Q^n$.

First we give a lower bound for ζ_0^ε. It is clear that ζ_0^ε is characterized by

$$\zeta_0^\varepsilon \equiv \sup_{\tilde{\phi} \in H_N^1(\tilde{I}_1),\ \| \tilde{\phi} \|_{L^2(\tilde{I}_1)} = 1} (-\langle \tilde{\phi}_y, \tilde{\phi}_y \rangle + \langle \tilde{f}_u^\varepsilon \tilde{\phi}_y, \tilde{\phi}_y \rangle). \tag{3.18a}$$

We know from Remark 3.7 that

$$0 = \sup_{\tilde{\phi} \in H^1(\mathbf{R}),\ \| \tilde{\phi} \|_{L^2(\mathbf{R})} = 1} (-\langle \tilde{\phi}_y, \tilde{\phi}_y \rangle + \langle \tilde{f}_u^* \tilde{\phi}_y, \tilde{\phi}_y \rangle) \tag{3.18b}$$

which is attained by $\hat{\phi}_0 \left(= \dfrac{\gamma^*}{c^*} \dfrac{d}{dy} \tilde{u}^* \right)$. Taking a constant multiple of $\hat{\phi}_0^*$ (restricted to $\tilde{I}_1$) as a test function for (3.18a), and using the facts that $\lim_{\varepsilon \downarrow 0} \tilde{f}_u^\varepsilon = \tilde{f}_u^*$ in $C^2_{c.u.}(\mathbf{R})$-sense and $\dfrac{d}{dy}\tilde{u}^*$ decays exponentially as $|y| \to \infty$, we see that, for arbitrary small $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that

$$\zeta_0^\varepsilon > -\delta \quad \text{for} \quad 0 < \varepsilon < \varepsilon_\delta$$

holds. Also it is clear from (3.18a) that $\zeta_0^\varepsilon < \max_y \tilde{f}_u^\varepsilon$. These inequalities imply that

$$q^\varepsilon(y) \equiv \tilde{f}_u^\varepsilon - \zeta_0^\varepsilon \tag{3.19}$$

also satisfies the inequality (3.17) for small ε. Using this property, we can show the exponentially decaying property of $\tilde{\phi}_{01}^\varepsilon$:

$$|\tilde{\phi}_{01}^\varepsilon| \leq C\exp(-C_1|y|), \tag{3.20}$$

which will be proved in Lemma 3.11 in more general setting. Multiplying $\tilde{\phi}_{01}^\varepsilon$ on both sides of (3.16) and integrating over $\tilde{I}_1$, we see from (3.19) that $\| \tilde{\phi}_{01}^\varepsilon \|_{H^1(\tilde{I}_1)} < M$ independently of ε, which implies via Sobolev imbedding theorem that $\tilde{\phi}_{01}^\varepsilon$ has a convergent subsequence $\tilde{\phi}_{01}^\varepsilon$ (hereafter we use the same notation for subsequences) on any compact subset $K(\subset \mathbf{R})$ in $C^0(K)$-topology. Using (3.16) again, this becomes a convergent sequence in $C^2(K)$-topology. Using a diagonal argument on an expanding sequence of compact intervals to $\mathbf{R}$ and recalling the boundedness of ζ_0^ε, we can find a convergent subsequence $(\tilde{\phi}_{01}^\varepsilon, \zeta_0^\varepsilon)$ in $C_{c.u.}^2(\mathbf{R}) \times \mathbf{R}$-topology. The limiting function denoted by $(\tilde{\Phi}_{01}^*, \zeta_0^*)$ satisfies

$$\frac{d^2}{dy^2}\tilde{\Phi}_{01}^* + \tilde{f}_u^*\tilde{\Phi}_{01}^* = \zeta_0^*\tilde{\Phi}_{01}^*. \tag{3.21}$$

We shall show that

$$\lim_{\varepsilon\downarrow 0}(\tilde{\phi}_{01}^\varepsilon, \zeta_0^\varepsilon) = (\tilde{\phi}_{01}^*, \zeta_0^*) = (\hat{\phi}_{01}^*, 0) \quad \text{in} \quad C^2(\mathbf{R}) \times \mathbf{R}\text{-sense.} \tag{3.22}$$

Here we use the convention of Remark 3.5 for $\tilde{\phi}_{01}^\varepsilon$. First, because of (3.20), we see that $\tilde{\phi}_{01}^\varepsilon \not\equiv 0$ and satisfies (3.21), moreover $\tilde{\phi}_{01}^* > 0$ since all $\tilde{\phi}_{01}^\varepsilon$ are positive principal eigenfunctions. On the other hand, we know (see Remark 3.7) that (3.21) has zero as the principal eigenvalue with $\hat{\phi}_0^*$ being the associated eigenfunction. Hence, because of the simplicity of the principal eigenvalue, we obtain (3.22) in $C_{c.u.}^2(\mathbf{R}) \times \mathbf{R}$-sense for a chosen subsequence. Apparently the limiting function does not depend on the choice of the subsequence, and taking into account (3.20), we can conclude that $(\tilde{\phi}_{01}^\varepsilon, \zeta_0^\varepsilon)$ itself converges to $(\hat{\phi}_0^*, 0)$ in $C^2(\mathbf{R}) \times \mathbf{R}$-topology, which completes the proof of (3.22).

All the above discussions remain valid when we change the boundary conditions to Dirichlet ones (with replacing $H_N^1(\tilde{I}_1)$ by $H_0^1(\tilde{I}_1)$ in (3.16)). In particular, the principal eigenvalue ζ_{01D}^ε of $(3.15)_D$ (or $(3.16)_D$) also satisfies

$$\lim_{\varepsilon\downarrow 0}\zeta_{01D}^\varepsilon = 0, \tag{3.23}$$

where the subscript D represents that the concerned quantity is considered under Dirichlet boundary conditions. Note that, by comparison theorem,

$$\zeta_{01D}^\varepsilon < \zeta_0^\varepsilon \qquad \text{for} \quad \varepsilon > 0. \tag{3.24}$$

We denote by ϕ_{01D}^ε the principal eigenfunction of $(3.15)_D$.

Now we return to the problem (3.14) for n-layered solution. Flipping ϕ_{01}^{ε} of (3.15) n times in an even way, the resulting function (denoted by ϕ_0^{ε}) defined on I, becomes a principal eigenfunction (i.e., nodal zero) of (3.14). Note that the eigenvalue ζ_0^{ε} remains the same as before by folding operation. On the other hand, by flipping ϕ_{01D}^{ε} n-times in an odd way, we obtain the eigenfunction (denoted by $\phi_{n-1,D}^{\varepsilon}$) of $(3.14)_D$ which has $n-1$ nodal zeros inside of I. Hence $\zeta_{01D}^{\varepsilon}$ becomes the n-th eigenvalue $\zeta_{n-1,D}^{\varepsilon}$ of L^{ε} under Dirichlet boundary conditions. Making use of the comparison theorem and the nodal property of Sturm-Liouville operator, we see that

$$\zeta_{n-1,D}^{\varepsilon} = \zeta_{01D}^{\varepsilon} < \zeta_{n-1}^{\varepsilon} < \cdots < \zeta_0^{\varepsilon}$$

holds, namely the eigenvalues $\zeta_1^{\varepsilon}, \cdots, \zeta_{n-1}^{\varepsilon}$ are sandwiched by ζ_0^{ε} and $\zeta_{01D}^{\varepsilon}$. The asymptotic behaviors (3.22) and (3.23) lead to the first part of Lemma 3.10. □

As for the second part, first note that the second eigenvalue ζ_{11}^{ε} (nodal one) of (3.15) is equal to the $(n+1)$ -th eigenvalue ζ_n^{ε} (nodal n) of (3.14). Since the principal eigenvalue ζ_0^{ε} of (3.15), which is simple up to $\varepsilon = 0$, converges to zero as $\varepsilon \downarrow 0$, ζ_{11}^{ε} and hence ζ_n^{ε} must be strictly negative for small ε. This completes the proof of Lemma 3.10.

In order to know the precise asymptotic behaviors of $\zeta_i^{\varepsilon,\sigma}$ $(i = 0, \cdots, n-1)$, we need to find the asymptotic forms of $\phi_i^{\varepsilon,\sigma}$ $(i = 0, \cdots, n-1)$. As we saw in the proof of Lemma 3.10, the stretched function $\tilde{\phi}_{01}^{\varepsilon,\sigma}$ (on unit subinterval) converges to the principal eigenfunction of (3.11). Noting the relation $\int_{I_1} \left|\frac{\phi}{\sqrt{\varepsilon}}\right|^2 dx = \int_{\tilde{I}_1} |\tilde{\phi}|^2 dy$ between a function $\phi(x)(x \in I_1)$ and its stretched one $\tilde{\phi}(y)(y \in \tilde{I}_1)$, we see that the L^2-normalized function $\phi_{01}^{\varepsilon,\sigma}(x)$ has a sharp peak of height $O(1/\sqrt{\varepsilon})$, and decays quickly outside of this peak as $\varepsilon \downarrow 0$. Hence it is indispensable to use *stretching* to characterize the asymptotic behavior of critical eigenfunctions. Since there are n layers, it is convenient to introduce the following notation.

$$\phi_{ij}^{\varepsilon,\sigma} \equiv \phi_i^{\varepsilon,\sigma}\big|_{I_j}\,;\text{restriction to the subinterval } I_j \text{ of the } i\text{-th eigenfunction},$$

$$\tilde{\phi}_{ij}^{\varepsilon,\sigma} = \tilde{\phi}_{ij}^{\varepsilon,\sigma}(y) \equiv \phi_{ij}^{\varepsilon,\sigma}(x_j^* + \varepsilon y);\text{stretched function of } \ y \in \tilde{I}_j,$$

where $y = (x - x_j^*)/\varepsilon$ and $\tilde{I}_j$ denotes $(-\ell/\varepsilon, r/\varepsilon)$ or $(-r/\varepsilon, \ell/\varepsilon)$. It holds that

$$\int_I |\phi_i^{\varepsilon,\sigma}|^2 dx = \sum_{j=1}^{n} \int_{I_j} |\phi_{ij}^{\varepsilon,\sigma}|^2 dx = \sum_{j=1}^{n} \int_{\tilde{I}_j} |\sqrt{\varepsilon}\tilde{\phi}_{ij}^{\varepsilon,\sigma}|^2 dy. \tag{3.25}$$

We define $\hat{\phi}_{ij}^{\varepsilon,\sigma}$ by

$$\hat{\phi}_{ij}^{\varepsilon,\sigma}(y) \equiv \sqrt{\varepsilon}\tilde{\phi}_{ij}^{\varepsilon,\sigma}, \qquad y \in \tilde{I}_j. \tag{3.26}$$

We call $\hat{\phi}_{ij}^{\varepsilon,\sigma}$ the *normalized j-th stretched function* of $\phi_i^{\varepsilon,\sigma}$. It is obvious from (3.25) that

$$\int_I |\phi_i^{\varepsilon,\sigma}|^2 dx = \sum_{j=1}^{n} \int_{\tilde{I}_j} |\hat{\phi}_{ij}^{\varepsilon,\sigma}|^2 dy \tag{3.27}$$

A key result for the critical eigenfunctions is that, for any i and j, $\hat{\phi}_{ij}^{\varepsilon,\sigma}$ converges to a constant multiple of $\frac{d}{dy}\tilde{u}^*$ in $C^2(\mathbf{R})$-sense as $\varepsilon \downarrow 0$ (Lemma 3.12). Namely, if we look at a critical eigenfunction in a stretched form, it is very close to the derivative of the stretched layer function $\tilde{u}^*(y)$ of Theorem 3.3. To this end, we need the following lemma.

Lemma 3.11 (Exponential decaying property). *Let $\hat{\phi}_{ij}^{\varepsilon,\sigma}$ be the normalized j-th stretched eigenfunction of $\phi_i^{\varepsilon,\sigma}$ $(0 \le i \le n-1, 1 \le j \le n)$. Then there is a finite interval I_0, which is independent of ε, σ, i, and j, such that*

$$\left| \frac{d^k}{dy^k} \hat{\phi}_{ij}^{\varepsilon,\sigma} \right| \le C \exp(-C_1 |y|), \qquad y \in \tilde{I}_j \backslash I_0 \tag{3.28}$$

hold for $k = 0, 1, 2$, where C and C_1 are positive constants independent of ε, σ, i, and j.

Proof. In view of Lemma 3.10 and (3.19), we have for small ε

$$-\gamma < \tilde{f}_u^\varepsilon - \zeta_i^\varepsilon < -\mu < 0, \qquad y \in \tilde{I}_j \backslash I_0, \tag{3.29}$$

where μ, γ, and I_0 are same as in (3.17). We decompose $\tilde{I}_j$ into three parts (see Figure 3.1): $\tilde{I}_j = \tilde{I}_{j,\ell} \cup I_0 \cup \tilde{I}_{j,r}$.

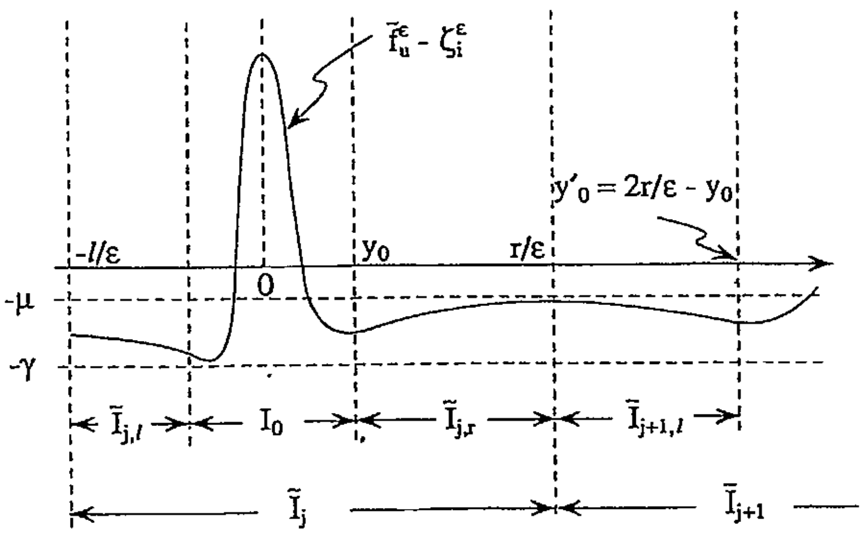

Figure 3.1

Without loss of generality, we prove (3.28) on the interval $\tilde{I}_{j,r} \equiv (y_0, r/\varepsilon)$. We consider the problem on the extended interval $\tilde{I}_R \equiv \tilde{I}_{j,r} \cup \tilde{I}_{j+1,\ell}$. If $\tilde{I}_{j,r}$ is the right-end interval, we extend everything to the right in an even manner. Note that $\tilde{f}_u^\varepsilon - \zeta_i^\varepsilon$ has reflectional symmetry at $y = r/\varepsilon$ and (3.29) holds on $\tilde{I}_R$. By using (3.29), it is not difficult to show that there exist two linearly independent solutions $\psi_+(y)$ and $\psi_-(y)$ of

$$\frac{d^2}{dy^2}\tilde{\phi} + (\tilde{f}_u^\varepsilon - \zeta_i^\varepsilon)\tilde{\phi} = 0 \tag{3.30}$$

on $\bar{I}_R$ satisfying

$$\psi_+(y_0) = 1, \qquad \frac{d}{dy}\psi_+(y_0') = 0, \tag{3.31a}$$

$$C_2 \exp(-\gamma y) \leq \frac{d^k}{dy^k}\psi_+(y) \leq C_3 \exp(-\mu y), \qquad k = 0, 1, 2, \tag{3.31b}$$

$$\psi_-(y) \equiv \psi_+(2r/\varepsilon - y), \qquad y \in \bar{I}_R, \tag{3.31c}$$

where $y_0' = 2r/\varepsilon - y_0$, C_2 and C_3 are positive constants independent of ε, σ, i and j. Here we use the fact that $\bar{f}_u^\varepsilon - \zeta_i^\varepsilon$ has reflectional symmetry at $y = r/\varepsilon$ which implies that $\psi_-(y)$ defined by (3.31c) becomes a solution of (3.30). It is convenient to introduce the following pair of linearly independent solutions:

$$\begin{aligned} \Psi_s &\equiv \frac{1}{2}(\psi_+ + \psi_-) \\ \Psi_a &\equiv \frac{1}{2}(\psi_+ - \psi_-) \end{aligned} \tag{3.32}$$

where Ψ_s (resp. Ψ_a) is an even (resp. odd) function at $y = r/\varepsilon$. $\hat{\phi}_{ij}^\varepsilon$ can be expressed by

$$\hat{\phi}_{ij}^\varepsilon = c_s\Psi_s + c_a\Psi_a \qquad \text{on } \bar{I}_R. \tag{3.33}$$

Since $\| \phi_i^\varepsilon \|_{L^2(I)} = 1$, we see from (3.27), (3.33) and $(\Psi_s, \Psi_a)_{\bar{I}_R} = 0$ that

$$\int_{\bar{I}_R} |\hat{\phi}_{ij}^\varepsilon|^2 dy = c_s^2 \| \Psi_s \|^2_{L^2(\bar{I}_R)} + c_a^2 \| \Psi_a \|^2_{L^2(\bar{I}_R)} \leq 1.$$

It is clear from (3.31b), that

$$m_1 < \| \Psi_s \|_{L^2(\bar{I}_R)}, \quad \text{and} \quad \| \Psi_a \|_{L^2(\bar{I}_R)} < m_2,$$

where $m_i (i = 1, 2)$ are positive constants independent of ε, σ, i and j. These two inequalities lead to the following

$$|c_s|, \quad |c_a| \leq M,$$

where M does not depend on parameters. In terms of $\psi_\pm$, $\hat{\phi}_{ij}^\varepsilon$ can be written as $\hat{\phi}_{ij}^\varepsilon = (c_s + c_a)\psi_+/2 + (c_s - c_a)\psi_-/2$, where the coefficients are bounded due to the above estimate. This implies the required estimate for $\hat{\phi}_{ij}^\varepsilon$, since we see from (3.31) that ψ_+ (and its derivatives) decays exponentially and the contribution of ψ_- on $\bar{I}_{j,r}$ is exponentially small. □

Now we are ready to prove the following.

Lemma 3.12 (Precompactness of $\hat{\phi}_{ij}^\varepsilon$). *There exists a subsequence* $\{\hat{\phi}_{ij}^{\varepsilon_m,\sigma}\}_{m=1}^\infty$ *with* $\lim_{m\uparrow\infty} \varepsilon_m = 0$ *from* $\{\hat{\phi}_{ij}^{\varepsilon,\sigma}\}$ *such that*

$$\lim_{m\uparrow\infty} \hat{\phi}_{ij}^{\varepsilon_m,\sigma} = \kappa_j^i \hat{\phi}_L^* \qquad \textit{in } C^2(\mathbf{R})\textit{-sense (recall Remark 3.5)} \tag{3.34}$$

hold simultaneously for all i and $j(0 \leq i \leq n-1, 1 \leq j \leq n)$. *The resulting vectors* $\{c^*\mathbf{k}^i\}_{i=0}^{n-1}$ *defined by*

$$c^*\mathbf{k}^i \equiv c^*(\kappa_1^i, \cdots, \kappa_n^i)$$

$$c^* \equiv \|\hat{\phi}_L^*\|_{L^2(\mathbf{R})}$$

form an othonormal set in $\mathbf{R}^n$, *i.e.,*

$$(c^*)^2 \sum_{j=1}^{n} \kappa_j^\ell \kappa_j^k = \delta_{\ell k}, \qquad \ell, k \in \{0, 1, \cdots, n-1\}, \tag{3.35}$$

where $\delta_{\ell k}$ *denotes the Kronecker's* δ.

Remark 3.13. *The coefficients* κ_j^i *in Lemma 3.12 except* $i = 0$ *may depend on the choice of subsequence, however it does not affect later discussions. In fact we will see in Section 3.3 that the final form* (3.75) *of the SLEP system is independent of the choice of subsequence. It is conjectured that* $\hat{\phi}_{ij}^{\varepsilon,\sigma}$ *has a unique limit and* κ_j^i *does not depend on subsequences.*

Proof. Using Lemma 3.11, the proof of (3.34) for a fixed i and j proceeds in a quite similar way as in the proof of Lemma 3.10 where $(\tilde{\phi}_{01}^\varepsilon, \zeta_0^\varepsilon)$ was shown to converge to $(\hat{\phi}_0^*, 0)$ in $C^2(\mathbf{R}) \times \mathbf{R}$-sense when $\varepsilon \downarrow 0$. However, because of the lack of knowledge about the convergence of $\| \hat{\phi}_{ij}^\varepsilon \|_{L^2(\bar{I}_j)}$ $(i \geq 1)$ as $\varepsilon \downarrow 0$ (although it is uniformly bounded), we have to choose a subsequence, and this cause the above anbiguity, i.e., κ_j^i may depend on the choice of subsequence. Using a diagonal argument for i and j, we can find a subsequence ε_m such that (3.34) holds for all i and j. The orthogonality (3.35) comes from that of the eigenfunctions of the Sturm-Liouville operator, i.e., $\langle \phi_i^\varepsilon, \phi_j^\varepsilon \rangle = \delta_{ij}$. □

Corollary 3.14.

$$\int_I |\phi_i^{\varepsilon,\sigma}| dx = L_i(\varepsilon, \sigma)\sqrt{\varepsilon},$$

where $L_i(\varepsilon, \sigma)$ *is a positive continuous function in* Q^n *and satisfies*

$$L_i^* \equiv \lim_{m \uparrow \infty} L_i(\varepsilon_m, \sigma) = \sum_{j=1}^{n} |\kappa_j^i|. \tag{3.36}$$

Proof. On each subinterval I_j, we have

$$\begin{aligned} \int_{I_j} \phi_{ij}^\varepsilon dx &= \sqrt{\varepsilon} \int_{\bar{I}_j} \sqrt{\varepsilon} \tilde{\phi}_{ij}^\varepsilon dy \\ &= \sqrt{\varepsilon} \int_{\bar{I}_j} \hat{\phi}_{ij}^\varepsilon dy. \end{aligned}$$

On the other hand, it follows from (3.34) and $\int_{\mathbf{R}} \hat{\phi}_L^* = 1$ that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\tilde{I}_j} |\hat{\phi}_{ij}^{\varepsilon}| dy &= |\kappa_j^i| \int_{\mathbf{R}} \hat{\phi}_L^* dy \\ &= |\kappa_j^i|. \end{aligned} \tag{3.37}$$

Hence

$$\int_{I_j} |\phi_i^{\varepsilon}| dx = \sqrt{\varepsilon} \sum_{j=1}^{n} \int_{\tilde{I}_j} |\hat{\phi}_{ij}^{\varepsilon}| dy.$$

We define $L_i(\varepsilon, \sigma)$ by

$$L_i(\varepsilon, \sigma) = \sqrt{\varepsilon} \sum_{j=1}^{n} \int_{\tilde{I}_j} |\hat{\phi}_{ij}^{\varepsilon}| dy.$$

Since the integral term behaves continuously for $\varepsilon > 0$, so does $L_i(\varepsilon, \sigma)$ in Q^n. the equality (3.36) is a direct consequence of (3.37). □

Lemma 3.15 (Asymptotic Behaviors of Eigenvalues of $L^{\varepsilon,\sigma}$). *Let $\{\zeta_i^{\varepsilon,\sigma}, \phi_i^{\varepsilon,\sigma}\}_{i=0}^{\infty}$ be the complete orthonormal set of eigenvalues and eigenfunctions of*

$$L^{\varepsilon,\sigma} \equiv \left(\varepsilon^2 \frac{d^2}{dx^2} + f_u^{\varepsilon,\sigma} \right) \phi = \zeta \phi,$$

subject to Neumann boundary conditions, where $f_u^{\varepsilon,\sigma}$ denotes the partial derivative f_u evaluated at the normal n-layered solution. The first n eigenvalues $\zeta_0^{\varepsilon,\sigma}, \cdots, \zeta_{n-1}^{\varepsilon,\sigma}$ ($\zeta_0^{\varepsilon,\sigma} > \cdots > \zeta_{n-1}^{\varepsilon,\sigma}$) are critical eigenvalues of $L^{\varepsilon,\sigma}$, which are positive for $\varepsilon > 0$ and satisfy the asymptotic formula as $\varepsilon \downarrow 0$ (see Figure 3.2)

$$\zeta_i^{\varepsilon,\sigma} = \hat{\xi}_i(\varepsilon, \sigma) \varepsilon \sigma + e_i(\varepsilon, \sigma), \tag{3.38}$$

Here $\hat{\xi}_i$ and $e_i (i = 0, \cdots, n-1)$ are positive continuous functions in Q^n (see Corollary 3.9 and Theorem 3.3) which are continuously extendable to $\varepsilon = 0$ and satisfy the following:

$$\begin{aligned} \hat{\xi}^{*,\sigma} &\equiv \lim_{\varepsilon \downarrow 0} \hat{\xi}_i(\varepsilon, \sigma) \\ &= \frac{1}{n} \left(\frac{\gamma^*}{c^*} \right)^2 \frac{dJ}{dv}(v^*) \int_0^{x_1^*(\frac{\sigma}{n^2})} g\left(U^{*,\sigma/n^2}, V^{*,\sigma/n^2} \right) dx > 0, \end{aligned} \tag{3.39}$$

$$|e_i(\varepsilon, \sigma)| \le C \exp\left(-\frac{\gamma}{\varepsilon} \right), \tag{3.40}$$

where γ^* and c^* are positive constants defined in Remark 3.7, $\left(U^{*,\sigma/n^2}, V^{*,\sigma/n^2} \right)$ is the basic pattern in Theorem 3.3 with replacing σ by σ/n^2, and C and γ are positive constants independent of $(\varepsilon, \sigma) \in Q^n$ and i. Note that the asymptotic limit $\hat{\xi}^{*,\sigma}$ does not depend on i $(0 \le i \le n-1)$. The rest of the eigenvalues $\zeta_i^{\varepsilon,\sigma} (i \ge n)$ are negative and *uniformly bounded away from zero with respect to small ε, namely, it holds that*

$$0 > -\Delta^* > \zeta_n^{\varepsilon,\sigma} > \zeta_{n+1}^{\varepsilon,\sigma} > \cdots$$

for small ε, where Δ^* is a positive constant independent of $(\varepsilon, \sigma) \in Q^n$.

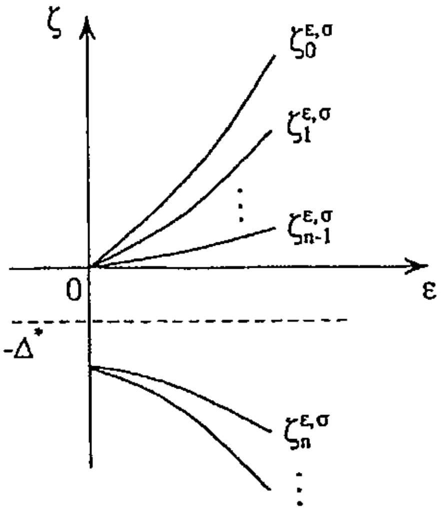

Figure 3.2.

Remark 3.16. *Lemma 3.15 also holds under homogeneous Dirichlet boundary conditions without essential changes. In particular, the asymptotic limit $\hat{\zeta}^{*,\sigma}$ remains the same as the Neumann case.*

Proof. Let $\phi_i^\varepsilon (0 \le i \le n-1)$ be the L^2-normalized i-th eigenfunction of L^ε, and recall that

$$\hat{\phi}_{ij}^\varepsilon(y) \equiv \sqrt{\varepsilon}\phi_i^\varepsilon(x_j^* + \varepsilon y), \qquad y \in \bar{I}_j \equiv \left(-\frac{\ell}{\varepsilon}, \frac{r}{\varepsilon}\right),$$

i.e., the stretched function of ϕ_i^ε with center at x_j^* on j-th subinterval $(1 \le j \le n)$, and

$$\int_{I_j} |\phi_i^\varepsilon|^2 dx = \int_{\bar{I}_j} |\hat{\phi}_{ij}^\varepsilon|^2 dy.$$

It is clear that $\hat{\phi}_{ij}^\varepsilon$ satisfies

$$\frac{d^2}{dy^2}\hat{\phi}_{ij}^\varepsilon + \tilde{f}_u^\varepsilon \hat{\phi}_{ij}^\varepsilon = \zeta_i^\varepsilon \hat{\phi}_{ij}^\varepsilon \qquad \text{on } \bar{I}_j \tag{3.41}$$

and from (3.34)

$$\lim_{\varepsilon \downarrow 0} \hat{\phi}_{ij}^\varepsilon = \kappa_j^i \hat{\phi}_{ij}^\varepsilon \qquad \text{in } C^2(\mathbf{R})\text{-sense} \tag{3.42}$$

holds for an appropriately chosen subsequence. Here we keep the notation ε instead of ε_m, since the final result (3.39) does not depend on the choice of subsequence. We take j such that $\kappa_j^i \neq 0$. On the other hand, the y-derivative of the stretched normal n-layered solution satisfies

$$\frac{d^2}{dy^2}\tilde{u}_y^\varepsilon + \tilde{f}_u^\varepsilon \tilde{u}_y^\varepsilon = -\tilde{f}_v^\varepsilon \tilde{v}_y^\varepsilon \qquad y \in \tilde{I}_j. \tag{3.43}$$

Multiplying $\tilde{u}_y^\varepsilon$ on both sides of (3.41) and integrating by parts twice, we have

$$\langle \hat{\phi}_{ij}^\varepsilon, -\tilde{f}_v^\varepsilon \tilde{v}_y^\varepsilon \rangle - \hat{\phi}_{ij}^\varepsilon \frac{d}{dy}\tilde{u}_y^\varepsilon \Big|_{-\ell/\varepsilon}^{r/\varepsilon} = \zeta_i^\varepsilon \langle \hat{\phi}_{ij}^\varepsilon, \tilde{u}_y^\varepsilon \rangle. \tag{3.44}$$

Here we use (3.43) and the fact that $\tilde{u}_y^\varepsilon = 0$ at $y = -\ell/\varepsilon, r/\varepsilon$. Using Lemma 3.11, we see that

$$|\text{the second term on the left-hand side of (3.44)}| \leq C \exp\left(-\frac{\gamma}{\varepsilon}\right), \tag{3.45}$$

where C, γ are positive constants independent of ε and σ. In order to compute the first term of (3.44), first note that $\tilde{v}^\varepsilon$ satisfies

$$\frac{1}{\sigma}\frac{d^2}{dy^2}\tilde{v}^\varepsilon + \varepsilon^2 g(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon) = 0. \tag{3.46}$$

Integrating (3.46) from $-\ell/\varepsilon$ to y and using $\tilde{v}_y^\varepsilon(-\ell/\varepsilon) = 0$, we have

$$\tilde{v}_y^\varepsilon = -\sigma\varepsilon\tilde{\theta}_j^\varepsilon(y), \tag{3.47}$$

where

$$\tilde{\theta}_j^\varepsilon(y) \equiv \varepsilon \int_{-\ell/\varepsilon}^{y} g(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)dy.$$

Here $\tilde{\theta}_j^\varepsilon(y)$ can be obtained by stretching the following function at $x = x_j^*$:

$$\theta_j^\varepsilon(x) \equiv \int_{-\ell+x_j^*}^{x} g(u^\varepsilon, v^\varepsilon)dx.$$

θ_j^ε is C^1 for $\varepsilon > 0$ and its C^1-norm is uniformly bounded with respect to ε, and hence we see from Remark 3.4 that $\tilde{\theta}_j^\varepsilon(y)$ converges to a constant function as $\varepsilon \downarrow 0$. More precisely,

$$\lim_{\varepsilon\downarrow 0}\tilde{\theta}_j^\varepsilon(y) = \int_{-\ell+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx. \tag{3.48}$$

holds in $C_{c.u.}^2(\mathbf{R})$-sense, where (U_n^*, V_n^*) is the reduced solution of the normal n-layered solution. In view of Lemma 3.12 and Remark 3.7, we see that

$$\begin{aligned} \lim_{\varepsilon\downarrow 0}\hat{\phi}_{ij}^\varepsilon &= \kappa_j^i \hat{\phi}_L^* = \gamma^* \frac{d}{dy}\tilde{u}^* \\ \lim_{\varepsilon\downarrow 0}\tilde{u}_y^\varepsilon &= \frac{c^*}{\gamma^*}\hat{\phi}_0^* \end{aligned} \tag{3.49}$$

hold in $C^2_{c.u.}(\mathbb{R})$-sense. Substituting (3.47) into (3.44), ζ_i^ε is expressed by

$$\zeta_i^\varepsilon = \hat{\xi}_i(\varepsilon,\sigma)\sigma\varepsilon + e_i(\varepsilon,\sigma), \tag{3.50a}$$

where

$$\hat{\xi}_i(\varepsilon,\sigma) \equiv \frac{\langle \tilde{f}_v^\varepsilon \hat{\phi}_{ij}^\varepsilon, \tilde{\theta}_j^\varepsilon\rangle}{\langle \hat{\phi}_{ij}^\varepsilon, \tilde{u}_y^\varepsilon\rangle} \tag{3.50b}$$

$$e_i(\varepsilon,\sigma) \equiv -\hat{\phi}_{ij}^\varepsilon \tilde{u}_{yy}^\varepsilon \Big|_{-\ell/\varepsilon}^{r/\varepsilon} / \langle \hat{\phi}_{ij}^\varepsilon, \tilde{u}_y^\varepsilon\rangle. \tag{3.50c}$$

It is clear from (3.45) that $e_i(\varepsilon,\sigma)$ satisfy (3.40). We compute the asymptotic form of $\hat{\xi}_i(\varepsilon,\sigma)$ as $\varepsilon \downarrow 0$. Using (3.48) and (3.49), we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\langle \tilde{f}_v^\varepsilon \hat{\phi}_{ij}^\varepsilon, \tilde{\theta}_j^\varepsilon\rangle &= \gamma^* \kappa_j^i \int_{-\infty}^{+\infty} f_v(\tilde{u}^*, v^*)\frac{d}{dy}\tilde{u}^* dy \\ &\times \int_{-\ell+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx.\end{aligned} \tag{3.51}$$

Note that $\tilde{u}^*(y)$ is strictly monotone increasing (resp. decreasing) depending on the quantity (3.48) being negative (resp. positive). For the increasing case, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} f_v(\tilde{u}^*, v^*)\frac{d}{dy}\tilde{u}^* dy &= \frac{d}{dv}\int_{h_-(v^*)}^{h_+(v^*)} f(s, v^*)ds \\ &= \frac{dJ}{dv}(v^*) < 0 \qquad \text{(see (A.2))}.\end{aligned} \tag{3.52}$$

For the decreasing case, the sign becomes opposite. On the other hand, we have from Remark 3.7 and (3.42),

$$\lim_{\varepsilon\downarrow 0}\langle \hat{\phi}_{ij}^\varepsilon, \tilde{u}_y^\varepsilon\rangle = \kappa_j^i \frac{(c^*)^2}{\gamma^*}. \tag{3.53}$$

Substituting (3.51), (3.52) and (3.53) into (3.50b), we finally obtain for the increasing case

$$\hat{\xi}_i^* \equiv \lim_{\varepsilon\downarrow 0}\hat{\xi}_i(\varepsilon,\sigma) = -\left(\frac{\gamma^*}{c^*}\right)^2 \frac{dJ}{dv}(v^*)\int_{-\ell+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx. \tag{3.54}$$

For the decreasing case, the integral part should be replaced by $\int_{-r+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx$ which is equal to $-\int_{-\ell+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx$. However, the sign of $\frac{dJ}{dv}(v^*)$ (see (3.52)) also changes at the same time, hence $\hat{\xi}_i^*$ is equal to (3.54) in either case. Recalling Proposition

3.8, we see that $R^n(U^{*,\sigma/n^2}) = U_n^*$ and $R^n(V^{*,\sigma/n^2}) = V_n^*$, and hence we have

$$\int_{-\ell+x_j^*}^{x_j^*} g(U_n^*, V_n^*)dx = \frac{1}{n}\int_0^{x_j^*(\sigma/n^2)} g(U^{*,\sigma/n^2}, V^{*,\sigma/n^2})dx,$$

which completes the proof of (3.39). The final part of Lemma 3.15 was already shown in Lemma 3.10. □

3.3. Derivation of the SLEP System

The most delicate and crucial part of the linearized spectral analysis for layered solutions is to find all critical eigenvalues of (3.12) and clarify their asymptotic behaviors as $\varepsilon \downarrow 0$. The SLEP method gives us a unified tool to deal with this problem. The basic idea of it is to find a *nice scaling* which blows up the degenerate situation of $\mathcal{L}^{\varepsilon,\sigma}$ (see (3.12)) as $\varepsilon \downarrow 0$. It turns out that the study of asymptotic behaviors of critical eigenvalues is reduced to solving a linear eigenvalue problem of $n \times n$ *symmetric* matrix called the *SLEP system*, although (3.12) is not self-adjoint. The aim of this section is to show how the linearized problem (3.12) is reduced to the SLEP system with respect to the *scaled* critical eigenvalues. Without loss of generality, we can restrict the region of λ to Λ_1 defined by

$$\Lambda_1 \equiv \{\lambda \mid \mathrm{Re}\lambda > -\mu_1 > \max(-\Delta^*, -\mu)\} \tag{3.55}$$

for some fixed $\mu_1 > 0$, where $-\Delta^*$ and $-\mu$ are negative constants appeared in Lemma 3.15 and (3.17). First note the following lemma.

Lemma 3.17. *The first n eigenvalues $\{\zeta_i^{\varepsilon,\sigma}\}_{i=0}^{n-1}$ of Sturm-Liouville operator $L^{\varepsilon,\sigma}$ do not belong to the spectra of $\mathcal{L}^{\varepsilon,\sigma}$ for small ε.*

Proof. See Appendix B. □

Remark 3.18. *Lemma 3.17 combined with Lemma 3.15 implies that, if $\lambda \in \Lambda_1$ is an eigenvalue of $\mathcal{L}^{\varepsilon,\sigma}$, the resolvent $(L^{\varepsilon,\sigma} - \lambda)^{-1}$ exists for all small ε.*

Solving the first equation of (3.12) with respect to w and substituting it into the second equation after expanding it by using the complete orthonormal set of $L^{\varepsilon,\sigma}$, we have the equivalent eigenvalue problem containing only z:

$$\begin{aligned}&\tfrac{1}{\sigma}\tfrac{d^2}{dx^2}z + \textstyle\sum_{i=0}^{n-1} \frac{\langle -f_v^{\varepsilon,\sigma}z, \phi_i^{\varepsilon,\sigma}\rangle}{\zeta_i^{\varepsilon,\sigma}-\lambda} g_u^{\varepsilon,\sigma}\phi_i^{\varepsilon,\sigma} + g_u^{\varepsilon,\sigma}(L^{\varepsilon,\sigma} - \lambda)^{\dagger}(-f_v^{\varepsilon,\sigma}z) + g_v^{\varepsilon,\sigma}z = \lambda z\\ &\lambda \in \Lambda_1,\end{aligned} \tag{3.56}$$

where the *reduced resolvent* $(L^{\varepsilon,\sigma} - \lambda)^{\dagger}$ is defined by

$$(L^{\varepsilon,\sigma} - \lambda)^{\dagger}(\cdot) \equiv \sum_{i\geq n} \frac{\langle \cdot, \phi_i^{\varepsilon,\sigma}\rangle}{\zeta_i^{\varepsilon,\sigma} - \lambda}\phi_i^{\varepsilon,\sigma} : L^2(I) \to L^2(I) \cap \{\phi_0^{\varepsilon,\sigma}, \cdot, \phi_{n-1}^{\varepsilon,\sigma}\}^{\perp}. \tag{3.57}$$

It is clear from (3.57) that $(L^{\varepsilon,\sigma}-\lambda)^{\dagger}$ is uniformly L^2-bounded operator, more precisely, we have

$$\| (L^{\varepsilon,\sigma}-\lambda)^{\dagger} \| \le \frac{1}{|\zeta_n^{\varepsilon,\sigma}-\lambda|} \tag{3.58}$$

for $\lambda \in \Lambda_1$. Note that the denominator $|\zeta_n^{\varepsilon,\sigma}-\lambda|$ is strictly bounded away from zero (see Lemma 3.15 and (3.55)). On the other hand, recalling Lemma 3.15, all the denominators of the second term of (3.56) must go to zero as $\varepsilon \downarrow 0$ if λ is a critical eigenvalue of (3.12). Hence more careful treatment is needed to handle this term. Let us begin with the study of the reduced resolvent.

Lemma 3.19 (Asymptotic Limit of $(L^{\varepsilon,\sigma}-\lambda)^{\dagger}$). *$(L^{\varepsilon,\sigma}-\lambda)^{\dagger}$ is a uniform L^2-bounded operator with respect to ε and becomes a multiplication operator in the limit of $\varepsilon \downarrow 0$. Namely,*

$$\lim_{\varepsilon\downarrow 0}(L^{\varepsilon,\sigma}-\lambda)^{\dagger}(F^{\varepsilon,\sigma}h) = \frac{F^{*,\sigma}h}{f_u^{*,\sigma}-\lambda}$$

in strongly L^2-sense for any $h \in L^2(I) \cap L^{\infty}(I)$, smooth function $F(u,v)$, and $\lambda \in \Lambda_1$, where $F^{\varepsilon,\sigma} = F(D^n(\varepsilon,\sigma))$ and $F^{*,\sigma} = F(U_n^{*,\sigma}(x), V_n^{*,\sigma}(x))$ (i.e., evaluated at the reduced solution of $D^n(\varepsilon,\sigma)$). Moreover, if h belongs to $H^1(I)$, the above convergence is uniform on any bounded set in $H^1(I)$.

Proof. Let $S_\delta(x)$ be a smooth cut-off function defined on I satisfying

$$S_\delta(x) = \begin{cases} 1 & \text{if} \quad |x| \ge \delta/2 \\ 0 & \text{if} \quad |x| \le \delta/4 \end{cases}$$

with

$$0 \le S_\delta(x) \le 1, \qquad \sup_{x\in I}\left|\frac{d^i}{dx^i}S_\delta(x)\right| \le M_\delta \quad (i=1,2),$$

where M_δ is a positive constant which tends to $+\infty$ as $\delta \downarrow 0$. We assume for simplicity that $F^{\varepsilon,\sigma} \equiv 1$, $\lambda = 0$, and h is smooth. It is not difficult to extend it for the general case by using approximation and density argument.

Define S_δ^i by $S_\delta^i \equiv S_\delta(x - x_i^*)$, then

$$h_\delta \equiv S_\delta^1 \cdot S_\delta^2 \cdots S_\delta^n \cdot h$$

is a punctured function of h at layer positions, i.e., $h_\delta \equiv 0$ in a small neighbourhood of $x_i^*(i = 1, \cdots, n)$. In view of (3.58), we see that it suffices to prove the lemma for h_δ since h_δ approaches h in L^2-sense as $\delta \downarrow 0$. In order to show that $h_\delta/f_u^{*,\sigma}$, which we denote by u_δ, is equal to $\lim_{\varepsilon\downarrow 0}(L^\varepsilon)^\dagger h_\delta$, we first apply $L^{\varepsilon,\sigma}$ to u_δ:

$$\begin{aligned} L^{\varepsilon,\sigma}u_\delta &= \varepsilon^2\frac{d^2}{dx^2}\left(\frac{h_\delta}{f_u^{*,\sigma}}\right)+f_u^{\varepsilon,\sigma}\left(\frac{h_\delta}{f_u^{*,\sigma}}\right) \\ &= h_\delta+\frac{(f_u^{\varepsilon,\sigma}-f_u^{*,\sigma})}{f_u^{*,\sigma}}h_\delta+\varepsilon^2 H_\delta, \end{aligned} \tag{3.59}$$

where $H_\delta \equiv \dfrac{d^2}{dx^2}\left(\dfrac{h_\delta}{f_u^{*,\sigma}}\right)$. Note that H_δ is well-defined since h_δ becomes zero in a neighbourhood of each x_i^*. Applying $(L^{\varepsilon,\sigma})^\dagger$ to both sides of (3.59), we have

$$u_\delta-\sum_{i=0}^{n-1}\langle u_\delta,\phi_i^{\varepsilon,\sigma}\rangle\phi_i^{\varepsilon,\sigma}=(L^{\varepsilon,\sigma})^\dagger h_\delta+(L^{\varepsilon,\sigma})^\dagger\left\{\frac{(f_u^{\varepsilon,\sigma}-f_u^{*,\sigma})}{f_u^{*,\sigma}}h_\delta+\varepsilon^2 H_\delta\right\}. \tag{3.60}$$

Recalling Corollary 3.14 and $u_\delta \equiv 0$ in a neighbourhood of each x_i^*, the second term on the left-hand side tends to zero in L^2-sense as $\varepsilon \downarrow 0$. Since $f_u^{\varepsilon,\sigma}-f_u^{*,\sigma}$ goes to zero in L^2-sense as $\varepsilon \downarrow 0$ and H_δ remains bounded in L^2-sense, the second term of the right-hand side of (3.60) also tends to zero when $\varepsilon \downarrow 0$. Hence we obtain

$$u_\delta=\lim_{\varepsilon\downarrow 0}(L^{\varepsilon,\sigma})^\dagger h_\delta.$$

The final part of the lemma can be proved in the same way as in the proof of Lemma 2.2 in [44], so we omit the details. □

It is convenient to classify the spectrum of (3.12) into two classes; the *critical* eigenvalues which tend to zero as $\varepsilon \downarrow 0$, and *noncritical* eigenvalues which are bounded away from the imaginary axis for small ε. Making use of Lemma 3.19, we can show that *noncritical* eigenvalues are *not* dangerous to the stability of $D^n(\varepsilon,\sigma)$, namely, they have strictly negative real parts independent of ε.

Proposition 3.20 (A Priori Bound for Noncritical Eigenvalues). *Let B_δ be a closed ball with center at the origin and radius δ in the complex plane* **C**. *Suppose that λ is an arbitrary noncritical eigenvalue of (3.12) which stays outside of B_δ for all small ε. Then there exist positive constants μ^* and ε_δ such that*

$$Re\lambda<-\mu^*<0 \quad for \quad 0<\varepsilon<\varepsilon_\delta, \tag{3.61}$$

where μ^* does not depend on δ and ε.

Proof. There is no essential difference in proofs between mono-layer and multi-layer cases. So we delegate it to that of Proposition 2.1 in [44]. □

Now we can concentrate on the behavior of critical eigenvalues. Let $\lambda=\lambda(\varepsilon)$ be an arbitrary critical eigenvalue of (3.12), and assume that λ varies in the ball $B_\delta=\{\lambda \mid |\lambda|<\delta\}$ for some $\delta>0$. We shall specify the size of δ later. In view of Corollary 3.14, Lemma 3.15, and $z\in H_N^1(I)$, we see that both the denominator and the numerator of the second term on the left-hand side of (3.56) tend to zero as $\varepsilon \downarrow 0$. Here the scaling technique comes up to convert it into more tractable and nondegenerate form. Before that, we first rewrite (3.56) in the form of a *finite dimensional eigenvalue problem* by using the following operator $K^{\varepsilon,\sigma,\lambda}$.

Lemma 3.21 (Operator $K^{\varepsilon,\sigma,\lambda}$). *Let $\hat{B}^{\varepsilon,\sigma,\lambda}$ be a bilinear form defined by*

$$\hat{B}^{\varepsilon,\sigma,\lambda}(z^1,z^2)=\frac{1}{\sigma}\langle z^1_x,z^2_x\rangle-\langle\{g^{\varepsilon,\sigma}_u(L^{\varepsilon,\sigma}-\lambda)^{\dagger}(-f^{\varepsilon,\sigma}_v\cdot)+g^{\varepsilon,\sigma}_v-\lambda\}z^1,z^2\rangle$$

$$\text{for}\quad z^i\in H^1_N(I)\ (i=1,2).$$

Then, for a given $h\in H^{-1}(I)$, the equation for $z\in H^1_N(I)$

$$\hat{B}^{\varepsilon,\sigma,\lambda}(z,\psi)=\langle h,\psi\rangle\ \text{ for any }\ \psi\in H^1_N(I)$$

has a unique solution $z=z(h)$ for small ε (including $\varepsilon=0$) and $\lambda\in B_\delta$. Define the mapping $K^{\varepsilon,\sigma,\lambda}$ by

$$K^{\varepsilon,\sigma,\lambda}h=z(h);\ H^{-1}(I)\longrightarrow H^1_N(I).$$

$K^{\varepsilon,\sigma,\lambda}$ is a bounded operator from $H^{-1}(I)$ to $H^1_N(I)$, and depends continuously on (ε,σ) and analytically on λ in operator norm sense, respectively. The limiting form $K^{*,\sigma,0}\equiv\lim_{\varepsilon\downarrow 0}K^{\varepsilon,\sigma,0}$ is given by (3.68) and (3.69).

Proof. See Lemma 3.1 of [44]. □

Applying $K^{\varepsilon,\sigma,\lambda}$ to (3.56), we have

$$z=\sum_{i=0}^{n-1}\frac{\langle -f^{\varepsilon,\sigma}_v z,\phi^{\varepsilon,\sigma}_i\rangle}{\zeta^{\varepsilon,\sigma}_i-\lambda}K^{\varepsilon,\sigma,\lambda}(g^{\varepsilon,\sigma}_u\phi^{\varepsilon,\sigma}_i). \tag{3.62}$$

This shows that z is a linear combination of $K^{\varepsilon,\sigma,\lambda}(g^{\varepsilon,\sigma}_u\phi^{\varepsilon,\sigma}_i)$ $(i=0,\cdots,n-1)$ yielding

$$z=\sum_{i=0}^{n-1}\alpha_i K^{\varepsilon,\sigma,\lambda}(g^{\varepsilon,\sigma}_u\phi^{\varepsilon,\sigma}_i), \tag{3.63}$$

where $\mathbf{a}=(\alpha_0,\cdots,\alpha_{n-1})$ is a real vector. Note that $K^{\varepsilon,\sigma,\lambda}(g^{\varepsilon,\sigma}_u\phi^{\varepsilon,\sigma}_i)$ $(i=0,\cdots,n-1)$ are linearly independent since $\{\phi^{\varepsilon,\sigma}_i\}_{i=0}^{n-1}$ are linearly independent. Substituting (3.63) into (3.62), we obtain an *n-dimensional* matrix eigenvalue problem:

$$M^{\varepsilon,\sigma}\mathbf{a}=\begin{pmatrix}(\zeta^{\varepsilon,\sigma}_0-\lambda)\alpha_0\\ \vdots\\ (\zeta^{\varepsilon,\sigma}_{n-1}-\lambda)\alpha_{n-1}\end{pmatrix}, \tag{3.64}$$

where $M^{\varepsilon,\sigma}=\{\langle -f^{\varepsilon,\sigma}_v\phi^{\varepsilon,\sigma}_i,K^{\varepsilon,\sigma,\lambda}(g^{\varepsilon,\sigma}_u\phi^{\varepsilon,\sigma}_j)\rangle\}_{i,j=0}^{n-1}$. Note that $M^{\varepsilon,\sigma}$ also depends on λ through $K^{\varepsilon,\sigma,\lambda}$. This problem is highly degenerated, since all the elements of $M^{\varepsilon,\sigma}$ and $\zeta^{\varepsilon,\sigma}_i-\lambda$ $(i=0,\cdots,n-1)$ tend to zero as $\varepsilon\downarrow 0$ (recall Corollary 3.14, Lemma 3.15, Lemma 3.21, and that λ is a critical eigenvalue). The following characterization of the asymptotic form of $\phi^{\varepsilon,\sigma}_i$ by $\sqrt{\varepsilon}$ -scaling plays a key role to unfold this degenerate situation.

Lemma 3.22 (Asymptotic Form of $\phi_i^{\varepsilon,\sigma}/\sqrt{\varepsilon}$). *Let $\{\phi_i^{\varepsilon_m,\sigma}\}_{m=1}^{\infty}$ be an arbitrary convergent sequence in the sense of Lemma 3.12 on each stretched subinterval $\tilde{I}_j (j = 1, \cdots, n)$ for $i \in (0, \cdots, n-1)$. Then it holds that*

$$\lim_{m\uparrow\infty} -f_v^{\varepsilon_m,\sigma}\frac{\phi_i^{\varepsilon_m,\sigma}}{\sqrt{\varepsilon}} = c_1^*\Delta_i \equiv c_1^*\sum_{j=1}^{n}\kappa_j^i\delta(x - x_j^*(\sigma)), \tag{3.65a}$$

$$\lim_{m\uparrow\infty} g_u^{\varepsilon_m,\sigma}\frac{\phi_i^{\varepsilon_m,\sigma}}{\sqrt{\varepsilon}} = c_2^*\Delta_i \equiv c_2^*\sum_{j=1}^{n}\kappa_j^i\delta(x - x_j^*(\sigma)) \tag{3.65b}$$

both in $H^{-1}(I)$ -sense, where $c_1^* \equiv -\gamma^*\frac{dJ}{dv}(v^*)$, $c_2^* \equiv \gamma^*\{g(h_+(v^*), v^*) - g(h_-(v^*), v^*)\}$, and $\delta(x - x_j^*(\sigma))$ denotes the Dirac's δ-function at $x = x_j^*(\sigma)$. The vectors $\mathbf{k}^i = (\kappa_1^i, \cdots, \kappa_n^i)$ $(i = 0, \cdots, n-1)$ satisfy the orthogonal relation (3.35).

Proof. Recallng (3.26) and Lemma 3.11, we see that this is essentially a restatement of Lemma 3.12 in the original x-coordinate. The only difference is the coefficients $c_i^*(i = 1, 2)$ which appear due to the existence of $-f_v^{\varepsilon_m,\sigma}$ and $g_u^{\varepsilon_m,\sigma}$. To show (3.65a), it suffices to compute the integral

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\int_{\tilde{I}_j}\sqrt{\varepsilon}(-\tilde{f}_v^{\varepsilon_m,\sigma})\tilde{\phi}_{ij}^{\varepsilon_m,\sigma}dy &= \int_{-\infty}^{\infty} -\tilde{f}_v^{\varepsilon_m,\sigma}\kappa_j^i\hat{\phi}_L^* dy \\
&= \int_{-\infty}^{\infty} -\tilde{f}_v^{\varepsilon_m,\sigma}\kappa_j^i\gamma^*\frac{d}{dy}\tilde{u}^* dy \\
&= -\gamma\kappa_j^i\int_{h_-(v^*)}^{h_+(v^*)} f_v(s, v^*)ds \\
&= -\kappa_j^i\gamma\frac{d}{dv}J(v^*) \\
&= c_1^*\kappa_j^i,
\end{aligned}$$

which implies (3.65a). Similar computation leads to (3.65b). □

Hereafter, we fix a convergent subsequence $\{\phi_i^{\varepsilon_m,\sigma}/\sqrt{\varepsilon}\}_{m=1}^{\infty}$, and for simplicity of notation, we simply write ε instead of ε_m keeping in mind that ε actually means a discrete parameter ε_m. In view of Lemma 3.15 and Lemma 3.22, we see that ε-scaling is the most suitable to blow up (3.64). In fact, dividing (3.64) by ε on both sides, we have

$$\tilde{M}^{\varepsilon,\sigma}\mathbf{a} = \begin{pmatrix} (\zeta_0^{\varepsilon,\sigma}/\varepsilon - \lambda/\varepsilon)\alpha_0 \\ \vdots \\ (\zeta_{n-1}^{\varepsilon,\sigma}/\varepsilon - \lambda/\varepsilon)\alpha_{n-1} \end{pmatrix}, \tag{3.66}$$

where $\tilde{M}^{\varepsilon,\sigma} = \{(-f_v^{\varepsilon,\sigma}\phi_i^{\varepsilon,\sigma}/\sqrt{\varepsilon}, K^{\varepsilon,\sigma,\lambda}(g_u^{\varepsilon,\sigma}\phi_j^{\varepsilon,\sigma}/\sqrt{\varepsilon}))\}_{i,j=0}^{n-1}$. The problem (3.66) is non-degenerate and well-defined continuously up to $\varepsilon = 0$. In fact, in the limit of $\varepsilon \downarrow 0$, we see from Lemmas 3.15, 3.21, and 3.22 that (3.66) becomes

$$\{\tilde{M}^{*,\sigma} + (\tau^{*,\sigma} - \sigma\hat{\zeta}^{*,\sigma})I\}\mathbf{a}^* = 0 \qquad \textit{(SLEP system)} \qquad (3.67)$$

where $\tilde{M}^{*,\sigma} \equiv \lim_{\varepsilon\downarrow 0}\tilde{M}^{\varepsilon,\sigma} = \{c_1^* c_2^*\langle\Delta_i, K^{*,\sigma}\Delta_j\rangle\}_{i,j=0}^{n-1}$, $K^{*,\sigma} \equiv K^{0,\sigma,0}$, $\tau^{*,\sigma} \equiv \lim_{\varepsilon\downarrow 0}\lambda(\varepsilon)/\varepsilon$, $\sigma\hat{\zeta}^{*,\sigma} \equiv \lim_{\varepsilon\downarrow 0}\zeta_i^{\varepsilon,\sigma}/\varepsilon$ $(i = 0, \cdots, n-1)$, and $\mathbf{a}^* = \lim_{\varepsilon\downarrow 0}\mathbf{a}$. Here we use the fact that the critical eigenvalue $\lambda = \lambda(\varepsilon)$ can be written in the following form:

Lemma 3.23. *Any critical eigenvalue λ must have the form*

$$\lambda = \varepsilon\tau(\varepsilon, \sigma),$$

where τ is a bounded function up to $\varepsilon = 0$. The problem (3.66) depends on τ smoothly.

Proof. Suppose that there is a critical eigenvalue $\lambda_c(\varepsilon)$ which tends to zero strictly slower than $O(\varepsilon)$. Then the associated eigenvalue problem (3.66) must have an arbitrary large (in modulus) eigenvalue when $\varepsilon \downarrow 0$. However this is not possible since both $\tilde{M}^{\varepsilon,\sigma}$ and $\hat{\zeta}^{\varepsilon,\sigma}$ are uniformly bounded and have definite limits as $\varepsilon \downarrow 0$. This also implies the boundedness of τ up to $\varepsilon = 0$. The last part is clear from Lemma 3.21. □

The limiting problem (3.67) is called the *SLEP system* of (3.12) with respect to the scaled eigenvalues $\tau^{*,\sigma}$. From now on, we focus on the SLEP system (3.67), since all information on the asymptotic behaviors of critical eigenvalues for $\varepsilon > 0$ can be derived from (3.67). However the only defect of (3.67) is that it does not look free from the choice of the subsequence, i.e., $\tilde{M}^{*,\sigma}$ may depend on Δ_i. In order to get rid of this ambiguity, we shall apply a change of bases to $\tilde{M}^{*,\sigma}$ which depends on $\{\Delta_i\}_{i=0}^{n-1}$. It turns out that the resulting matrix denoted by $\tilde{\mathbf{G}}_N$ is *independent* of the choice of the subsequence and has n real distinct eigenvalues, which leads to our main result (Theorem 3.25). For this purpose, we introduce the Green function $G_N = G_N(x, y; \sigma)$ associated with the operator $K^{*,\sigma,0}$ (see Lemma 3.21) defined as follows:

$$K^{*,\sigma,0}\phi = \langle G_N(x, y; \sigma), \phi\rangle, \quad \text{for any} \quad \phi \in H^{-1}(I). \qquad (3.68)$$

More explicitly,

$$G_N(x, y; \sigma) = -\frac{\sigma}{W(h, k)} \times \begin{cases} h(x)k(y), & 0 \le x \le y \le 1, \\ h(y)k(x), & 0 \le y \le x \le 1, \end{cases} \qquad (3.69a)$$

where h and k satisfy the equation

$$\left(-\frac{1}{\sigma}\frac{d^2}{dx^2} - \frac{\det^{*,\sigma}}{f_u^{*,\sigma}}\right)\phi = 0, \qquad \phi \in H^2(I) \qquad (3.69b)$$

with the boundary conditions

$$h(0) = 1, h'(0) = 0 \;\text{ and }\; k(1) = 1, k'(1) = 0, \qquad (3.69c)$$

where $\det^{*,\sigma} \equiv f_u^{*,\sigma} g_v^{*,\sigma} - f_v^{*,\sigma} g_u^{*,\sigma} > 0$ (see (A.4)) and $W(h, k)$ denotes the Wronskian of h and k. Note that h (resp. k) is strictly positive and increasing (resp. decreasing), respectively, since $-\det^{*,\sigma}/f_u^{*,\sigma}$ is strictly positive from (A.3) and (A.4). It follows from (3.68) that

$$\langle \delta(x - x_i^*(\sigma)), K^{*,\sigma,0}\delta(x - x_j^*(\sigma)) \rangle = G_N(x_i^*(\sigma), x_j^*(\sigma); \sigma).$$

Therefore, recalling that $\Delta_i = \sum_{j=1}^{n} \kappa_j^i \delta(x - x_j^*(\sigma))$, we have

$$\langle \Delta_i, K^{*,\sigma,0}\Delta_j \rangle = {}^t\mathbf{k}^i \mathbf{G}_N \mathbf{k}^j, \tag{3.70}$$

where $\mathbf{G}_N$ is an $n \times n$ symmetric matrix with positive components defined by

$$\mathbf{G}_N \equiv \{G_N(x_i^*(\sigma), x_j^*(\sigma); \sigma)\}_{i,j=1}^{n}. \tag{3.71}$$

Let us define matrix P by

$$P \equiv c^*(\mathbf{k}^0, \mathbf{k}^1, \cdots, \mathbf{k}^{n-1}) \tag{3.72}$$

which becomes orthogonal from (3.35). Then the matrix $\tilde{M}^{*,\sigma}$ in (3.67) can be rewritten as

$$\begin{aligned} \tilde{M}^{*,\sigma} &= \{c_1^* c_2^* {}^t\mathbf{k}^i \mathbf{G} \mathbf{k}^j\}_{i,j=0}^{n-1} \\ &= \frac{c_1^* c_2^*}{(c^*)^2} {}^tP \mathbf{G}_N P \\ &= {}^tP \tilde{\mathbf{G}}_N P, \end{aligned} \tag{3.73}$$

where

$$\tilde{\mathbf{G}}_N \equiv \frac{c_1^* c_2^*}{(c^*)^2} \mathbf{G}_N. \qquad \textit{(SLEP matrix)} \tag{3.74}$$

This shows that $\tilde{M}^{*,\sigma}$ is an orthogonal transformation of the real symmetric matrix $\tilde{\mathbf{G}}_N$ which clearly does *not* depend on the choice of the subsequence, and hence neither do the eigenvalues of $\tilde{M}^{*,\sigma}$. We reach the final form by multiplying both side of (3.67) by P:

$$\{\tilde{\mathbf{G}}_N + (\tau^{*,\sigma} - \sigma\hat{\zeta}^{*,\sigma})I\}\mathbf{b}^* = 0, \tag{3.75}$$

where $\mathbf{b}^* \equiv P\mathbf{a}^*$. This is, what we call, *the normal SLEP system* of (3.12). Since the normal SLEP system (3.75) is the common limit of all subsequences, the eigenvalues of (3.66) (without taking subsequence) converge to those of (3.75) when $\varepsilon \downarrow 0$. Moreover, since $\tilde{\mathbf{G}}_N$ has n real distinct eigenvalues (see Theorem 3.24), the associated eigenfunctions as well as eigenvalues of (3.66) are uniquely determined as continuous functions of ε by usual *regular* perturbation. What we have to do is to show that $\tilde{\mathbf{G}}_N$ has n distinct real eigenvalues and determine their signs. Especially, we are interested in the minimum eigenvalue of $\tilde{\mathbf{G}}_N$, since it determines the maximum value of the scaled critical eigenvalues $\tau^{*,\sigma}$. If the maximum value of $\tau^{*,\sigma}$ is negative (resp. positive), the normal n-layered solution becomes asymptotically stable (resp. unstable). We have the following result for the eigenvalue problem (3.75), the proof of which is delegated to Section 3.4.

Theorem 3.24 (Eigenvalues of the SLEP System). *The set of eigenvalues of* $\tilde{\mathbf{G}}_N$

$$\tilde{\mathbf{G}}_N\theta = \gamma\theta \tag{3.76}$$

consists of n real distinct positive eigenvalues

$$0 < \gamma_{n-1} < \gamma_{n-2} < \cdots < \gamma_0. \tag{3.77}$$

Namely, in term of $\tau^{*,\sigma}$, *(3.75) has n distinct real eigenvalues*

$$\tau_0^{*,\sigma} < \tau_1^{*,\sigma} < \cdots < \tau_{n-1}^{*,\sigma}, \tag{3.78}$$

where $\tau_i^{*,\sigma} = \sigma\hat{\xi}^{*,\sigma} - \gamma_i$ $(0 \le i \le n-1)$. *Moreover, it holds that*

$$\tau_{n-1}^{*,\sigma} < 0. \tag{3.79}$$

Theorem 3.24 leads us to the following main result by a regular perturbation. Asymptotic forms of critical eigenfunctions are also given in the next theorem.

Theorem 3.25. *There is a positive constant* $\hat{\varepsilon}$ *such that the critical eigenvalues of* $\mathcal{L}^{\varepsilon,\sigma}$ *consist of n real distinct eigenvalues* $\lambda_0^c(\varepsilon), \cdots, \lambda_{n-1}^c(\varepsilon)$ *which are simple and continuous for* $0 \le \varepsilon < \hat{\varepsilon}$ *satisfing the asymptotic relations (see Figure 3.3)*

$$\lambda_k^c(\varepsilon) \simeq \tau_k^{*,\sigma}\varepsilon, \qquad k = 0, \cdots, n-1$$

as $\varepsilon \downarrow 0$, *where* $\tau_k^{*,\sigma}$ $(k = 0, \cdots, n-1)$ *are given in Theorem 3.24. The associated critical eigenfunction* $\Phi^k(\varepsilon) = {}^t(w^k(\varepsilon), z^k(\varepsilon))$ *has the following asymptotic form*

$$\lim_{\varepsilon\downarrow 0}\Phi^k(\varepsilon) \equiv \Phi^{k^*} = \begin{pmatrix} w^{k^*} \\ z^{k^*} \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{j=1}^n q_j^k\delta_{x_j} - \frac{f_v^{*,\sigma}}{f_u^{*,\sigma}}c_2^*\sum_{j=1}^n q_j^k K^{*,\sigma,0}\delta_{x_j} \\ \displaystyle c_2^*\sum_{j=1}^n q_j^k K^{*,\sigma,0}\delta_{x_j} \end{pmatrix}, \tag{3.80}$$

where $\mathbf{q}^k = (q_1^k, \cdots, q_n^k)$ $(\|q^k\| = 1)$ *is an eigenvector of* $\tilde{\mathbf{G}}_N$ *for* γ_k, *and we use the simple notation* δ_{x_j} *instead of* $\delta(x - x_j^*(\sigma))$. *The rest of the eigenvalues, i.e., noncritical ones, have strictly negative real parts uniformly for* $0 < \varepsilon < \hat{\varepsilon}$.

Remark 3.26. *Note that asymptotic forms (3.80) are free from the chosen subsequence in Lemma 3.22.*

Proof. Since we know from Theorem 3.24 that $\{\tau_i^{*,\sigma}\}_{i=0}^{n-1}$ are real and distinct, the first part of the theorem is easily obtained by applying the implicit function theorem to the characteristic equation of (3.66) with replacing λ by $\varepsilon\tau$. In fact there exist ε^* and κ^* such that (3.66) has a unique ε-family of solutions $\{\tau_k(\varepsilon,\sigma)\}_{k=0}^{n-1}$ for $0 < \varepsilon < \varepsilon^*$ and $|\tau - \tau_k^{*,\sigma}| < \kappa^*$ satisfying $\lim_{\varepsilon\downarrow 0}\tau_k(\varepsilon,\sigma) = \tau_k^{*,\sigma}$ $(k = 0, \cdots, n-1)$. Conversely we can find ε_{κ_0} $(\le \varepsilon^*)$ and κ_0 $(> |\tau_0^{*,\sigma}|)$ such that any solution of (3.66) with $0 < \varepsilon < \varepsilon_{\kappa_0}$ and $|\tau| < \kappa_0$ must coincide with one of $\{\tau_k(\varepsilon,\sigma)\}_{k=0}^{n-1}$. Let $\hat{\delta} = \varepsilon_{\kappa_0}\kappa_0$. Then, taking

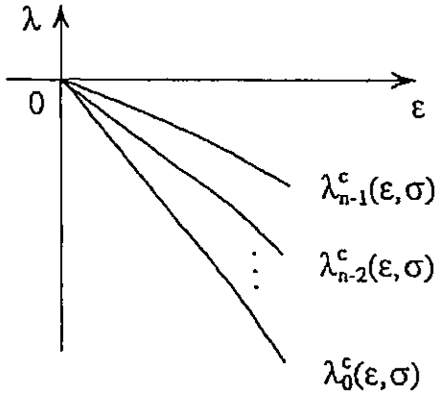

Figure 3.3.

$\delta = \hat{\delta}$ in Proposition 3.20, we see that noncritical eigenvalues of (3.12) satisfy (3.61) for $0 < \varepsilon < \hat{\varepsilon}$ where $\hat{\varepsilon} = \min\{\varepsilon_{\kappa_0}, \varepsilon_{\hat{\delta}}\}$. Simplicity of each critical eigenvalue can be proved in a parallel way as in the proof of Theorem 3.1 of [44]. The only remaining thing is to show (3.80). Denote by $Q = (\mathbf{q}^0, \cdots, \mathbf{q}^{n-1})$ the orthogonal matrix which diagonalizes $\bar{\mathbf{G}}_N$, i.e.,

$$ {}^tQ\bar{\mathbf{G}}_N Q = D_n, \tag{3.81} $$

where $D_n = diag(\gamma_0, \gamma_2, \cdots, \gamma_{n-1})$. Recalling that the matrix $\bar{\mathbf{G}}_N$ is represented with respect to the base $\{c_2^* K^{*,\sigma,0}(\delta_{x_i})\}_{i=1}^n$ (see (3.63), (3.72), (3.73)), it is clear that $z^{k^*} = c_2^* \sum_{j=1}^{n} q_j^k K^{*,\sigma,0}\delta_{x_j}$ is an eigenfunction associated with γ_k (and hence $\tau_k^{*,\sigma}$). As for the w-component, we have the expression for w by solving the first equation of (3.12)

$$ w = \sum_{i=0}^{n-1} \frac{\langle z, -f_v^{\varepsilon,\sigma}\frac{\phi_i^{\varepsilon,\sigma}}{\sqrt{\varepsilon}}\rangle}{\zeta_i^{\varepsilon,\sigma} - \tau} \frac{\phi_i^{\varepsilon,\sigma}}{\sqrt{\varepsilon}} + (L^{\varepsilon,\sigma} - \lambda)^{\dagger}(-f_v^{\varepsilon,\sigma} z). \tag{3.82a} $$

Using Lemma 3.19 and 3.22, we see that w^{k^*} can be represented as

$$ w^{k^*} = \sum_{i=0}^{n-1} \frac{c_1^* \langle z^{k^*}, \Delta_i\rangle}{\zeta_i^{*,\sigma} - \tau_k^{*,\sigma}} \Delta_i - \frac{f_v^{*,\sigma}}{f_u^{*,\sigma}} z^{k^*}. \tag{3.82b} $$

Substituting the above expression of z^{k^*} into (3.82b), we obtain

$$ \begin{aligned} w^{k^*} &= c_1^* c_2^* \sum_{i=0}^{n-1} \frac{\langle \sum_{j=0}^{n} q_j^k K^{*,\sigma,0}\delta_{x_j}, \sum_{\ell=0}^{n} \kappa_\ell^i \delta_{x_\ell}\rangle}{\zeta_i^{*,\sigma} - \tau_k^{*,\sigma}} \Delta_i \\ &\quad - c_2^* \frac{f_v^{*,\sigma}}{f_u^{*,\sigma}} \sum_{j=0}^{n} q_j^k K^{*,\sigma,0}\delta_{x_j}. \end{aligned} \tag{3.83} $$

Since we want to have an asymptotic form independent of $\{\mathbf{k}^i\}_{i=0}^{n-1}$, we rewrite the first term on the right-hand side of (3.83). First note that $G_{j\ell} = \langle K^{*,\sigma,0}\delta_{x_j}, \delta_{x_\ell}\rangle$, we see from (3.81) that the numerator becomes

$$c_1^* c_2^* \sum_{j,\ell=1}^{n} G_{j\ell} q_j^k \kappa_\ell^i = (c^*)^2 \gamma_k \sum_{\ell=1}^{n} q_\ell^k \kappa_\ell^i.$$

Hence the first term on the right-hand side of (3.83) becomes

$$\frac{(c^*)^2 \gamma_k}{\hat{\zeta}_i^{*,\sigma} - \tau_k^{*,\sigma}} \sum_{i,j,\ell=1} q_\ell^k \kappa_\ell^i \kappa_j^i \delta_{x_j}.$$

Recalling that $P^t P = I$, where $P = c^*(\mathbf{k}^0, \cdots, \mathbf{k}^{n-1})$, this is equal to

$$\sum_{j=1}^{n} q_j^k \delta_{x_j}.$$

Here we use the relation $\hat{\zeta}^{*,\sigma} - \tau_k^{*,\sigma} = \gamma_k$. Combining the above results, we conclude that

$$w^{k^*} = \sum_{j=1}^{n} q_j^k \delta_{x_j} - c_2^* \frac{f_v^{*,\sigma}}{f_u^{*,\sigma}} \sum_{j=0}^{n} q_j^k K^{*,\sigma,0} \delta_{x_j},$$

which completes the proof. □

Combining Theorem 3.25 with Corollary 3.9, we obtain the following result which is equivalent to Main Theorem in Section 1.

Theorem 3.27. *For any fixed σ and positive integer n with $0 < \sigma < \sigma_0$, there exists $\varepsilon_n(\sigma) > 0$ such that the normal n-layered solution is asymptotically stable for $0 < \varepsilon < \varepsilon_n(\sigma)$, where $\varepsilon_n(\sigma)$ depends on n and σ with $\lim_{n\uparrow\infty} \varepsilon_n(\sigma) = 0$. Hence the number of asymptotically stable normal multi-layered solutions tends to infinity as $\varepsilon \downarrow 0$.*

3.4. Eigenvalue Problem for the SLEP Matrix

We shall prove Theorem 3.24 by sequence of lemmas. A key trick is to consider the eigenvalue problem of the *inverse* of $\tilde{\mathbf{G}}_N$, not $\tilde{\mathbf{G}}_N$ itself. The main feature of $\tilde{\mathbf{G}}_N^{-1}$ is that it is a tri-diagonal symmetric matrix (Lemma 3.28), which is well studied, and a general theory (Lemma 3.29) can be applied to $\tilde{\mathbf{G}}_N^{-1}$ to conclude Theorem 3.24 except (3.79). In order to show (3.79), which is crucial for the stability, we need to introduce the auxiliary matrix $\tilde{\mathbf{G}}_D$ which is similarly defined as $\tilde{\mathbf{G}}_N$ with replacing the Neumann boundary conditions by Dirichlet ones. Namely,

$$\tilde{\mathbf{G}}_D \equiv \frac{c_1^* c_2^*}{(c^*)^2} \mathbf{G}_D \equiv \frac{c_1^* c_2^*}{(c^*)^2} \{G_D(x_i^*(\sigma), x_j^*(\sigma); \sigma)\}_{i,j=1}^{n}, \tag{3.84}$$

where $G_D(x, y; \sigma)$ is the Green function of the operator (3.69b) under homogeneous Dirichlet boundary conditions. A key observation for $\tilde{\mathbf{G}}_D$ is that the minimum eigenvalue of it is equal to $\sigma\hat{\zeta}^{*,\sigma}$, or, in terms of $\tau^{*,\sigma}$, the maximum eigenvalue of (3.75) with replacing $\tilde{\mathbf{G}}_N$ by $\tilde{\mathbf{G}}_D$ is equal to zero (Lemma 3.31). This comes from the fact that the x-derivative of the normal n-layered solution is always an eigenfunction of (3.12) under Dirichlet boundary conditions associated with the *zero* eigenvalue, i.e.,

$$\begin{aligned} &L^{\varepsilon,\sigma}u_x^\varepsilon + f_v^{\varepsilon,\sigma}v_x^\varepsilon = 0 \\ &M^{\varepsilon,\sigma}v_x^\varepsilon + g_u^{\varepsilon,\sigma}u_x^\varepsilon = 0 \\ &u_x^\varepsilon = 0 = v_x^\varepsilon \quad \text{on } \partial I. \end{aligned} \tag{3.85}$$

A comparison between components of $\tilde{\mathbf{G}}_N^{-1}$ and $\tilde{\mathbf{G}}_D^{-1}$ (Lemma 3.32) leads to the conclusion (3.79). In view of (3.74) and (3.84), it suffices to compare $\mathbf{G}_N^{-1}$ with $\mathbf{G}_D^{-1}$.

Lemma 3.28 (Inverse of the SLEP matrix). *The inverse of* $\mathbf{G}_N$ *exists and* $\mathbf{G}_N^{-1}$ *is a tri-diagonal real symmetric matrix such that*

(*a*) *All diagonal elements are equal except* (1, 1)- *and* (*n*, *n*)-*components.*
(*b*) *Every other off-diagonal elements are equal.*
(*c*) *All diagonal* (*resp. off-diagonal*) *elements are positive* (*resp. negative*).

More precisely we have

$$\mathbf{G}_N^{-1} = \frac{W(h,k)}{\sigma}\begin{pmatrix} -\dfrac{h_2/h_1}{A_{12}} & \dfrac{1}{A_{12}} & & & \\ & -\dfrac{A_{13}}{A_{12}A_{23}} & \dfrac{1}{A_{23}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \dfrac{1}{A_{12}} \\ & & & & -\dfrac{h_2/h_1}{A_{12}} \end{pmatrix}, n: \textit{even}, \tag{3.86}$$

$$\mathbf{G}_N^{-1} = \frac{W(h,k)}{\sigma}\begin{pmatrix} -\dfrac{h_2/h_1}{A_{12}} & \dfrac{1}{A_{12}} & & & \\ & -\dfrac{A_{13}}{A_{12}A_{23}} & \dfrac{1}{A_{23}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \dfrac{1}{A_{23}} \\ & & & & -\dfrac{k_{n-1}/k_n}{A_{23}} \end{pmatrix}, n: \textit{odd}, \tag{3.87}$$

where

$$A_{ij} \equiv \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix}$$

$$h_i \equiv h(x_i^*(\sigma)), \qquad k_i \equiv k(x_i^*(\sigma)) \qquad (\textit{see } (3.69a) \textit{ and } (3.71)) .$$

Exactly the same formulae hold for $\mathbf{G}_D^{-1}$ *with replacing* h_i, k_j *by* h_i^d, k_j^d. *Here* h_i^d, k_j^d *are the correspondents of the Green function for the Dirichlet boundary conditions.*

Proof. See Appendix C. □

The following result is basic for the study of the eigenvalue problem of tri-diagonal matrix. For the proof, see, for instance, Wilkinson [60; chapter 5].

Lemma 3.29 (Eigenvalues and Eigenfunctions of Tri-diagonal Symmetric Matrix). *Let* T *be a symmetric tri-diagonal matrix with non-zero off-diagonal elements of the form*

$$T = \begin{pmatrix} \alpha_1 & \beta_2 & & & O \\ \beta_2 & \alpha_1 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \beta_n \\ O & & & \beta_{n-1} & \alpha_n \end{pmatrix}, \qquad \beta_i \neq 0.$$

Then, T *has* n *real, distinct, and simple eigenvalues* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. *The corresponding eigenvector* $\mathbf{x}^k = {}^t(x_1^k, \cdots, x_n^k)$ *to* λ_k *is expressed by*

$$x_1^k = 1,\ x_r^k = (-1)^{r-1} p_{r-1}(\lambda_k)/\beta_2\beta_3\cdots\beta_r\ (2 \le r \le n), \tag{3.88}$$

where $p_r(\lambda)$ *denotes the leading principal minor of order* r *of* $(T - \lambda I)$ *with* $p_0(\lambda) \equiv 1$. *Finally, the polynomials* $p_0(\lambda), p_1(\lambda), \cdots, p_n(\lambda)$ *satisfy the Sturm sequence property. Namely, let the quantities* $p_0(\mu), p_1(\mu), \cdots, p_n(\mu)$ *be evaluated for some value of* μ. *Then* $s(\mu)$, *the number of agreements in sign of consecutive members of this sequence, is equal to the number of eigenvalues of* T *which are strictly greater than* μ

Corollary 3.30 (Positivity of Eigenvalues of $\mathbf{G}_N$). *All eigenvalues of* $\mathbf{G}_N^{-1}$ *(and hence* $\mathbf{G}_N$ *also) are strictly positive. This is also true for* $\mathbf{G}_D^{-1}$.

Proof. It suffices to show that the minimum eigenvalue of $\mathbf{G}_N^{-1}$ is strictly positive. First, noting that all elements of $\mathbf{G}_N$ are non-negative, we see from the Perron-Frobenius Theorem (see Varga [59; chapter 2]) that the largest eigenvalue $\gamma_{\max}$ of it is real, simple, and positive with a positive eigenvector. Hence $\gamma_{\max}^{-1}$ must be one of the eigenvalues of $\mathbf{G}_N^{-1}$. On the other hand, the only eigenvalue of $\mathbf{G}_N^{-1}$ which has a positive eigenvector is the minimum one, say λ_1, because of (3.88) and the Sturm sequence property of Lemma 3.29. This implies $\lambda_1 = \gamma_{\max}^{-1} > 0$ and completes the proof. □

We need two more lemmas to prove (3.79) of Theorem 3.24. The first one is a restatement of the remark at the beginning of this subsection.

Lemma 3.31. *The minimum eigenvalue of* $\tilde{\mathbf{G}}_D$ *is equal to* $\sigma\hat{\xi}^{*,\sigma}$. *In terms of* $\tau^{*,\sigma}$, *this is equivalent to say that the greatest eigenvalue of the problem (3.75) with replacing* $\tilde{\mathbf{G}}_N$ *by* $\tilde{\mathbf{G}}_D$ *is equal to zero.*

Proof. It is clear from (3.85) that $\sigma\hat{\xi}^{*,\sigma}$ (i.e., $\tau = 0$) is one of the eigenvalues of $\tilde{\mathbf{G}}_D$. The only thing we have to prove is that it is the minimum eigenvalue of $\tilde{\mathbf{G}}_N$. To do this, note that Lemma 3.29 also holds for $\tilde{\mathbf{G}}_D^{-1}$, and therefore, the eigenvector associated with the largest eigenvalue has no agreement in sign of consecutive numbers of components. On the other hand, it is not difficult to see that the eigenvector of $\tilde{\mathbf{G}}_D$ associated with $\sigma\hat{\xi}^{*,\sigma}$ also has the same property, since the original eigenvector (u_x^{ξ}, v_x^{ξ}) of (3.85) can be generated by flipping the unit form on the first subinterval $(0, 1/n)$ by $(n-1)$-times in odd symmetric way. Hence the largest eigenvalue of $\tilde{\mathbf{G}}_D^{-1}$ must be $(\sigma\hat{\xi}^{*,\sigma})^{-1}$, which implies from Corollary 3.30 that $\sigma\hat{\xi}^{*,\sigma}$ is the minimum eigenvalue of $\tilde{\mathbf{G}}_D$. □

A direct consequence of this lemma is that (3.79) is equivalent to

$$\text{Minimum eigenvalue of } \tilde{\mathbf{G}}_N > \text{Minimum eigenvalue of } \tilde{\mathbf{G}}_D = \sigma\hat{\xi}^{*,\sigma}. \tag{3.89}$$

In view of Corollary 3.30 this is equivalent to

$$\text{Maximum eigenvalue of } \tilde{\mathbf{G}}_N^{-1} < \text{Maximum eigenvalue of } \tilde{\mathbf{G}}_D^{-1}\ , \tag{3.90}$$

which is more convenient for us. To show (3.90), we make a comparison between the elements of $\tilde{\mathbf{G}}_N^{-1}$ and $\tilde{\mathbf{G}}_D^{-1}$.

Lemma 3.32 (Comparison between $\tilde{\mathbf{G}}_N^{-1}$ and $\tilde{\mathbf{G}}_D^{-1}$). *It holds that*
(i) All the components of $\tilde{\mathbf{G}}_N^{-1}$ *and* $\tilde{\mathbf{G}}_D^{-1}$ *are equal except* $(1, 1)$ *and* (n, n) *components.*
(ii) For $(1, 1)$ *and* (n, n) *components, the following inequalities hold*

$$(\tilde{\mathbf{G}}_N^{-1})_{11} < (\tilde{\mathbf{G}}_D^{-1})_{11}$$

and

$$(\tilde{\mathbf{G}}_N^{-1})_{nn} < (\tilde{\mathbf{G}}_D^{-1})_{nn}.$$

Proof. See Appendix C. □

Now we are ready to prove Theorem 3.24.

Proof of Theorem 3.24.
It is clear that Lemmas 3.28, 3.29, and Corollary 3.30 imply all the results of Theorem 3.24 except the stability inequality (3.79). Recalling the variational characterization of the principal eigenvalues of $\tilde{\mathbf{G}}_N^{-1}$ and $\tilde{\mathbf{G}}_D^{-1}$, the inequality (3.90) is a direct consequence of Lemma 3.32. Finally the inequality (3.90) combined with Lemma 3.31 leads us to the inequality (3.79). □

4. Recovery Process of Stability

— From the Shadow System to the Full System —

In Section 3 we showed that, for a *fixed* $\sigma > 0$, all normal n-layered solutions are stable for small $\varepsilon > 0$. However their stabilities are subtle in the sense that they have n real negative critical eigenvalues which tend to zero with order ε as $\varepsilon \downarrow 0$. On the other hand, a single reaction diffusion equation has also similar normal n-layered solutions, but they are all unstable as was mentioned in Section 1.

A naive question is that "Can we somehow interpolate these two opposite results from spectral point of view?" A key ingredient for this is the shadow system (see (1.16) and Section 4.1) which is obtained as the limiting system when $\sigma \downarrow 0$, i.e., the diffusivity of the controller v is extremely high, and hence v is reduced to a constant function $v \equiv \xi(t)$ in spatial direction. The shadow system is an intermediate system between the full system and the single equation, and plays a pivotal role to answer the above question. Namely the spectra of the linearized problem (3.12) converge to those of the shadow system as $\sigma \downarrow 0$, and if the scalar controller ξ of the shadow system is fixed to be a constant with respect to time, it reduces to a single reaction diffusion equation.

In Section 4.1 we shall prove that the normal n-layered solutions to the shadow system have $(n-1)$ real *positive* eigenvalues which tend to zero exponentially as $\varepsilon \downarrow 0$, and the associated eigenfunctions belong to the D^n-symmetry breaking space (see Lemma 4.3). In Section 4.2 we shall show that the $(n-1)$ critical eigenvalues $\{\lambda_k^c(\varepsilon)\}_{k=1}^{n-1}$ for the full system (see Theorem 3.25) converge to the above positive ones as $\sigma \downarrow 0$. The only one exceptional critical eigenvalue $\lambda_0^c(\varepsilon)$, which belongs to the D^n-symmetric space, does not change its sign when $\sigma \downarrow 0$, because the scalar controller ξ is enough to stabilize the perturbation in this D^n-symmetric space. It is clear from this that spatial variation of the controller v plays a key role in stabilizing multi-layered solutions. We shall study this transition of stability through the analysis of the SLEP matrix $\tilde{M}^{\varepsilon,\sigma}$ as $\sigma \downarrow 0$.

4.1. Instability of Multi-layered Solutions for the Shadow System

We consider the limiting system of (3.1) when $\sigma \downarrow 0$, i.e., the diffusivity of v is extremely large. As $\sigma \downarrow 0$, the second component v approaches a constant function in spatial direction under the assumption that (u, v) remains bounded in C^0-sense for all time which is guaranteed from our assumptions for (f, g) (existence of invariant rectangle). On the other hand, integrating both sides of the second equation of (3.1), and using the Neumann boundary conditions, we have

$$\int_I v_t dx = \int_I g(u, v) dx,$$

which holds independently of σ. Thus we have the following limiting system as $\sigma \downarrow 0$ (see [41] and [30] for more precise derivation);

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + f(u, \xi) \\ \xi_t &= \int_I g(u, \xi) dx \end{aligned} \tag{4.1}$$

subject to $u_x = 0$ on ∂I, where $\xi(t)$ is a constant function.

We call (4.1) the *shadow system* of (3.1). It is known (see Appendix 1 in [44]) that (4.1) has a unique ε-family of normal n-layered solutions $(u^n(x;\varepsilon,0), \xi^n(\varepsilon))$, and $D^n(\varepsilon,\sigma) = (u^n(x;\varepsilon,\sigma), v^n(x;\varepsilon,\sigma))$ converges to this solution as $\sigma \downarrow 0$ in $C^2_\varepsilon \times C^2$-topology. Since the degree of freedom of the controller ξ is equal to one, (4.1) cannot stabilize the normal n-layered solutions ($n \geq 2$) as observable ones. In terms of the spectral behavior, this can be represented in the next theorem. In this section we only deal with the solutions of (4.1), so we omit the superscript 0 for simplicity like f^ε_v, L^ε, τ^*_0 instead of $f^{\varepsilon,0}_v$, $L^{\varepsilon,0}$, $\tau^{*,0}_0$.

Theorem 4.1. *The following linearized eigenvalue problem at* $((u^n(x;\varepsilon,0), \xi^n(\varepsilon))$

$$\begin{cases} L^\varepsilon w + f^\varepsilon_v \eta = \lambda w \\ \displaystyle\int_I \{g^\varepsilon_u w + g^\varepsilon_v \eta\} = \lambda\eta \end{cases} \qquad (w,\eta) \in (H^2(I) \cap H^1_N(I)) \times \mathbf{C} \tag{4.2}$$

has exactly n critical eigenvalues $\{\lambda^s_0(\varepsilon), \lambda^s_1(\varepsilon), \cdots, \lambda^s_{n-1}(\varepsilon)\}$ *such that*

$$\lambda^s_0(\varepsilon) < 0 < \lambda^s_{n-1}(\varepsilon) < \cdots < \lambda^s_1(\varepsilon)$$

and satisfy

$$\lambda^s_0(\varepsilon) \simeq \tau^*_0 \varepsilon \qquad (\tau^*_0 < 0), \tag{4.3}$$

$$\lambda^s_i(\varepsilon) \leq C\exp(-\gamma/\varepsilon) \qquad (i = 1, \cdots, n-1) \tag{4.4}$$

when $\varepsilon \downarrow 0$, *where C and* γ *are positive constants independent of i. All the other noncritical eigenvalues of (4.2) have strictly negative real parts for small* ε. *Moreover the unstable eigenvalue* $\lambda^s_i(\varepsilon)$ *coincides with the i-th eigenvalue* ζ^ε_i *of the Sturm-Liouville operator* L^ε *for* $i = 1, \cdots, n-1$. *The associated eigenfunction with* $\lambda^s_0(\varepsilon)$ *belongs to the* D^n*-symmetric space* X^+, *and those for* $\{\lambda^s_i(\varepsilon)\}^{n-1}_{i=1}$ *are given by* $\{(\phi^\varepsilon_i, 0)\}^{n-1}_{i=1}$ *and belong to the* D^n*-breaking space* X^-, *where* ϕ^ε_i *is the eigenfunction of* L^ε *for* ζ^ε_i $(i = 1, \cdots, n-1)$. *See Lemma 4.3 for the definitions of* X^+ *and* X^-.

In order to prove this theorem, we need to show the following two lemmas.

Lemma 4.2. *Let* $\{\zeta^\varepsilon_i, \phi^\varepsilon_i\}^\infty_{i=0}$ *be the complete orthnormal set (in* L^2*-sense) of* L^ε. *Then* ϕ^ε_i *has exactly i internal simple zeros and the first n eigenvalues* ($\zeta^\varepsilon_0 > \zeta^\varepsilon_1 > \cdots > \zeta^\varepsilon_{n-1}$) *are positive for* $\varepsilon > 0$ *and tend to zero exponentially as* $\varepsilon \downarrow 0$, *i.e.*,

$$0 < \zeta^\varepsilon_i \leq C\exp(-\frac{\gamma}{\varepsilon}) \qquad i = 0, 1, \cdots, n-1, \tag{4.5}$$

where C and γ *are positive constants independent of i.*

Proof. The proof of Lemma 3.15 is also valid for this case. Since $v = \xi$ is a constant function, the first term of (3.44) does not appear, which leads us to the estimate (4.5). The details are left to the reader. □

Next we introduce the orthogonal decomposition of $L^2(I)$ which takes into account the symmetry of normal n-layered solutions due to the folding up principle (see Corollary 3.9). A function $u(x) \in L^2(I)$ is called D^n*-symmetric* if it is generated by $u(x)|_{[0,1/n]}$

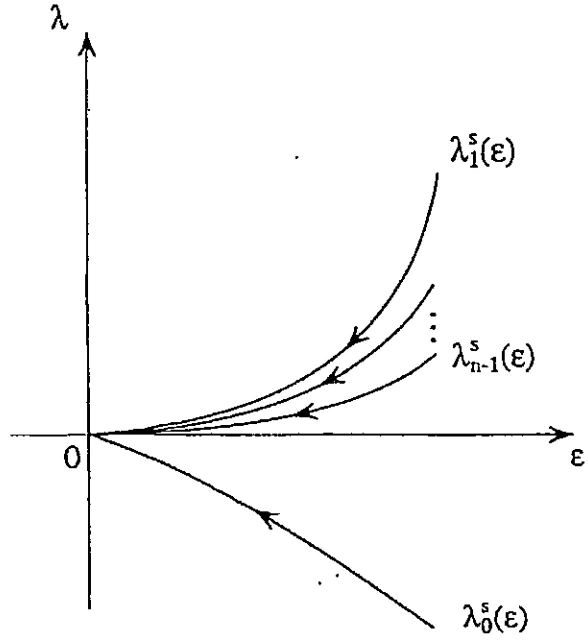

Figure 4.1.

(restriction to $[0, 1/n]$) by flipping it $(n-1)$-times in an even way. Apparently each component of the normal n-layered solution is D^n-symmetric. $L^2(I)$ can be decomposed into D^n-symmetric part and its orthogonal complement in the following way.

Lemma 4.3. *The space $L^2(I)$ has the following orthogonal decomposition*

$$L^2(I) = X^+ \oplus X^-,$$

where

$$X^+ \equiv \text{ closure of } span\{\phi^\varepsilon_{nk}\}^\infty_{k=0} \qquad \text{in } L^2(I)$$

$$X^- \equiv \text{ closure of } span\{\phi^\varepsilon_i\}_{i\neq nk} \qquad \text{in } L^2(I).$$

X^+ consists of the D^n-symmetric functions, in paticular, normal n-layered solutions and constant function belong to this space. We call X^+ (resp. X^-) the D^n-symmetric (resp. D^n-breaking)-space. Finally we denote the orthogonal projection onto X^+ by P^+.

Proof. Since the potential term f^ε_u of L^ε is D^n-symmetric and from the nodal property of eigenfunctions, $\{\phi^\varepsilon_{nk}\}^\infty_{k=0}$ is generated via folding from a complete orthogonal base of the restricted Sturm-Liouville problem L^ε to $(0, 1/n)$ with Neumann boundary conditions at $x = 0$ and $1/n$. This immediately leads us to the above lemma. □

Remark 4.4. *Let $\tilde{L}^2(I)$ be an extended space of $L^2(I)$ defined by*

$$\tilde{L}^2(I) \equiv \{\text{each element of } L^2(I) \text{ is extended to the double interval } (0,2)$$

$$\text{in an even way and identify both end points}\}.$$

Similar extension can be applied to X^+ and X^-, and we have an orthogonal decomposition

$$\bar{L}^2(I) = \bar{X}^+ \oplus \bar{X}^-.$$

Then the element of $\bar{X}^+$ is characterized as a function which is invariant under D^n-action, i.e., reflection and $2\pi/n$-rotation. Note that X^+ is invariant under usual algebraic operations such as addition, product and so on. For more systematic group theoretical approach to our problems, see [24] and [25].

Proof of Theorem 4.1. The eigenvalue problem (4.2) can be decomposed as follows by using the projection P^+ onto X^+:

$$\begin{cases} L^\varepsilon w^+ + f_v^\varepsilon \eta = \lambda w^+ \\ \int (g_u^\varepsilon w^+ + g_v^\varepsilon \eta) dx = \lambda \eta \end{cases} \tag{4.6}$$

$$\begin{cases} L^\varepsilon w^- = \lambda w^- \\ \int g_u^\varepsilon w^- dx = 0, \end{cases} \tag{4.7}$$

where $w = w^+ + w^-$, $w^+ \equiv P^+ w \in X^+$, and $w^- = (I - P^+)w \in X^-$. Here we use the fact that L^ε commutes with P^+, and that f_v^ε, g_u^ε, g_v^ε, and η belong to X^+. The system (4.7) is equivalent to

$$L^\varepsilon w^- = \lambda w^-, \tag{4.8}$$

since w^- ($\in X^-$) is orthogonal to g_u^ε ($\in X^+$). Since each component of the decomposition

$$L^2(I) \times \mathbf{R} = (X^+ \times \mathbf{R}) \oplus (X^- \times \{0\})$$

is invariant under the linearized operation (i.e., the left-hand side of (4.2)). It suffices to solve problems (4.6) and (4.8) separately.

First it is clear from (4.8), Lemmas 4.2 and 4.3 that the set of eigenvalues $\{\zeta_i^\varepsilon\}_{i=1}^{n-1}$ of L^ε are the required positive critical eigenvalues $\{\lambda_i^s(\varepsilon)\}_{i=1}^{n-1}$ in Theorem 4.1 and all the other spectra of (4.8) are noncritical and strictly negative for small ε. The remaining negative critical eigenvalue $\lambda_0^s(\varepsilon)$ can be obtained by solving (4.6) in the following way. Using Lemma 4.2, the first equation of (4.6) can be solved with respect to w^+:

$$\begin{aligned} w^+ &= (L^\varepsilon - \lambda)^{-1}(-f_v^\varepsilon \eta) \\ &= \frac{(-f_v^\varepsilon \eta, \phi_0^\varepsilon)}{\zeta_0^\varepsilon - \lambda}\phi_0^\varepsilon + (L^\varepsilon - \lambda)^\dagger(-f_v^\varepsilon \eta), \end{aligned} \tag{4.9}$$

where $(L^\varepsilon - \lambda)^\dagger \equiv \sum_{k=1}^{\infty} \frac{(-f_v^\varepsilon \eta, \phi_{kn}^\varepsilon)}{\zeta_{kn}^\varepsilon - \lambda}\phi_{kn}^\varepsilon$, the reduced resolvent defined on X^+, which satisfies all the properties in Section 3, especially Lemma 3.19. Also note that ζ_0^ε does not belong to the spectrum of (4.6) for small ε, which can be proved in an analogous way of Lemma 3.17. Substituting (4.9) into the second equation of (4.6), we have after dividing it by η

$$\int_I \left\{ \frac{\langle -f_v^\varepsilon, \phi_0^\varepsilon \rangle}{\zeta_0^\varepsilon - \lambda} g_u^\varepsilon \phi_0^\varepsilon + g_u^\varepsilon (L^\varepsilon - \lambda)^\dagger (-f_v^\varepsilon) + g_v^\varepsilon \right\} dx = \lambda. \tag{4.10}$$

When $\eta = 0$, it follows from (4.9) that $w \equiv 0$, hence it suffices to consider (4.10).

Suppose λ is a noncritical eigenvalue. It follows from Corollary 3.14 and the fact that the denominator is bounded away from zero that the first term on the left-hand side of (4.10) tends to zero. In view of Lemma 3.19, we see that the second term of (4.10) approaches $\dfrac{-g_u^* f_v^*}{f_u^* - \lambda}$ as $\varepsilon \downarrow 0$. Thus (4.10) becomes in the limit of $\varepsilon \downarrow 0$

$$\int_I \left\{ \frac{\det^* - \lambda g_v^*}{f_u^* - \lambda} \right\} dx = \lambda, \tag{4.11}$$

where $\det^* \equiv f_u^* g_v^* - g_u^* f_v^*$. Recalling the assumptions (A.3) $\sim$ (A.5), one can show after some computation that any solution λ of (4.11) must have strict negative real part, which is also true for small positive ε by continuity arguments. Thus noncritical eigenvalues are not dangerous to stability.

We next consider the case where λ is a critical eigenvalue. Since the same asymptotic characterization of Lemma 3.22 and 3.23 also holds for the shadow system, we rewrite it in the following form

$$\int_I \left\{ \frac{\langle -f_v^\varepsilon, \phi_0^\varepsilon/\sqrt{\varepsilon} \rangle}{\hat{\zeta}_0^\varepsilon - \tau} g_u^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} + g_u^\varepsilon (L^\varepsilon - \varepsilon\tau)^\dagger (-f_v^\varepsilon) + g_v^\varepsilon \right\} dx = \varepsilon\tau, \tag{4.12}$$

where $\tau \equiv \lambda/\varepsilon$ and $\hat{\zeta}_0^\varepsilon \equiv \zeta_0^\varepsilon/\varepsilon$. Note that $\hat{\zeta}_0^\varepsilon$ tends to zero exponentially (see Lemma 4.1). Hence, when $\varepsilon \downarrow 0$, (4.12) becomes

$$-\frac{c_1^* c_2^*}{\tau^*} + \int_I \frac{\det^*}{f_u^*} dx = 0,$$

which leads us to the expression

$$\tau_0^* \equiv \lim_{\varepsilon \downarrow 0} \tau(\varepsilon) = \frac{c_1^* c_2^*}{\displaystyle\int_I \left(\frac{\det^*}{f_u^*} \right) dx}. \tag{4.13}$$

Recalling (A.3) and (A.4b), we see that $\tau_0^* < 0$. Multiplying $\hat{\zeta}_0^\varepsilon - \tau$ on both sides of (4.12) and applying the implicit function theorem to it, we easily see that (4.12) has a unique continuous solution $\tau = \tau(\varepsilon)$ up to $\varepsilon = 0$ with $\tau(0) = \tau_0^*$ given by (4.13). This completes the proof of Theorem 4.1. □

4.2. Recovery of Stability (From the shadow system to the full system)

We shall fill the gap of the stability results between the shadow system (Theorem 4.1) and the full system (Theorem 3.27) by studying the eigenvalue problem (3.66) in the limit of $\sigma \downarrow 0$. The next lemma is a key observation for this purpose.

Lemma 4.5. *When $\sigma \downarrow 0$, the operator $K^{\varepsilon,\sigma,\lambda}$ defined in Lemma 3.21 converges to $K^{\varepsilon,0,\lambda}$ in operator norm sense, where $K^{\varepsilon,0,\lambda}$ maps $h \in H^{-1}(I)$ to the constant function $c(h)$ defined by*

$$c(h) \equiv -\langle h, 1\rangle \Big/ \langle g_u^\varepsilon (L^\varepsilon - \lambda)^\dagger (-f_v^\varepsilon) + g_v^\varepsilon - \lambda, 1\rangle \,. \tag{4.14}$$

For $\lambda = 0$, (4.14) *tends to the following as $\varepsilon \downarrow 0$*

$$c^*(h) = -\frac{\langle h, 1\rangle}{\langle \det^* / f_u^*, 1\rangle}. \tag{4.15}$$

Idea of proof. Although we refer the proof to that of Lemma 3.1 of [44], we give here an intuitive idea. Roughly speaking, $K^{\varepsilon,\sigma,\lambda}$ can be expressed by

$$K^{\varepsilon,\sigma,\lambda} = \left\{ -\frac{1}{\sigma}\frac{d^2}{dx^2} - g_u^{\varepsilon,\sigma}(L^{\varepsilon,\sigma} - \lambda)^\dagger (-f_v^{\varepsilon,\sigma}\cdot) - g_v^{\varepsilon,\sigma} + \lambda \right\}^{-1}.$$

We decompose h as

$$h = \hat{h} + \int_I h dx,$$

where the average of $\hat{h}$ is equal to zero, i.e., $\int_I \hat{h} dx = 0$. According to this orthogonal decomposition, the operator $K^{\varepsilon,\sigma,\lambda}$ is splitted into two parts; one acts on $\hat{h}$-space and the other on average-space. As $\sigma \downarrow 0$, $K^{\varepsilon,\sigma,\lambda}|_{\hat{h}-space}$ tends to zero in operator norm because of $\left\{ -\frac{1}{\sigma}\frac{d^2}{dx^2} \right\}^{-1}$. Hence only the average part remains nontrivial which is determined by

$$-c(h) \int_I \left\{ g_u^{\varepsilon,\sigma}(L^{\varepsilon,\sigma} - \lambda)^\dagger (-f_v^{\varepsilon,\sigma}) + g_v^{\varepsilon,\sigma} - \lambda \right\} dx = \int_I h dx.$$

This leads us to (4.14). □

Now we can prove the following.

Theorem 4.6. *When $\sigma \downarrow 0$, the eigenvalues of* (3.66)

$$\tilde{M}^{\varepsilon,\sigma}\mathbf{a} = \begin{pmatrix} (\zeta_0^{\varepsilon,\sigma}/\varepsilon - \lambda/\varepsilon)\alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ (\zeta_{n-1}^{\varepsilon,\sigma}/\varepsilon - \lambda/\varepsilon)\alpha_{n-1} \end{pmatrix} \tag{3.66}$$

tend to the critical eigenvalues $\{\lambda_0^s(\varepsilon), \lambda_1^s(\varepsilon), \cdots, \lambda_{n-1}^s(\varepsilon)\}$ for the shadow system in Theorem 4.1.

Proof. The (i, j)-component of $\tilde{M}^{\varepsilon,\sigma}$ is equal to $\langle -f_v^{\varepsilon,\sigma}\phi_i^{\varepsilon,\sigma}/\sqrt{\varepsilon}, K^{\varepsilon,\sigma,\lambda}(g_u^{\varepsilon,\sigma}\phi_j^{\varepsilon,\sigma}/\sqrt{\varepsilon})\rangle$ (see (3.66)). In view of Lemma 4.3, we see that both $-f_v^{\varepsilon,\sigma}\phi_i^{\varepsilon,\sigma}/\sqrt{\varepsilon}$ and $g_u^{\varepsilon,\sigma}\phi_j^{\varepsilon,\sigma}/\sqrt{\varepsilon}$ are orthogonal to constant function for $i, j \neq 0$, i.e., their averages are equal to zero, while

$\int_I g_u^{\varepsilon,\sigma}(\phi_0^{\varepsilon,\sigma}/\sqrt{\varepsilon})dx$ converges to a nonzero constant as $\sigma \downarrow 0$. Hence all components except (1, 1) of $\tilde{M}^{\varepsilon,\sigma}$ approach zero from Lemma 4.5 when $\sigma \downarrow 0$. More precisely, we have

$$\tilde{M}^{\varepsilon,0} \equiv \lim_{\sigma\downarrow 0} \tilde{M}^{\varepsilon,\sigma} = \begin{pmatrix} m(\varepsilon,\lambda) & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & \\ \cdot & & & O & \\ \cdot & & & & \\ 0 & & & & \end{pmatrix} \tag{4.16a}$$

where

$$m(\varepsilon,\lambda) \equiv \left\langle -f_v^\varepsilon \phi_0^\varepsilon/\sqrt{\varepsilon},\ -\int_I g_u^\varepsilon(\phi_0^\varepsilon/\sqrt{\varepsilon})dx \Big/ \int_I \{g_u^\varepsilon(L^\varepsilon-\lambda)^\dagger(-f_v^\varepsilon)+g_v^\varepsilon-\lambda\}dx \right\rangle \tag{4.16b}$$

and

$$m(0,0) = -c_1^* c_2^* \Big/ \int_I (\det{}^*/f_u^*)dx\,. \tag{4.16c}$$

In view of the eigenvalue problems (3.66), (3.67), and the proof of Theorem 4.1 (see (4.12) and (4.13)), the above asymptotic characterization (4.16) leads us to Theorem 4.6. □

Remark 4.7. *In Theorem 4.6, we used (3.66) which may depend on the choice of the subsequence, however the proof of it does not depend on such a choice. In fact, the principal eigenfunction* $\phi_0^{\varepsilon,\sigma}/\sqrt{\varepsilon}$ *is itself a convergent sequence and the orthogonal property used in the proof of Theorem 4.6 holds for any subsequence.*

5. Concluding Remarks

Up until now we have considered the reaction-diffusion systems (1.1) and applied the SLEP method to the layered solutions to study the stability properties. The basic idea of this method, however, has a wide range of applicability to various types of problems, which essentially comes from the universal structure of internal layers of this class. In what follows we shall discuss briefly about several topics to which the SLEP method is useful.

(i) *Systems with different scales of relaxation parameters*

If the relaxation parameter δ of (1.1a) is taken to be $\varepsilon\tau$ ($\tau = O(1)$), then the dynamics of (1.1) and its singular limit system drastically change. In fact, it is easily seen from (1.11) that the propagation speed of the internal layer is of $O(1)$ and no longer slow compared with the outer dynamics. A similar asymptotic analysis as in Section 1 yields the following singular limit dynamics:

$$(\varphi_i)_t = \frac{(-1)^{i-1}}{\tau} c(V(\varphi_i(t))) \tag{5.1a}$$

$$V_t = DV_{xx} + G_\Phi(V). \tag{5.1b}$$

The dynamics of layers ((5.1a)) evolves simultaneously with that of the outer part ((5.1b)). It is plausible that, when τ becomes small, the outer dynamics cannot catch up with the speed of layers, and hence is not able to settle them down to a steady state. This instability really occurs in the form of Hopf bifurcation for the original system (1.1) with $\delta = \varepsilon\tau$ as well as for (5.1), and we have *layer oscillations* (*breathers*) (see [46] and [42] for details). The well-posedness and asymptotic behaviors of (5.1) has been proven by [30] for mono-layer case. For multi-layer case, the dynamics becomes more rich and complicated such as synchronization, annihilation, and coalescence, the study of which is under progress (see [31] for the case of two breathers). Finally, suppose we consider the system (1.1) with $\delta = \varepsilon\tau$ on the entire line $\mathbf{R}$, instability also occurs for travelling front solutions as τ becomes small, however the structure of bifurcation is different from the finite interval case due to the translation invariance. See [47] and [35] for details.

(ii) *Neumann layered solutions and separators*

The Main Theorem in Section 1 shows the coexistence of arbitrary many stable steady states in the limit of $\varepsilon \downarrow 0$, however, in order for the system (1.1) to be self-consistent, there must exist other unstable steady states or invariant sets which play the role of *separators* among stable ones. In fact it is possible to construct such unstable layer solutions in a rather systematic way (see Fujii and Hosono [23], and Nishiura and Tsujikawa [50]) although it may not exhaust all types of unstable solutions. Typical profiles are illustrated in Figure 5.1. These solutions consist of normal n-layered solutions plus *Neumann layers* (i.e., boundary layers satisfying the Neumann boundary condition), and play the role of separators among normal layered solutions. For example, it is not difficult to imagine that the solution in Figure 5.1 (a) is the separator between normal 1-layer and 2-layer solutions (see Figure 5.2). More precisely, the dimension of the unstable manifold of the Neumann layered solution is equal to one, and it is connected to 1-layer and 2-layer solutions. However, in general, rigorous justification of the existence of these connecting orbits remains an unexplored field compared with the scalar case (see Brunovský and Fiedler [5] and references therein). This is partly because of the lack of powerful tools such as Lyapunov function and the lap number for the system (1.1).

(iii) *Heteroclinic and Homoclinic bifurcations*

Global bifurcation such as heteroclinic or homoclinic bifurcation is one of the most important issues in dynamical system theory, and one can find many applications to various fields. For instance, a creation of homoclinic loop from two heteroclinic orbits is quite interesting from a PDE view point, since it means that travelling pulse solutions are born from a pair of front and back solutions (see, for example, Rinzel and Terman [54]). However, when one tries to apply such tools to practical problems, one obviously has to check several transversal and generic conditions along *large amplitude* orbits (see, for example, Chow, Deng, and Terman [13], Deng [14], and Kokubu [34]). This is usually not an easy task without a good control of parametric dependency of generating orbits from which new kind of solutions emanate when parameters vary. The SLEP method is suitable for this purpose when such orbits can be constructed by singular perturbation, since the usual transversal conditions are related to the spectral behavior of the linearized problem at the generating orbits. In fact all the hypotheses imposed on heteroclinic and homoclinic bifurcation theorem are rigorously verified by Kokubu, Nishiura, and

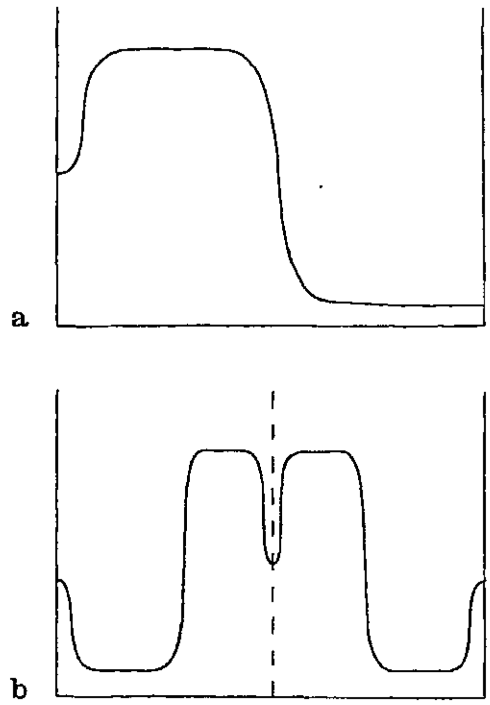

Figure 5.1.

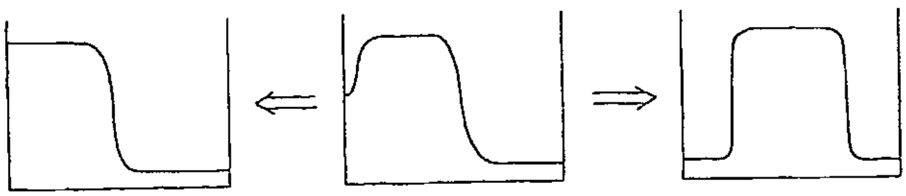

Figure 5.2.

Oka [35] for a system of bistable reaction diffusion equations with the aid of the SLEP method. A relation between the stability of front (or back) solutions and the intersecting manner of the stable and unstable manifolds is also given in [35].

(iv) *Higher dimensional cases*

The study of morphologies of interfaces in higher space dimensions such as dendrites in solidification problem (Langer [36]) and spirals in chemical reaction (Fife [19], Keener and Tyson [33]) is a central problem in pattern formation theory. Such phenomena can be modelled by reaction diffusion systems (see, for instance, Caginalp [6] and Fife [20]) containing a small parameter ε which represents the width of interface, and interfacial patterns can be constructed as singularly perturbed solutions to the model systems. Physically speaking, this singular perturbation could be explained as the introduction of surface tension effect (see Pelcé [53]). Stability analysis and bifurcation in these interfacial problems are quite important, since they are directly related to *pattern selection* problem. Unfortunately there are very few works on this issue, partly because it is, in general, extremely difficult to show the existence of the singularly perturbed solutions in a constructive way in higher dimensional space (for a single equation, see Fife and Greenlee [21]). For special domains like channels or spherical shapes, it is possible to answer, to some extent, for both existence and stability for reaction diffusion systems (see

Ohta, Mimura, and Kobayashi [52], Ohta and Mimura [51], and Taniguchi and Nishiura [57]). To proceed further, it seems reasonable to assume, as a working hypothesis, the existence of an ε-family of singularly perturbed solutions which has a smooth interface as $\varepsilon \downarrow 0$. Then the basic question is that how one can characterize the stability or instability of an interface by the *geometry* of it and *outer* solutions. And is it possible to derive the singular limit eigenvalue problem contracted on the interface?

Let us consider the reaction diffusion system in $\Omega \subset \mathbf{R}^n$:

$$\begin{cases} u_t = \varepsilon^2 \Delta u + f(u, v) & \\ & \text{in} \quad \Omega \subset \mathbf{R}^n \\ v_t = D\Delta v + g(u, v) & \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n} & \text{on} \quad \partial\Omega, \end{cases} \tag{5.2}$$

where Ω is a smooth bounded domain and $\partial/\partial n$ denotes the normal derivative, and assume that it has an ε -family of singularly perturbed steady state solutions with a smooth closed hypersurface Γ_0 as $\varepsilon \downarrow 0$. It turns out that the singular limit eigenvalue problem on Γ_0 is given by

$$\varepsilon\{\Delta_s + H\, \cdot\,\}\gamma + \frac{dc}{dv}(V^*\Big|_{\Gamma_0})\left\{(\nabla V^* \cdot n)\gamma + [G]\langle K^*(\delta_{\Gamma_0} \otimes \gamma), \delta_{\Gamma_0}\rangle\right\} = \tau\gamma, \tag{5.3}$$

where Δ_s represents the Laplace operator on Γ_0, $H\,\cdot$ is a bounded operator on $L^2(\Gamma_0)$ which depends on the geometry of Γ_0 but not on ε, $c(V)$ the velocity function as in (1.9), V^* the outer solution for v, $[G]$ the jump of value of g at Γ_0, δ_{Γ_0} the surface distribution of *Dirac*'s δ on Γ_0, K^* a compact operator similar to $K^{*,\sigma,0}$ (see (2.27) and Lemma 3.21), $\tau(\equiv \lim_{\varepsilon\downarrow 0} \lambda/\varepsilon)$ the scaled eigenvalue as in Lemma 3.23, and γ is an eigenfunction in $L^2(\Gamma_0)$ (see Nishiura [43] for a formal derivation of (5.3)). We call (5.3) *the SLEP equation on* Γ_0 for (5.2). Apparently (5.3) is itself a singular perturbation problem because of the first term on the left-hand side. The term $\varepsilon\Delta_s\gamma$ *cannot be neglected* since high frequency modes are stabilized by this term. It should be noted that the nonlocal term $[G]\langle K^*(\delta_{\Gamma_0} \otimes \gamma), \delta_{\Gamma_0}\rangle$ is responsible for the stabilization of low frequency modes. On the other hand, the second term $(\nabla V^* \cdot n)\gamma$ on the left-hand side is the principal part of destabilizing effect. Therefore only a finite band of modes could be destabilized in this system. In fact it can be proved that *any* stationary pattern of (5.2) with *smooth* limiting interface becomes *unstable* for small ϵ. Then what stable patterns look like and how they behave when ϵ tends to zero? One possibility is that the characteristic domain size of stable stationary patterns tends to zero as $\epsilon \downarrow 0$. This implies that stable patterns become finer and finer as $\epsilon \downarrow 0$. WE shall discuss more on these issues in Nishiura and Suzuki [49].

Appendix A

Proof of Lemma 2.7.

First we prove the equality

$$-\frac{dc}{dV}(v^*) = \frac{1}{\left\|\frac{d}{dy}\bar{u}^*\right\|_{L^2}^2}\frac{dJ}{dv}(v^*). \tag{1}$$

It follows from (1.8) and (1.9) that

$$u_{yy} + c(V)u_y + f(u, V) = 0 \tag{2a}$$

$$u(\pm\infty) = h_\pm(V). \tag{2b}$$

Differentiating (2a) by V, we have

$$(u')_{yy} + c(V)(u')_y + f_u(u, V) = -\frac{dc}{dV}(V)u_y - f_v(u, V)$$

where $u' = \dfrac{du}{dV}$. Let $V = v^*$, then

$$(u')_{yy} + f_u(u, v^*)u' = -\frac{dc}{dV}(v^*)u_y - f_v(u, v^*). \tag{3}$$

Here we used $c(v^*) = 0$. On the other hand, recalling Remark 3.7, $\dfrac{d}{dy}\bar{u}^*$ satisfies

$$W_{yy} + f_u(W, v^*)W = 0$$

with $\|W\|_{L^2} < \infty$, i.e., $\dfrac{d}{dy}\bar{u}^*$ belong to the kernel of the operator $\dfrac{d^2}{dy^2} + f_u(W, v^*)$ in $L^2(\mathbf{R})$. Applying the solvability condition to (3), we have

$$\begin{aligned} -\frac{dc}{dV}(v^*)\left\|\frac{d}{dy}\bar{u}^*\right\|^2 &= \int_{\infty}^{\infty} f_v(\bar{u}^*, v^*)\frac{d}{dy}\bar{u}^* dy \\ &= \frac{d}{dv}\int_{h_-(v^*)}^{h_+(v^*)} f(u, v^*)du, \end{aligned}$$

which shows (1).

Recalling the definitions of constants γ^*, c^*, c_1^* and c_2^* (see Remark 3.7 and Lemma 3.22), (1) implies (ii) of Lemma 2.7. As for (i), using the fact that V_2^* satisfies

$$\frac{1}{\sigma}(V_2^*)_{xx} + g(U_2^*, V_2^*) = 0$$

$$(V_2^*)_x = 0 \qquad \text{at} \quad x = 0, 1,$$

it is easily seen that

$$\frac{dV_2^*}{dx}(\varphi_1^*) = -\sigma \int_0^{\varphi_1^*} g(U_2^*, V_2^*)dx. \tag{4}$$

Combining (1), (4), with Lemma 3.15, we obtain (*i*) of Lemma 2.7. □

Appendix B

Proof of Lemma 3.17.

We shall prove a slightly more general statement: There are no eigenvalues of $\mathcal{L}^{\varepsilon,\sigma}$ which have the same asymptotic behaviour as (3.39) of Lemma 3.15. Let us prove this by contradiction. Suppose that there exists an eigenvalue $\lambda = \lambda(\varepsilon)$ of $\mathcal{L}^{\varepsilon,\sigma}$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda(\varepsilon)}{\varepsilon} = \zeta^{*,\sigma} \tag{1}$$

with the associated eigenfunction denoted by $(w(\varepsilon), z(\varepsilon))$. Hereafter we simply write $\lambda(\varepsilon)$, $w(\varepsilon)$, $z(\varepsilon)$ as λ, w, z. Solving the first equation of (3.12a) with respect to w, we have

$$w = \sum_{j=0}^{n-1} k_j(\varepsilon; z)\phi_j^{\varepsilon,\sigma} + (L^{\varepsilon,\sigma} - \lambda)^{\dagger}(-f_v^{\varepsilon,\sigma} z), \tag{2}$$

where

$$k_j(\varepsilon; z) \equiv \begin{cases} \dfrac{\langle -f_v^{\varepsilon,\sigma} z, \phi_j^{\varepsilon,\sigma}\rangle}{\zeta_j^{\varepsilon,\sigma} - \lambda} & \text{if } \lambda \neq \zeta_j^{\varepsilon,\sigma} \\ c_j \quad \text{arbitrary constant if} & \lambda = \zeta_j^{\varepsilon,\sigma}. \end{cases} \tag{3}$$

Note that if $\lambda = \zeta_j^{\varepsilon,\sigma}$, the solvability condition

$$\langle -f_v^{\varepsilon,\sigma} z, \phi_j^{\varepsilon,\sigma}\rangle = 0 \tag{4}$$

must be satisfied. Also, if necessary, by choosing an appropriate subsequence of $\lambda(\varepsilon)$ (we use the same notation for this), we can assume without loss of generality that either one of the cases of (3) occur when $\varepsilon \downarrow 0$. In what follows we treat only the first case since the second case can be dealt with in a similar way by using (4). Substituting (2) into the second equation of (3.12a) and applying $K^{\varepsilon,\sigma,\lambda}$ of Lemma 3.21 to it, we obtain

$$z = K^{\varepsilon,\sigma,\lambda}\left(\sum_{j=0}^{n-1} \sqrt{\varepsilon} k_j(\varepsilon; z) g_u^{\varepsilon,\sigma} \frac{\phi_j^{\varepsilon,\sigma}}{\sqrt{\varepsilon}}\right). \tag{5}$$

It is clear that not all k_j's are zero, otherwise $(w, z) \equiv 0$. Without loss of generality, we can normalize z as

$$\| z \|_{H_N^1(I)} = 1. \tag{6}$$

Recalling Lemma 3.21 and 3.22, we see as a necessary condition of (6) that $\sqrt{\varepsilon}k_j$ are uniformly bounded and not all of them go to zero as $\varepsilon \downarrow 0$. In view of (3), we see that

$$\sqrt{\varepsilon}\langle -f_v^{\varepsilon,\sigma}z, \phi_j^{\varepsilon,\sigma}\rangle = o(\varepsilon) \tag{7}$$

holds because $|\zeta_j^{\varepsilon,\sigma} - \lambda| = o(\varepsilon)$ as $\varepsilon \downarrow 0$ from our assumption (1). Dividing (7) by ε, we have

$$\lim_{\varepsilon\downarrow 0}\langle -f_v^{\varepsilon,\sigma}z, \frac{\phi_j^{\varepsilon,\sigma}}{\sqrt{\varepsilon}}\rangle = 0 \qquad \text{for all} \quad j. \tag{8}$$

Choosing an appropriate subsequence of $\sqrt{\varepsilon}k_j$ (with keeping the same notation), we obtain

$$\sqrt{\varepsilon}k_j(\varepsilon; z) \longrightarrow \hat{k}_j^* \qquad \text{for all} \quad j.$$

and the vector $\hat{\mathbf{k}}^* \equiv (\hat{k}_0^*, \cdots, \hat{k}_{n-1}^*)$ is not equal to zero. Therefore we have in the limit of $\varepsilon \downarrow 0$,

$$z^* \equiv \lim_{\varepsilon\downarrow 0} z(\varepsilon) = K^{*,\sigma,0}\left(c_2^* \sum_{j=0}^{n-1} \hat{k}_j^* \Delta_j\right). \tag{9}$$

On the other hand, it follows from (8) that

$$\langle z^*, \Delta_j\rangle = 0 \qquad \text{for all} \quad j,$$

which implies

$$\langle z^*, \sum_{j=0}^{n-1} \hat{k}_j^* \Delta_j\rangle = 0. \tag{10}$$

Substituting the expression (9) into (10), we have

$$\langle K^{*,\sigma,0}\hat{w}^*, \hat{w}^*\rangle = 0, \tag{11}$$

where $\hat{w}^* = \sum_{j=0}^{n-1} \hat{k}_j^* \Delta_j$. Using the positive definiteness of $K^{*,\sigma,0}$ (see Lemma 3.21), $\hat{w}^* = 0$ from (11), and hence $z^* = 0$, which is a contradiction. □

Appendix C

First we list up four sublemmas necessary to prove Lemmas 3.28 and 3.32. Recalling (3.69a) and (3.71), we can write $\mathbf{G}_N = -\dfrac{\sigma}{W(h,k)}G$, where G is a real symmetric matrix defined by $G = \{h_i k_j\}_{i,j=1}^n$ with $h_i = h(x_i^*(\sigma))$ and $k_j = k(x_j^*(\sigma))$. It suffices to consider the inverse of G for the proof.

Sublemma 1. *Let G be the matrix $\{h_i k_j\}_{i,j=1}^n$ defined as above and let ΔG_{ij} be the (i, j)-cofactor of G. Then we have*

$$\det G = (-1)^n A_{12} \cdot A_{23} \cdots A_{n-1,n} \cdot h_1 k_n \tag{a}$$

$$\Delta G_{ii} = (-1)^{n-2} A_{12} \cdots A_{i-1,i+1} \cdot A_{i+1,i+2} \cdots A_{n-1,n} \cdot h_1 k_n, \qquad i \neq 1, n \tag{b}$$

$$\Delta G_{11} = (-1)^{n-2} A_{23} \cdots A_{n-1,n} \cdot h_2 k_n \tag{c}$$

$$\Delta G_{nn} = (-1)^{n-2} A_{12} \cdots A_{n-2,n-1} \cdot h_1 k_{n-1} \tag{d}$$

$$\Delta G_{i+1,i} = (-1)^{n-1} A_{12} \cdots \check{A}_{i,i+1} \cdots A_{n-1,n} \cdot h_1 k_n, \qquad 1 \leq i \leq n-1 \tag{e}$$

$$\Delta G_{ij} = 0 \qquad (i, j) \notin \textit{tri-diagonal}, \tag{f}$$

where

$$A_{ij} \equiv \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix}$$

and ˇ *means that the right-hand side of (e) lacks its term.*

Sublemma 2.

$$A_{ij} = \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix} < 0, \qquad (i < j) \tag{a}$$

$$A_{ij} = A_{i+2,j+2} \tag{b}$$

$$A_{13} = A_{24} \tag{c}$$

Sublemma 3.

$$\frac{W(h,k)}{A_{12}} = \frac{W(h^d,k^d)}{A_{12}^d} \tag{a}$$

$$\frac{W(h,k)}{A_{23}} = \frac{W(h^d,k^d)}{A_{23}^d} \tag{b}$$

$$\frac{W(h,k)}{A_{13}} = \frac{W(h^d,k^d)}{A_{13}^d} \tag{c}$$

Sublemma 4.

$$\frac{h_2}{h_1} < \frac{h_2^d}{h_1^d} \tag{a}$$

$$\frac{k_{n-1}}{k_n} < \frac{k_{n-1}^d}{k_n^d} \tag{b}$$

Proof of Lemma 3.28.
The existence of $\mathbf{G}_N^{-1}$ is clear from Sublemma 1(a) and Sublemma 2(a). Since $\mathbf{G}_N$ is real symmetric, so is $\mathbf{G}_N^{-1}$. Tri-diagonality is a direct consequence of Sublemma 1(f). Using Sublemma 1, it is clear that $\mathbf{G}_N^{-1}$ takes the form (3.86) or (3.87). Properties (a)~(c) are the direct consequences of Sublemma 2. □

Proof of Lemma 3.32.
Sublemmas 3 and 4 imply Lemma 3.32. □

In what follows we shall prove Sublemmas 1 - 4.

Proof of Sublemma 1.
We prove only (a) and (f). The remaining part can be shown in a similar way. Recall that G takes the form

$$G \equiv \begin{pmatrix} h_1k_1 & h_1k_2 & h_1k_3 & \cdots & h_1k_n \\ h_1k_2 & h_2k_2 & h_2k_3 & \cdots & h_2k_n \\ h_1k_3 & h_2k_3 & h_3k_3 & \cdots & h_3k_n \\ \vdots & \vdots & \vdots & \ddots & \\ h_1k_n & & & & h_nk_n \end{pmatrix}. \tag{1}$$

Note that although the original (i, j)-component of G for $i > j$ is given by h_jk_i, we rewrite it as above by using the symmetry of G. Hence, at the i-th row, the subscript of h increases up to the diagonal part and keeps the number i after that like $(1, 2, \cdots, i, i, \cdots, i)$. We shall make G into a triangular matrix to compute its determinant. Dividing each i-th row by h_i, and substracting from the i-th column the $(i+1)$-th column multiplied by h_i/h_{i+1}, we obtain an upper triangular matrix whose product of diagonal elements gives

$$\begin{aligned} \det G &= \left(k_1 - \frac{h_1k_2}{h_2}\right)\left(k_2 - \frac{h_2k_3}{h_3}\right)\cdots\left(k_{n-1} - \frac{h_{n-1}k_n}{h_n}\right)k_n \times (h_1 \cdots h_n) \\ &= (-1)^{n-1}A_{12}A_{23}\cdots A_{n-1,n} \cdot h_1 \cdot k_n, \end{aligned}$$

which shows (a).

As for (f), it suffices to consider the case $i < j$. Let G_{ij} denote the submatrix of G which lacks i-th row and j-th column. We decompose G_{ij} into four block matrices as

$$G_{ij} = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix},$$

where B_1 is a square matrix of size $j-1$. Dividing i-th row by h_i $(1 \le i \le j-1)$, we easily see that B_3 can be reduced to zero matrix by fundamental transformations. Moreover the last two row vectors of B_1 are linearly dependent, since they are of the form

$$j-2 : \frac{k_{j-1}}{h_{j-1}}(h_1, h_2, \cdots, h_{j-1})$$

$$j-1:\frac{k_j}{h_j}(h_1,h_2,\cdots,h_{j-1}).$$

Combining these two results, we can conclude (f). □

In order to prove Sublemmas 2-4, we need to prepare the solution of (3.69b) which takes into account the periodic structure of the potential term $\det^{*,\sigma}/f_u^{*,\sigma}$.

First let us define the fundamental solutions on the unit periodic interval $(0,\omega)$, where $\omega=2/n$ (see Fig. A.1). Let $Y_+(x)$ and $Y_-(x)$ denote the solutions of (3.69b) on $(0,\omega)$ satisfying

$$Y_+(0)=1=Y_-(\omega)$$
$$Y'_+(0)=1=Y'_-(\omega), \tag{2}$$

where $'$ denotes d/dx. It is clear that $Y_+(x)$ (resp. $Y_-(x)$) is strictly monotone increasing (resp. decreasing) and satisfies

$$Y_+(\omega-x)=Y_-(x)$$
$$Y'_+(\omega)=-Y'_-(0). \tag{3}$$

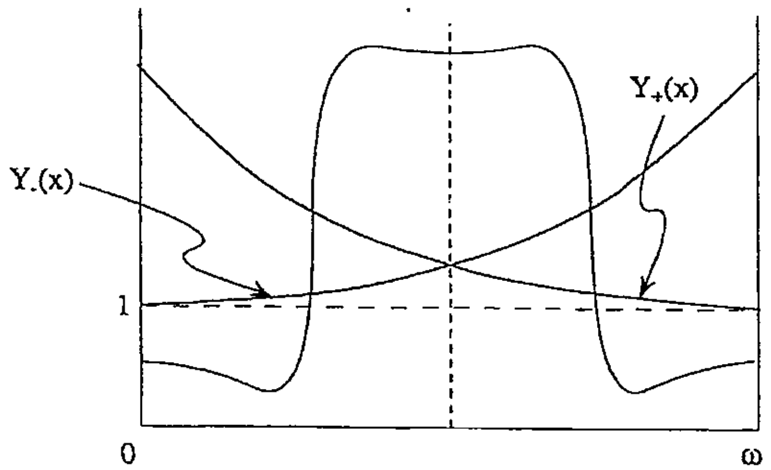

Figure A.1.

Lemma A.1. *There exist two linearly independent solution $F_\pm(x)$ of (3.69b) such that*

$$F_\pm(x)=e^{\pm\rho x}p_\pm(x), \tag{4}$$

where ρ is a real positive constant defined by

$$e^{\pm\rho\omega}=Y_+(\omega)\pm\left(Y_+(\omega)^2-1\right)^{\frac{1}{2}}, \tag{5}$$

and $p_\pm(x)$ are ω-periodic C^1-functions satisfying

$$p_+(0)=1=p_-(0),\qquad p_+(\tfrac{\omega}{2})=p_-(\tfrac{\omega}{2}) \tag{6}$$

$$p'_+(0)+p'_-(0)=0, \tag{7}$$

$$p'_+(\frac{\omega}{2}) + p'_-(\frac{\omega}{2}) = 0, \tag{8}$$

$$p_+(x_1^*)p_-(x_1^*) = p_+(x_2^*)p_-(x_2^*). \tag{9}$$

As a direct consequence of this lemma, we have

Corollary A.2.

$$F_\pm(0) = 1, \qquad F'_\pm(0) = \pm\rho + p'_\pm(0) \tag{i}$$

$$F'_+(0) + F'_-(0) = 0$$

$$h(x) = c^1_+ F_+(x) + c^1_- F_-(x) \tag{ii}$$

$$k(x) = c^2_+ F_+(x) + c^2_- F_-(x),$$

where

$$\mathbf{c}^1 = \begin{pmatrix} c^1_+ \\ c^1_- \end{pmatrix} = \frac{1}{W_F}\begin{pmatrix} F'_-(0) \\ -F'_+(0) \end{pmatrix},$$

$$\mathbf{c}^2 = \begin{pmatrix} c^2_+ \\ c^2_- \end{pmatrix} = \frac{1}{W_F}\begin{pmatrix} F'_-(1) \\ -F'_+(1) \end{pmatrix}.$$

Here W_F denotes the Wronskian of F_+ and F_-.

$$h^d(x) = d^1_+ F_+(x) + d^1_- F_-(x) \tag{iii}$$

$$k^d(x) = d^2_+ F_+(x) + d^2_- F_-(x),$$

where

$$\mathbf{d}^1 = \begin{pmatrix} d^1_+ \\ d^1_- \end{pmatrix} = \frac{1}{W_F}\begin{pmatrix} -F_-(0) \\ F_+(0) \end{pmatrix},$$

$$\mathbf{d}^2 = \begin{pmatrix} d^2_+ \\ d^2_- \end{pmatrix} = \frac{1}{W_F}\begin{pmatrix} -F_-(1) \\ F_+(1) \end{pmatrix}.$$

Proof of Lemma A.1.
We seek the solution of Floquet form (4) by using Y_+ and Y_-:

$$F(x) = c_+Y_+ + c_-Y_-. \tag{10}$$

We determine the coefficients $c_\pm$ so that, after one period ω, F and F' are multiplied by some constant γ, namely

$$\begin{aligned} c_+Y_+(\omega)+c_-Y_-(\omega) &= \gamma(c_+Y_+(0)+c_-Y_-(0)) \\ c_+Y'_+(\omega)+c_-Y'_-(\omega) &= \gamma(c_+Y'_+(0)+c_-Y'_-(0)). \end{aligned} \tag{11}$$

Using (2) and (3), (11) becomes

$$\begin{pmatrix} Y_+(\omega)-\gamma & 1-\gamma Y_+(\omega) \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

Apparently (12) has nontrivial solutions if and only if γ is a root of

$$\gamma^2-2Y_+(\omega)\gamma+1=0. \tag{13}$$

Since $Y_+(\omega)>1$, (13) has two real positive distinct roots $\gamma_\pm$ such that

$$\gamma_\pm = Y_+(\omega)\pm\left(Y_+(\omega)^2-1\right)^{\frac{1}{2}}, \quad 0<\gamma_-<1<\gamma_+.$$

We define ρ by $\rho\equiv(\log\gamma_+)/\omega$, then we have

$$e^{\pm\rho\omega}=\gamma_\pm, \qquad e^{\rho\omega}+e^{-\rho\omega}=2Y_+(\omega). \tag{14}$$

Computing the eigenvectors $(c_+, c_-)^t$ associated with $\gamma_\pm$, the resulting solutions of the form (10) become

$$\begin{aligned} F_\pm(x) &\equiv \left(Y_+^2(\omega)-1\right)^{-\frac{1}{2}}\{\pm e^{\pm\rho\omega}Y_+(x)\mp Y_-(x)\} \\ &= e^{\pm\rho x}p_\pm(x), \end{aligned} \tag{15a}$$

where $p_\pm(x)$ are defined by

$$p_\pm(x)\equiv\left(Y_+^2(\omega)-1\right)^{-\frac{1}{2}}\{\pm e^{\pm\rho(\omega-x)}Y_+(x)\mp e^{\mp\rho x}Y_-(x)\}. \tag{15b}$$

It is easy to verify that $p_\pm(x)$ are ω-periodic functions of C^1-class. The properties (6) $\sim$ (9) are the direct consequences of the expression (15b) and the reflectional symmetry of Y_+ and Y_- at $x=\omega/2$, so the details are left to the reader. □

Proof of Sublemma 2.
(a) Since $0<h_i<h_j$ and $k_i>k_j>0$, it is clear that $A_{ij}=h_ik_j-h_jk_i<0$.

Properties (b) and (c) can be verified in a similar way with the aid of Lemma A.1 and Corollary A.2, so we prove only (b) in what follows.

(b) It follows from Corollary A.2 that h_i and k_j have expressions as

$$h_i={}^t\mathbf{c}^1\cdot\mathbf{F}_i, \qquad k_j={}^t\mathbf{c}^2\cdot\mathbf{F}_j,$$

where $\mathbf{c}^i = {}^t(c^i_+, c^i_-)$ and $\mathbf{F}_i = {}^t(F_+(x^*_i), F_-(x^*_i))$. We rewrite $A_{i+2,j+2}$ as

$$A_{i+2,j+2} = \begin{vmatrix} h_{i+2} & k_{i+2} \\ h_{j+2} & k_{j+2} \end{vmatrix} = \begin{vmatrix} {}^t\mathbf{c}^1 \cdot \mathbf{F}_{i+2} & {}^t\mathbf{c}^2 \cdot \mathbf{F}_{i+2} \\ {}^t\mathbf{c}^1 \cdot \mathbf{F}_{j+2} & {}^t\mathbf{c}^2 \cdot \mathbf{F}_{j+2} \end{vmatrix}. \tag{16}$$

On the other hand, we see from Lemma A.1 that

$$\mathbf{F}_{i+2} = T\mathbf{F}_i, \tag{17}$$

where $T = \begin{pmatrix} e^{\rho\omega} & 0 \\ 0 & e^{-\rho\omega} \end{pmatrix}$. Note that $|T| = 1$. Substituting (17) into (16), we have

$$\begin{aligned} A_{i+2,j+2} &= \begin{vmatrix} {}^t\mathbf{c}^1 T\mathbf{F}_i & {}^t\mathbf{c}^2 T\mathbf{F}_i \\ {}^t\mathbf{c}^1 T\mathbf{F}_j & {}^t\mathbf{c}^2 T\mathbf{F}_j \end{vmatrix} \\ &= \left| \begin{pmatrix} {}^t\mathbf{c}^1 \\ {}^t\mathbf{c}^2 \end{pmatrix} T(\mathbf{F}_i\mathbf{F}_j) \right| \\ &= \begin{vmatrix} {}^t\mathbf{c}^1 \cdot \mathbf{F}_i & {}^t\mathbf{c}^1 \cdot \mathbf{F}_j \\ {}^t\mathbf{c}^2 \cdot \mathbf{F}_i & {}^t\mathbf{c}^2 \cdot \mathbf{F}_j \end{vmatrix} \\ &= A_{ij}, \end{aligned}$$

which completes the proof. □

Proof of Sublemma 3.

We shall prove only (*a*), since the remaining ones can be shown in a similar way. First Wronskians are given by

$$\begin{aligned} W(h,k) &= \begin{vmatrix} h(0) & k(0) \\ h'(0) & k'(0) \end{vmatrix} = \begin{vmatrix} 1 & k(0) \\ 0 & k'(0) \end{vmatrix} = k'(0), \\ W(h^d,k^d) &= \begin{vmatrix} h^d(0) & k^d(0) \\ (h^d)'(0) & (k^d)'(0) \end{vmatrix} = \begin{vmatrix} 0 & k^d(0) \\ 1 & (k^d)'(0) \end{vmatrix} = -k^d(0). \end{aligned} \tag{18}$$

In view of Corollary A.2, we see that

$$\begin{aligned} k'(0) &= F'_+(0)(c^2_+ - c^2_-) = (\rho + p'_+(0))(c^2_+ - c^2_-) \\ -k^d(0) &= -(d^2_+ + d^2_-). \end{aligned} \tag{19}$$

Using Corollary A.2, (18) and (19), we see that

$$
\begin{aligned}
I^N &\equiv \frac{A_{12}}{W(h,k)} = \frac{1}{F'_+(0)(c_+^2 - c_-^2)} \Big[\{c_+^1 F_+(x_1^*) + c_-^1 F_-(x_1^*)\} \\
&\times \{c_+^2 F_+(x_2^*) + c_-^2 F_-(x_2^*)\} - \{c_+^1 F_+(x_2^*) + c_-^1 F_-(x_2^*)\} \\
&\times \{c_+^2 F_+(x_1^*) + c_-^2 F_-(x_1^*)\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
I^D &\equiv \frac{A_{12}^d}{W(h^d,k^d)} = -\frac{1}{d_+^2 + d_-^2} \Big[\{d_+^1 F_+(x_1^*) + d_-^1 F_-(x_1^*)\} \\
&\times \{d_+^2 F_+(x_2^*) + d_-^2 F_-(x_2^*)\} - \{d_+^1 F_+(x_2^*) + d_-^1 F_-(x_2^*)\} \\
&\times \{d_+^2 F_+(x_1^*) + d_-^2 F_-(x_1^*)\}\Big] .
\end{aligned}
$$

Substituting the expression of Corollary A.2 into the I^N and I^D, we obtain

$$
\begin{aligned}
I^N &= \frac{1}{(F'_-(1) + F'_+(1))W_F} \Big[-\{F_+(x_1^*) + F_-(x_1^*)\} \\
&\times \{F'_-(1)F_+(x_2^*) - F'_+(1)F_-(x_2^*)\} + \{F_+(x_2^*) + F_-(x_2^*)\} \\
&\times \{F'_-(1)F_+(x_1^*) - F'_+(1)F_-(x_1^*)\}\Big] ,
\end{aligned}
\tag{20a}
$$

$$
\begin{aligned}
I^D &= \frac{1}{(F_+(1) - F_-(1))W_F} \Big[\{-F_+(x_1^*) + F_-(x_1^*)\} \\
&\times \{-F_-(1)F_+(x_2^*) + F_+(1)F_-(x_2^*)\} - \{-F_+(x_2^*) + F_-(x_2^*)\} \\
&\times \{-F_-(1)F_+(x_1^*) + F_+(1)F_-(x_1^*)\}\Big] .
\end{aligned}
\tag{20b}
$$

Next we shall express $F_\pm(1)$ and $F'_\pm(1)$ in terms of $p_+(1)$ and $p'_+(1)$. Since $\omega n/2 = 1$, we see from (4) that

$$
\begin{aligned}
F_\pm(1) &= e^{\pm\rho\omega n/2} p_\pm(1) \\
F'_\pm(1) &= \pm\rho e^{\pm\rho\omega n/2} p_\pm(1) + e^{\pm\rho\omega n/2} p'_\pm(1).
\end{aligned}
\tag{21}
$$

Recalling the ω -periodicity of $p_\pm(x)$, we have from (6) $\sim$ (8) that

$$
p_+(1) = p_-(1) \quad \text{and} \quad p'_+(1) + p'_-(1) = 0. \tag{22}
$$

Substituting (22) into (21), we have

$$
\begin{aligned}
F_\pm(1) &= e^{\pm\rho\omega n/2} p_+(1) \\
F'_\pm(1) &= \pm(\rho p_+(1) + p'_+(1)) e^{\pm\rho\omega n/2}.
\end{aligned}
\tag{23}
$$

Inserting (23) into (20), we obtain

$$I^N = \frac{1}{EW_F}\Big[\{F_+(x_1^*)+F_-(x_1^*)\}\{e^{-\rho\omega n/2}F_+(x_2^*)+e^{\rho\omega n/2}F_-(x_2^*)\}$$
$$-\{F_+(x_2^*)+F_-(x_2^*)\}\{e^{-\rho\omega n/2}F_+(x_1^*)+e^{\rho\omega n/2}F_-(x_1^*)\}\Big],$$

$$I^D = \frac{1}{EW_F}\Big[\{F_+(x_1^*)-F_-(x_1^*)\}\{-e^{-\rho\omega n/2}F_+(x_2^*)+e^{\rho\omega n/2}F_-(x_2^*)\}$$
$$-\{F_+(x_2^*)-F_-(x_2^*)\}\{-e^{-\rho\omega n/2}F_+(x_1^*)+e^{\rho\omega n/2}F_-(x_1^*)\}\Big],$$

where $E \equiv e^{\rho\omega n/2} - e^{-\rho\omega n/2}$. Expanding these expressions, we easily see that both I^N and I^D are equal to $\{F_+(x_1^*)F_-(x_2^*) - F_-(x_1^*)F_+(x_2^*)\}/W_F$, which completes the proof. □

Proof of Sublemma 4.

We shall prove only (a). First noting that $h(x)$ is equal to the fundamental solution $Y_+(x)$ on $(0,\omega)$ (see (2)), we can write

$$\frac{h_2}{h_1} = \frac{h(x_2^*)}{h(x_1^*)} = \frac{Y_+(x_2^*)}{Y_+(x_1^*)}. \tag{24}$$

We set for simplicity

$$\alpha \equiv Y_+(x_1^*), \qquad \beta \equiv Y_+(x_2^*), \qquad \Omega \equiv Y_+(\omega). \tag{25}$$

Since Y_+ is strictly monotone increasing, it is obvious that

$$0 < \alpha < \beta < \Omega.$$

By a simple computation, we easily see that $h^d(x)$ can be written as

$$h^d(x) = \frac{1}{Y_-'(0)}\{-Y_-(0)Y_+(x) + Y_-(x)\}. \tag{26}$$

Using this expression, we have

$$\frac{h_2^d}{h_1^d} = \frac{h^d(x_2^*)}{h^d(x_1^*)} = \frac{-Y_-(0)Y_+(x_2^*) + Y_-(x_2^*)}{-Y_-(0)Y_+(x_1^*) + Y_-(x_1^*)}. \tag{27}$$

Because of symmetry of Y_+ and Y_- (see Figure A.1), we have $Y_-(x_1^*) = \beta$, $Y_-(x_2^*) = \alpha$, and $Y_-(0) = \Omega$. Hence (27) becomes

$$\frac{h_2^d}{h_1^d} = \frac{\alpha - \Omega\beta}{\beta - \Omega\alpha}.$$

On the other hand, it is clear from (24) and (25) that $h_2/h_1 = \beta/\alpha$. Thus we have

$$\frac{h_2^d}{h_1^d} - \frac{h_2}{h_1} = \frac{\alpha^2 - \beta^2}{\alpha(\beta - \alpha\Omega)}.$$

Noting that $\beta - \alpha\Omega = Y_-'(0)h^d(x_1^*) < 0$ (see (26)), this implies the conclusion. □

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