ASYMPTOTIC CONFIGURATION OF
STATIONARY INTERFACIAL PATTERNS FOR REACTION
DIFFUSION SYSTEMS

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Abstract

Asymptotic configuration of interfacial patterns of reaction diffusion systems is considered when the interfacial thickness tends to zero. Under several hypotheses derived by formal asymptotic analysis, it is shown that there are no smooth limiting configuration of interface for generic domains. This partially explains the fact that interfacial patterns become fine and complicated as in the micro-phase separation of block copolymer in this limit.

1. Introduction

Morphology of final patterns in phase transition are usually simple ones: only one phase dominates the whole domain (non-conserved) or it is decomposed into simple subdomains (conserved) after coarsening process. This is due to the tendency to minimize the area of interface. However, if there is a microscopic constraint to the system, the final pattern becomes much richer and has in general a variety of morphologies from lamellar to labyrinthine patterns. Block copolymer is one of such materials where two monomers (say, A and B) are connected at some point (constraint), and this is responsible for the formation of very fine and complicated structures depending on the ratio of composite monomers in the process of micro-phase separation [3][2][13]. Locally each monomer moves in a random way and tends to segregate each other (bistability), however connectivity does not allow them to form a large domain consisting of only one monomer (nonlocality). Ohta and Kawasaki [9] proposed the following model system to describe such a phenomenon.

\[
\begin{aligned}
&u_t = \varepsilon^2 \Delta u + f(u,v) & \text{in } \Omega, \\
&0 = D \Delta v + u & \\
&\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} & \text{on } \partial \Omega.
\end{aligned}
\]

(1.1)

where \(u\) is the order parameter indicating A-rich or B-rich phase, \(v\) represents the nonlocal effect due to connectivity, \(\varepsilon (\ll 1)\) corresponds to the interfacial thickness and \(D (\gg 1)\) is proportional to the square of the polymerization index.

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(namely, the length of block copolymer) which is usually quite large, and \( f(u, v) \) is a cubic nonlinearity (typically of the form \( u - u^3 - v \)). It is anticipated that many other phenomena could be described by similar models to (1.1), since the basic mechanism creating a variety of patterns is due to the competition between local dynamics and nonlocal effect. In fact similar patterns are observed in liquid crystal, magnetic thin film, and so on. The arguments in this note is valid to slightly more general system:

\[
\begin{align*}
    u_t &= \varepsilon^2 \Delta u + f(u, v) & \text{in } \Omega, \\
    \delta v_t &= D \Delta v + g(u, v) \\
    \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} & \text{on } \partial \Omega.
\end{align*}
\]

(1.2)

where \( \delta \) is a nonnegative constant. Although the precise assumptions for \((f, g)\) are delegated to [7], they are qualitatively the same as (1.1). A naive approach to find nontrivial patterns of (1.2) is to consider the limiting case either \( \varepsilon \downarrow 0 \) or \( D \uparrow \infty \). For the latter case it is known (see [5][4]) that the resulting equations become a scalar equation with a constraint of integral type and it is unlikely to have a stable complicated pattern for such a system, since there are no stable multi-layered solutions even in 1D case [6]. On the other hand we know very little about the former case in higher space dimensions, since it has been regarded to be extremely difficult to find the first approximate stationary solutions in the limit of \( \varepsilon \downarrow 0 \). Especially we are interested in the behavior of the asymptotic configuration of the interface \( \Gamma^\varepsilon \) (see (2.1)). The aim of this note is to answer (at least partially) the following question.

Does (1.2) has an \( \varepsilon \)-family of stationary layered solutions with smooth interface \( \Gamma^\varepsilon \) up to \( \varepsilon = 0 \)?

The answer is obviously affirmative, since we know planar and spherical layered solutions (see [12][10]). However those domains have very special geometries, i.e., rectangles and spheres and it is not a priori clear that such smooth interfaces persist up to \( \varepsilon = 0 \) for generic domains. It turns out that the answer is negative for generic ones under several hypotheses derived by the formal asymptotic analysis. Here we only consider the case where \( \Gamma^\varepsilon \) is a simple closed curve inside of \( \Omega \). \((U^\varepsilon, V^\varepsilon)\) is called an \( \varepsilon \)-family of matched asymptotic solutions to (1.2), if it has an matched asymptotic expansion mentioned in Section 2.

**Main Theorem**

(a) (Disk Symmetry) Suppose that (1.2) with \( n = 2 \) has an \( \varepsilon \)-family of matched asymptotic solutions with simple closed smooth interfaces \( \Gamma^\varepsilon \) (see (2.1)) up to \( \varepsilon = 0 \) and that the matching condition \((MC)\) holds (see Lemmas 3.4.1 and 3.4.2), then \( \Gamma^0 \) must be a circle.

(b) (Non-existence) Moreover under Hypothesis 4.1 in Section 4, (a) implies that
the reduced problem (2.26) has no solutions for generic domains \( \Omega \), and hence there does not exist associated \( \varepsilon \)-family of matched asymptotic solutions.

Remark 1.1 The hypothesis in Main Theorem (a) can be weakened, in fact, it suffices to assume the existence of principal orders of outer and inner expansions. The extension to \( n \geq 3 \)-case can be done without difficulty. More precisely, see the forthcoming paper [8].

The above non-existence result is not a disappointing result and, in fact, it suggests an important thing about the behavior of the interface as \( \varepsilon \to 0 \). Namely, if some stationary pattern of (1.2) exists up to \( \varepsilon = 0 \), but does not have a smooth limiting interface, then the configuration of the interface must become fine and complicated as \( \varepsilon \to 0 \). In order to understand the morphology of the complicated patterns, it seems necessary to apply an appropriate rescaling to blow up the degenerate situation since there are no well-defined asymptotic limit of interfaces in the original framework. The study in this direction is now currently being done and will be reported elsewhere. The outline of this note is as follows. In Section 2 we shall display the expansion of \( \varepsilon \)-family of matched asymptotic solutions. In section 3 we shall derive \( (v_0)_y = \text{constant} \) along the interface, which is a byproduct of the analysis of the formal matched asymptotic expansions of the eigenvalue problem of Allen-Cahn operator associated with the original system (1.2). Based on these observations we shall prove the Main Theorem in Section 4. The key ingredient is the Serrin's result [11] (and its generalizations) for the overdetermined Poisson equation.

2. Matched asymptotic expansion of singularly perturbed stationary solutions

In this section, we summarize the hypothesis for the stationary solutions which have interior transition layers, especially, a precise definition of matched asymptotic expansion of them is presented. We assume that there exist an \( \varepsilon \)-family of smooth stationary solutions \((U^\varepsilon(x), V^\varepsilon(x))\) to (1.2) with interior transition layers such that the interface \( \Gamma^\varepsilon \):

\[
\Gamma^\varepsilon \equiv \{x \in \Omega | U^\varepsilon(x) = \frac{1}{2}(h_+(v^\ast) + h_-(v^\ast))\}
\]

is a smooth simple closed curve in \( \mathbb{R}^2 \) and have a definite limit \( \Gamma_0 \) with same properties as \( \varepsilon \to 0 \), where \( u_\pm \) are two stable branches of \( f(u, v) = 0 \) (see [7]). Let \( \Omega_0^+ \) be the region surrounded by \( \Gamma_0 \) and \( \Omega_0^- \equiv \Omega \setminus \bar{\Omega}_0^+ \). In some tubular neighbourhood \( T(\Gamma_0) \) of \( \Gamma_0 \), local coordinate system \((s, y)\) is defined and for \( x \in T(\Gamma_0) \)

\[
x = \Gamma_0(s(x)) + y(x)v(s(x))
\]

holds, where \( s(x) \) is the parameter measuring the arclength along \( \Gamma_0 \) to the point on \( \Gamma_0 \) closest to \( x \), \( v(s) \) is outward unit normal vector at \( s \), and \( y(x) \) is signed
distance from \( x \) to \( \Gamma_0 \) that is positive if \( x \in \Omega_0^- \). We use the notation \( u(s, y) \) for the representation of \( u(x) \) by the local coordinate system. Using this local coordinate system, \( \Gamma^\varepsilon \) can be expanded into

\[
(2.3) \quad \Gamma^\varepsilon = \Gamma_0(s) + \gamma(s, \varepsilon) \nu(s), \quad \gamma(s, \varepsilon) = \sum_{k=1}^{\infty} \gamma_k(s) \varepsilon^k + \varepsilon^m \gamma_{m+1}(s, \varepsilon).
\]

\((U^\varepsilon(x), V^\varepsilon(x))\) is called an \( \varepsilon \)-family of matched asymptotic solutions when it has the following expansion (2.4) (matched asymptotic expansion (MAE) procedure). Roughly speaking, \((U^\varepsilon(x), V^\varepsilon(x))\) is expanded separately in two regions \( \Omega_\pm^\varepsilon \) and they are matched smoothly at \( \Gamma^\varepsilon \). More precisely we have

\[
(2.4) \quad \begin{cases}
U^\pm_\varepsilon(x) = \sum_{k=0}^{\infty} u_k^\pm(x, \varepsilon) \varepsilon^k + \Phi_\varepsilon^\pm(x, \varepsilon) + \varepsilon^m R^\pm(x, \varepsilon) & x \in \Omega_\pm^\varepsilon \\
V_\varepsilon^\pm(x) = \sum_{k=0}^{\infty} v_k^\pm(x, \varepsilon) \varepsilon^k + \varepsilon^2 \Psi_\varepsilon^\pm(x, \varepsilon) + \varepsilon^m S^\pm(x, \varepsilon)
\end{cases}
\]

where

\[
(2.5) \quad \Phi_\varepsilon^\pm(x, \varepsilon) = \begin{cases}
\omega\left(\frac{y(x) - \gamma(s, \varepsilon)}{d}\right) \sum_{n=0}^{\infty} \phi_n^\pm(s(x), \frac{y(x) - \gamma(s, \varepsilon)}{\varepsilon}) \varepsilon^n & |y(x) - \gamma(s, \varepsilon)| \leq d, \\
0, & |y(x) - \gamma(s, \varepsilon)| > d,
\end{cases}
\]

\[
(2.6) \quad \Psi_\varepsilon^\pm(x, \varepsilon) = \begin{cases}
\omega\left(\frac{y(x) - \gamma(s, \varepsilon)}{d}\right) \sum_{k=0}^{n-2} \psi_k^\pm(s(x), \frac{y(x) - \gamma(s, \varepsilon)}{\varepsilon}) \varepsilon^k & |y(x) - \gamma(s, \varepsilon)| \leq d, \\
0, & |y(x) - \gamma(s, \varepsilon)| > d,
\end{cases}
\]

\(\omega(\tau) \in C^\infty(\mathbb{R})\) is a cut off function such that

\[
(2.7) \quad \omega(\tau) = 1 \text{ for } |\tau| \leq \frac{1}{2}, \omega(\tau) = 0 \text{ for } |\tau| \geq 1, 0 \leq \omega(\tau) \leq 1.
\]

\(d > 0\) is some small constant, and \(R(x, \varepsilon)\) and \(S(x, \varepsilon)\) are remainders. \(\phi_k^\pm\) and \(\psi_k^\pm\) are functions of \( s \) and \( \xi \), and \( \xi \) is stretched variable \( \xi \equiv (y - \gamma(s, \varepsilon))/\varepsilon \). The coefficients \( u_k^\pm, v_k^\pm, \phi_k^\pm, \) and \( \psi_k^\pm\) satisfy some equations and relations. We can obtain these equations by making outer and inner expansions and equating the same powers of \( \varepsilon^k \). The resulting relations are, what we call, matching conditions between inner and outer expansions and \( C^1 \)-matching conditions between
\((U^\pm, V^\pm)\) and \((U^\pm, V^\pm)\). Let \(\beta_\epsilon(s) = v^* + \sum_{k=1}^{m} \beta_k(s) \epsilon^k + \epsilon^m \beta_m(s, \epsilon)\) be the value of \(V^*(x)\) on \(\Gamma^\epsilon\), \(\Omega^\pm_\epsilon\) be the region surrounded by \(\Gamma^\epsilon\), and \(\Omega^\pm_\epsilon = \Omega \setminus \bar{\Omega}^\pm_\epsilon\).

We display the equations and the relations up to order \(O(\epsilon)\). Higher order terms are given by the same procedures.

\[O(\epsilon^0):\]
\[
\begin{aligned}
&\begin{cases}
  u^\pm_0 = h^\pm(v_0^\pm) \\
  D\Delta v^\pm_0 + g(h^\pm(v_0^\pm), v^\pm_0) = 0
\end{cases}
\quad x \in \Omega^\pm_0 \\
&v^\pm_0(s, 0) = v^*, \quad \frac{\partial v^\pm_0}{\partial n} = 0 \text{ on } \partial \Omega,
\end{aligned}
\]

\[O(\epsilon^1):\]
\[
\begin{aligned}
&\begin{cases}
  f^0_u u^\pm_1 + f^0_v \dot{v}^\pm_1 = 0 \\
  D\Delta \dot{v}^\pm_1 + g^{0_u} u^\pm_1 + g^{0_v} \dot{v}^\pm_1 = 0
\end{cases}
\quad x \in \Omega^\pm_0 \\
v^\pm_1(s, 0) = \beta_1(s) - (v_0^\pm)_y(s, 0) \gamma_1(s), \quad \frac{\partial v^\pm_1}{\partial n} = 0 \text{ on } \partial \Omega,
\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  \ddot{\phi}_0^\pm + f(h^\pm(v^*), v^*) = 0 \\
  D\ddot{\psi}_0^\pm = g(h^\pm(v^*), v^*) - g(h^\pm(v^*) + \phi_0^\pm, v^*)
\end{cases}
\quad \xi \in I^\pm, \quad 0 \leq s \leq \ell \\
&(v_0^\pm)_y(s, 0) \gamma_1(s) + (v^\pm_1)_y(s, 0) + \dot{\psi}_0^\pm(s, 0) = \\
&\quad (v_0^\pm)_y(s, 0) \gamma_1(s) + (v^\pm_1)_y(s, 0) + \dot{\psi}_0(s, 0)
\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  \ddot{\phi}_1^\pm + f^0 u^\pm \phi_1^\pm = -D_1 \phi_0^\pm - \bar{F}^1\pm \\
  D\ddot{\psi}_1^\pm = -DD_1 \phi_0^\pm - E^1\pm(s, \xi) - \bar{G}^1\pm
\end{cases}
\quad \xi \in I^\pm, \quad 0 \leq s \leq \ell
\end{aligned}
\]
\[ \phi_1^\pm(s, 0) = -(u_0^\pm)_y(s, 0) \gamma_1(s) - u_1^\pm(s, 0), \quad \phi_1^+(s, \pm\infty) = 0, \]
\[ \psi_1^\pm(s, \pm\infty) = 0 = \dot{u}_1^\pm(s, \pm\infty) \]

\[ (u_0^+)_y(s, 0) + \dot{\phi}_1^+(s, 0) = (u_0^-)_y(s, 0) + \dot{\phi}_1^-(s, 0), \]

where \( \cdot = \frac{\partial}{\partial \xi}, \quad I^+ \equiv (-\infty, 0), \quad I^- \equiv (0, \infty), \quad \ell \) is the total arclength of \( \Gamma_0 \),
\[ f_0^\pm \equiv \frac{\partial}{\partial s} f(u_0^\pm, v_0^\pm), \quad f_0^\pm \equiv \frac{\partial}{\partial s} f(h_0^\pm, \phi_0^\pm, v^\pm), \quad \text{and} \quad D_k \ (k = 0, 1, \cdots, m) \] are the coefficients of the expansion of Laplacian \( \Delta \) in the local coordinate system \((s, \xi)\);

\[ \Delta_x \equiv \frac{1}{\varepsilon^2} \sum_{k=0}^m \varepsilon^k D_k. \]

Here \( D_k \) is at most second order differential operator in \( s \) and \( \xi \). For example,

\[ D_0 \equiv \frac{\partial^2}{\partial \xi^2}, \quad D_1 \equiv -\kappa(s) \frac{\partial}{\partial \xi}, \quad D_2 \equiv \Lambda_0 - \kappa(s)^2 (\xi + \gamma_1(s)) \frac{\partial^2}{\partial \xi^2}, \]

\[ \Lambda_0 \equiv \frac{\partial^2}{\partial s^2} - 2 \gamma_1 \frac{\partial}{\partial s} - \gamma_1'' \frac{\partial}{\partial \xi} + (\gamma_1')^2 \frac{\partial^2}{\partial \xi^2}, \]

where \( \gamma_1' = \frac{\partial}{\partial s} \) and \( \kappa(s) \) is curvature of \( \Gamma_0(s) \). \( \tilde{F}^{k\pm}, \tilde{G}^{k\pm}, \) and \( E^{k\pm} \) are defined by

\[ \tilde{F}^{k\pm}(s, \xi) \equiv \frac{1}{k!} \frac{d^k}{d \varepsilon^k} \left[ \left( \sum_{i=0}^m u_i^\pm(s, \varepsilon, \xi + \gamma(s, \varepsilon)) \varepsilon^i + \sum_{i=0}^m \phi_i^\pm(s, \xi) \varepsilon^i \right) \right] \bigg|_{\varepsilon=0} \]

\[ \tilde{G}^{k\pm}(s, \xi) \equiv \frac{1}{k!} \frac{d^k}{d \varepsilon^k} \left[ \left( \sum_{i=0}^m u_i^\pm(s, \varepsilon, \xi + \gamma(s, \varepsilon)) \varepsilon^i + \sum_{i=0}^m \phi_i^\pm(s, \xi) \varepsilon^i \right) \right] \bigg|_{\varepsilon=0} \]

\[ E^{k\pm}(s, \xi) \equiv \frac{1}{k!} \frac{d^k}{d \varepsilon^k} \left[ \left( \sum_{i=0}^m u_i^\pm(s, \varepsilon, \xi + \gamma(s, \varepsilon)) \varepsilon^i + \sum_{i=0}^m \phi_i^\pm(s, \xi) \varepsilon^i \right) \right] \bigg|_{\varepsilon=0} \]

Here we present the precise form of \( \tilde{F}^{1\pm} \) for later use in Section 3;

\[ \tilde{F}^{1\pm} \equiv \{ f_0^\pm(u_0^\pm)_y(s, 0) + f_0^\pm(v_0^\pm)_y(s, 0) \} (\gamma_1(s) + \xi) \]

\[ -f_0^\pm u_1^\pm(s, 0) - f_0^\pm v_1^\pm(s, 0). \]
We call the second equation of (2.8) with (2.9) and (2.10) the reduced problem, that is

\[
\begin{cases}
\Delta v_{0}^\pm + g(h, v_{0}^\pm) = 0, & \text{in } \Omega_{0}^\pm \\
v_{0}^\pm(s, 0) = v^*, & \frac{\partial v_{0}^-}{\partial n} = 0 \text{ on } \partial \Omega,
\end{cases}
\]

(2.26)

\[
(v_{0}^+)_{y}(s, 0) = (v_{0}^-)_{y}(s, 0)
\]

3. Asymptotic formula for critical eigenvalues of Allen-Cahn operator associated with Activator-Inhibitor systems

In this section, we consider the following eigenvalue problem

\[
\begin{cases}
L^\varepsilon w = \lambda w & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(3.1)

where

\[
L^\varepsilon \equiv \varepsilon^2 \Delta + f_u^\varepsilon,
\]

\[
f_u^\varepsilon = \frac{\partial}{\partial w} f(U^\varepsilon, V^\varepsilon), \quad \text{and } (U^\varepsilon, V^\varepsilon) \text{ is a stationary solution of (1.2) having the expansion as in Section 2. The information on the spectral behavior of (3.1) is basic for the study of the stability of } (U^\varepsilon, V^\varepsilon). \text{ Especially the behavior of critical eigenvalues (i.e., those which tend to zero as } \varepsilon \downarrow 0) \text{ play the key role to investigate the spectrum of the linearized problem for the full system. However in this short note, we only focus on the asymptotic expansions of eigen-pairs of (3.1) and their matching conditions, which gives us an important result Proposition 3.4 to prove the Main theorem.}

Our approach is the matched asymptotic expansion method. We divide (3.1) into two problems as follows;

\[
\begin{cases}
\varepsilon^2 \Delta w^- + f_u^\varepsilon w^- = \lambda^\varepsilon w^- & \text{in } \Omega_{\varepsilon}^-,
\\
\frac{\partial w^-}{\partial n} = 0 & \text{on } \partial \Omega, \quad w^- = \Theta^\varepsilon \text{ on } \Gamma_{\varepsilon}.
\end{cases}
\]

(3.3)_-

\[
\begin{cases}
\varepsilon^2 \Delta w^+ + f_u^\varepsilon w^+ = \lambda^\varepsilon w^+ & \text{in } \Omega_{\varepsilon}^+,
\\
w^+ = \Theta^\varepsilon & \text{on } \Gamma_{\varepsilon},
\end{cases}
\]

(3.3)_+

where

\[
\Theta^\varepsilon \equiv \sum_{k=0}^{m} \varepsilon^k \Theta_k(s), \quad \lambda^\varepsilon \equiv \sum_{k=1}^{m} \varepsilon^k \lambda_k.
\]

(3.4)
\( \Omega^\pm_\varepsilon \) and \( \Gamma^\varepsilon \) are defined in Section 2. \( \Theta^\varepsilon \) and \( \lambda^\varepsilon \) are determined by \( C^1 \)-matching condition of \( w^-_\varepsilon \) and \( w^+_\varepsilon \) on \( \Gamma^\varepsilon \).

**Outer expansion** Let

\[
(3.5) \quad w^\pm = \sum_{k=0}^m \varepsilon^k w^\pm_k(x) \quad \text{and} \quad f^\varepsilon_u = \sum_{k=0}^m \varepsilon^k F^\varepsilon_k \]

where

\[
(3.6) \quad F^\varepsilon_k = \frac{1}{k!} \frac{d^k}{d\varepsilon^k} f_u \left( \sum_{i=0}^m \varepsilon^i u^+_i(x), \sum_{i=0}^m \varepsilon^i v^-_i(x) \right) \bigg|_{\varepsilon=0}.
\]

Substituting (3.5) and (3.6) into (3.3)\(_\pm\), we have

\[
\sum_{k=2}^m \varepsilon^k \Delta w^\pm_{k-2} + \sum_{k=0}^m \left( \varepsilon^k \sum_{i+j=k} F^\varepsilon_i \varepsilon^\pm_j \right) = \sum_{k=1}^m \left( \varepsilon^k \sum_{i+j=k, i \geq 1} \lambda_i \varepsilon^\pm_j \right).
\]

Equating like power of \( \varepsilon^k \), we obtain the following equations:

\[
k = 0: \quad F^\varepsilon_0 = 0,
\]

\[
k = 1: \quad F^\varepsilon_1 w^+_1 + F^\varepsilon_1 w^-_1 = \lambda_1 w^\pm_0,
\]

\[
2 \leq k \leq m: \quad \Delta w^\pm_{k-2} + \sum_{i+j=k} F^\varepsilon_i \varepsilon^\pm_j = \sum_{i+j=k, i \geq 1} \lambda_i \varepsilon^\pm_j,
\]

By using induction arguments, we see \( w^\pm_k = 0 \quad (0 \leq k \leq m) \).

**Inner expansion** We introduce the stretched variable \( \xi \) in the neighbourhood of \( \Gamma^\varepsilon \) as in Section 2. Substituting

\[
\tilde{w}^\pm = \sum_{k=0}^m \varepsilon^k \tilde{w}_k^\pm(s, \xi) \quad \text{and} \quad \tilde{f}^\varepsilon_u = \sum_{k=0}^m \tilde{F}^\varepsilon_k \tilde{\varepsilon}^k
\]

into (3.3)\(_\pm\)

\[
\sum_{k=0}^m \left( \varepsilon^k \sum_{i+j=k} \tilde{D}_i \tilde{w}_j^\pm \right) + \sum_{k=0}^m \left( \varepsilon^k \sum_{i+j=k} \tilde{F}^\varepsilon_i \tilde{w}_j^\pm \right) = \sum_{k=1}^m \left( \varepsilon^k \sum_{i+j=k, i \geq 1} \lambda_i \tilde{w}_j^\pm \right),
\]

where

\[
\tilde{F}^\varepsilon_k = \frac{1}{k!} \frac{d^k}{d\varepsilon^k} f_u \left( \sum_{i=0}^m \varepsilon^i u^+_i(s, \xi + \gamma(s, \varepsilon)) + \sum_{i=0}^m \varepsilon^i \phi_i^\pm(s, \xi), \right. \left. \sum_{i=0}^m \varepsilon^i v^-_i(s, \xi) + \sum_{i=2}^m \varepsilon^i \psi_i^\pm(s, \xi) \right) \bigg|_{\varepsilon=0}.
\]
Equating like powers of $\varepsilon^k$, we have the following problems for $\zeta_k^\pm$.

\begin{equation}
(3.7) \quad \left\{
\begin{array}{l}
\zeta_0^\pm + F_u^0 \zeta_0^\pm = 0, \\
\zeta_0^\pm(s, \pm \infty) = 0, \\
\zeta_0^\pm(s, 0) = \Theta_0(s),
\end{array}
\right. \quad \xi \in I^\pm, \quad 0 \leq s \leq \ell
\end{equation}

\begin{equation}
(3.8) \quad \left\{
\begin{array}{l}
\tilde{\zeta}_k^\pm + F_u^0 \zeta_k^\pm = \sum_{i=1}^k R_i^\pm \zeta_{k-i}^\pm, \\
\zeta_k^\pm(s, \pm \infty) = 0, \\
\zeta_k^\pm(s, 0) = \Theta_k(s),
\end{array}
\right. \quad \xi \in I^\pm, \quad 0 \leq s \leq \ell
\end{equation}

where

\begin{equation}
(3.9) \quad R_k^\pm(s, \xi) \equiv \lambda_i - D_i - F_u^i.\end{equation}

By the similar argument in Fife [1], we can see that the inhomogeneous terms of (3.8) and their derivatives of any order with respect to $s$ and $\xi$ decay exponentially as $|\xi| \to \infty$. Then noting that $\phi_0^\pm (>0)$ are fundamental solutions of (3.7) and (3.8), the solutions of them are given by

\begin{equation}
(3.10) \quad \zeta_0^\pm = \frac{\phi_0(\xi)}{\phi_0(0)} \Theta_0(s)
\end{equation}

\begin{equation}
(3.11) \quad \zeta_k^\pm(s, \xi) = \frac{\phi_0(\xi)}{\phi_0(0)} \Theta_k(s) + \phi_0(\xi) \int_0^\xi (\phi_0(\tau))^{-2} \times \int_{\tau}^\xi \left( \sum_{i=1}^k \phi_0(\tau) R_i^\pm(s, \tau) \zeta_{k-i}^\pm(s, \tau) \right) d\tau dt, \quad (1 \leq k \leq m).
\end{equation}

Finally we define the formal approximation to the solution $W_m^\pm(x)$ of (3.3)$\pm$. We take a sufficiently small $\varepsilon_0 > 0$ and let $\kappa_\varepsilon(s)$ be the curvature of $l^\xi$. Define $d > 0$ by

\[ \max_{0 \leq s \leq \ell, 0 \leq \varepsilon \leq \varepsilon_0} |\kappa_\varepsilon(s)| \leq \frac{1}{2d} \]

and $W_m^\pm(x)$ on $\Omega_\varepsilon^\pm$ by

\begin{equation}
(3.12) \quad W_m^\pm(x) = \begin{cases} 
\omega \left( \frac{y(x) - \gamma(s; \varepsilon)}{d} \right) \sum_{k=0}^m \varepsilon^k \zeta_k^\pm(s, \frac{y(x) - \gamma(s; \varepsilon)}{\varepsilon}), & |y(x) - \gamma(s; \varepsilon)| \leq d, \\
0, & |y(x) - \gamma(s; \varepsilon)| > d,
\end{cases}
\end{equation}
Lemma 3.1. There exist $K > 0$ independent of $\varepsilon$, such that for all $x \in \Omega^\pm$ and sufficiently small $\varepsilon$,

$$\left| (\varepsilon^2 \Delta + f_u^e - \lambda^e) [W_m^\pm] \right| \leq K \varepsilon^{m+1}$$

holds.

In order to determine $\Theta^e$ and $\lambda^e$, $W_m^\pm$ must satisfy the $C^1$-matching conditions

$$(3.13) \quad \varepsilon(W_m^+(s, \gamma(s, \varepsilon))) - \varepsilon(W_m^-(s, \gamma(s, \varepsilon))) = O(\varepsilon^{m+1}),$$

which can be rewritten as

$$(3.14) \quad \varepsilon(W_m^+(s, \gamma(s, \varepsilon))) - \varepsilon(W_m^-(s, \gamma(s, \varepsilon))) = \sum_{k=0}^{m} \varepsilon^k \{ \hat{\zeta}_k^+(s, 0) - \hat{\zeta}_k^-(s, 0) \} + O(\varepsilon^{m+1}).$$

Noting (2.12), we see that $\hat{\zeta}_0^+(s, 0) - \hat{\zeta}_0^-(s, 0) = 0$ is already satisfied. For (3.14), we see that

Lemma 3.2. Each $C^1$-matching condition

$$(3.15) \quad \hat{\zeta}_k^+(s, 0) - \hat{\zeta}_k^-(s, 0) = 0 \quad (1 \leq k \leq m)$$

is equivalent to the formal solvability condition for (3.8):

$$(MC)_k \quad \left\langle \sum_{i=1}^{k} \hat{R}_i(s, \xi) \hat{\zeta}_{k-i}^+, \hat{\phi}_0(\xi) \right\rangle_\xi = 0,$$

where $\hat{R}_i(s, \xi)$ and $\hat{\zeta}_i(s, \xi)$ are

$$\hat{R}_i(s, \xi) = \begin{cases} R_i^+(s, \xi) & \xi \in (-\infty, 0) \\ R_i^-(s, \xi) & \xi \in [0, \infty) \end{cases}$$

and

$$\hat{\zeta}_i(s, \xi) = \begin{cases} \zeta_i^+(s, \xi) & \xi \in (-\infty, 0) \\ \zeta_i^-(s, \xi) & \xi \in [0, \infty) \end{cases}$$

respectively, and $\langle \cdot, \cdot \rangle_\xi$ denotes $L^2$-inner product with respect to $\xi$.

Proof. Differentiating (3.11) with respect to $\xi$ at $\xi = 0$, we have

$$\hat{\zeta}_k^+(0, \xi) = \frac{\hat{\phi}_0(\xi)}{\lambda_0(0)} \Theta_k(s) + \frac{1}{\lambda_0(0)} \sum_{i=1}^{k} \int_{-\infty}^{0} \hat{\phi}_0(\tau) R_i^+(s, \tau) \zeta_{k-i}^+(s, \tau) d\tau dt.$$
Then,
\[
0 = \dot{\zeta}_k^+(s,0) - \dot{\zeta}_k^-(s,0) = \frac{1}{\phi_0(0)} \left[ \int_{-\infty}^{0} \sum_{i=1}^{k} \dot{\phi}_0(\tau) R_i^+(s,\tau) \zeta_{k-i}^+(s,\tau) d\tau dt \right. \\
\left. - \int_{0}^{\infty} \sum_{i=1}^{k} \dot{\phi}_0(\tau) R_i^-(s,\tau) \zeta_{k-i}^-(s,\tau) d\tau dt \right] = \left( \sum_{i=1}^{k} \bar{R}_i(s,\xi) \zeta_{k-i}^- \phi_0(\xi) \right)_\xi.
\]

In the following of this section, we study how $\Theta^\varepsilon$ and $\lambda^\varepsilon$ are determined. When $k = 1$, we have the following result.

**Lemma 3.3.** $\lambda_1$ is determined by $(MC')_1$ and is given by

\[(3.16) \quad \lambda_1 = (v_0)_y(s,0) \int_{h^-_0(t^*)}^{h^+_0(t^*)} f_v(u,\beta^*) du / \int_{-\infty}^{+\infty} (\phi_0(t))^2 dt\]

**Proof.** $\zeta_1^\pm$ satisfies

\[
\ddot{\zeta}_1^\pm + \bar{F}_u^0 \zeta_1^\pm = \frac{\Theta_0}{\phi_0(0)} (\lambda_1 \phi_0^\pm + H^\pm), \quad H^\pm \equiv \kappa \dot{\phi}_0 - \bar{F}_u^0 \phi_0
\]

and $(MC')_1$ is rewritten as

\[
\dot{\zeta}_1^+(s,0) - \dot{\zeta}_1^-(s,0) = \frac{\Theta_0(s)}{(\phi_0(0))^2} \left[ \lambda_1 \int_{-\infty}^{+\infty} (\phi_0(t))^2 dt \right. \\
\left. + \int_{-\infty}^{0} H^+ \phi_0(t) dt + \int_{0}^{+\infty} H^- \phi_0(t) dt \right] = 0.
\]

We see that $r^\pm \equiv \dot{\phi}_1$ satisfy the next equation;

\[
r^\pm + \bar{F}_u^0 r^\pm = \Omega^\pm
\]

\[
\Omega^\pm(s,\xi) \equiv H^\pm(s,\xi) - \{(v_0^\pm)_y(s,0) \bar{F}_u^0 + (v_0^\pm)_y(s,0) \bar{F}_u^0 \}.
\]

This can be solved as

\[(3.17) \quad r^\pm(s,\xi) = \frac{r^\pm(0)}{\phi_0(0)} \phi_0^\pm(\xi) + \frac{\Theta_0(\xi)}{\phi_0(0)} \int_{0}^{\xi} (\phi_0(t))^{-2} \int_{\pm\infty}^{t} \Omega^\pm(s,\tau) \phi_0(\tau) d\tau dt.
\]

Differentiating (3.17) with respect to $\xi$ at $\xi = 0$, we have

\[(3.18) \quad \dot{r}^\pm(s,0) = \frac{r^\pm(0)}{\phi_0(0)} \phi_0^\pm(0) + \frac{1}{\phi_0(0)} \int_{\pm\infty}^{0} \Omega^\pm(s,\tau) \phi_0(\tau) d\tau.
\]
Using

\[ \int_{-\infty}^{0} \int_{u}^{0} \dot{\phi}_0(\tau)d\tau = \dot{\phi}_0(0), \quad \int_{-\infty}^{0} \int_{u}^{0} \dot{\phi}_0(\tau)d\tau = \int_{\beta_0^+}^{\beta_0^-} f_u(u, \beta_0^+)du \]

and after some computation, (3.18) can be rewritten as

\[ \int_{-\infty}^{0} H^\pm(s, \tau)\dot{\phi}_0(\tau)d\tau = \dot{\phi}_1^\pm(s, 0)\dot{\phi}_0(0) - \dot{\phi}_0(0)\{ \dot{\phi}_1^\pm(s, 0) + (v_0^\pm)_y(s, 0) \} + (v_0^\pm)_y(s, 0) \int_{\beta_0^+}^{\beta_0^-} f_u(u, \beta_0^+)du, \]

where \( \alpha = (h_+^{(\beta_0^+)} + h_-^{(\beta_0^-)})/2 \). Moreover, noting that \( \dot{\phi}_1^- (s, 0) = \dot{\phi}_1^+(s, 0), v_0^\pm \) are \( C^1 \)-matched on \( \Gamma_0 \) (see (2.10)) (so we omit the superscript \( \pm \) of \( (v_0)_y \) and (2.19), we have

\[ \dot{\zeta}_1^+(s, 0) - \dot{\zeta}_1^-(s, 0) = \frac{\Theta_0(s)}{(\phi_0(0))^2} \left( \lambda_1 \int_{-\infty}^{+\infty} (\phi_0(t))^{-2} dt + (v_0)_y(s, 0) \int_{h_+^{(\beta_0^+)}}^{h_-^{(\beta_0^-)}} f_u(u, \beta_0^+)du \right), \]

which implies (3.16).

Noting that the inhomogeneous terms of the equation \( \dot{\zeta}_1^+ \) and \( \dot{\zeta}_1^- \) are continuous at \( \xi = 0 \), once \( \lambda_1 \) is determined, we can regard \( \dot{\zeta}_1 \) as \( C^2 \)-solution on \( \mathbb{R} \). So we omit \( \cdot \) and write \( \zeta_1 \) as

\[ \zeta_1(s, \xi) = \frac{\phi_0(\xi)}{\phi_0(0)} \Theta_1(s) + \frac{\phi_0(\xi)}{\phi_0(0)} \left[ \int_{0}^{\xi} (\phi_0(t))^{-2} dt \right] \Theta_0(s). \]

As a corollary of Lemma 3.3, we have an important consequence.

**Proposition 3.4.**

\( (v_0)_y \equiv \text{constant on } \Gamma_0 \)

Next, for \( (MC)_2 \) we have

**Lemma 3.5.** \( (MC)_2 \) is equivalent to the following problem;

\[ \begin{cases} 
\frac{d^2}{ds^2} \Theta_0 + P(s)\Theta_0 = \lambda_2 \Theta_0, & s \in (0, \xi), \\
\Theta_0(0) = \Theta_0(\xi), \quad \frac{d\Theta_0}{ds}(0) = \frac{d\Theta_0}{ds}(\xi).
\end{cases} \]

where, \( P(s) \) is a smooth bounded function of \( s \).

**Proof.** First note that \( (MC)_2 \) is rewritten as

\[ 0 = (R_1 \zeta_1 + R_2 \zeta_0, \dot{\phi}_0)_{\xi} = (R_1 \zeta_1, \dot{\phi}_0)_{\xi} + (R_2 \zeta_0, \dot{\phi}_0)_{\xi} \]
Substituting the representation of $\zeta_1$ into the first term of (3.20), we have
\[ \langle R_1 \zeta_1, \dot{\phi}_0 \rangle_\xi = \frac{1}{\dot{\phi}_0(0)} \left[ \Theta_1 (R_1 \dot{\phi}_0, \dot{\phi}_0) + \Theta_0 \int_{-\infty}^{+\infty} \dot{\phi}_0(\xi) (R_1(s, \xi) \dot{\phi}_0(\xi)) \right] \]
\[ \times \int_{-\infty}^{\xi} (R_1(s, \tau) \dot{\phi}_0(\tau)) d\tau d\xi = \mathcal{G}(\Theta_0), \]
where
\[ \mathcal{G}(\Theta_0) = \frac{\Theta_0}{\dot{\phi}_0(0)} \int_{-\infty}^{+\infty} \dot{\phi}_0(\xi) (R_1(s, \xi) \dot{\phi}_0(\xi)) \int_{0}^{\xi} (\dot{\phi}_0(t))^{-2} \]
\[ \times \int_{-\infty}^{\xi} \dot{\phi}_0(\xi) (R_1(s, \tau) \dot{\phi}_0(\tau)) d\tau d\xi. \]
On the other hand, recalling (2.21), (3.9) and (3.10), the second term of (3.20) is computed as follows
\[ \langle R_2 \zeta_0, \dot{\phi}_0 \rangle_\xi = \frac{\Theta_0}{\dot{\phi}_0(0)} \int_{-\infty}^{+\infty} (\lambda_2 - \tilde{F}_u^2)(\dot{\phi}_0(\xi))^2 d\xi - \frac{1}{\dot{\phi}_0(0)} \int_{-\infty}^{+\infty} \dot{\phi}_0(\xi) \partial_{ss} \Theta_0 \]
\[ -2\gamma_1 \dot{\phi}_0(\xi) \partial_\xi \Theta_0 - \gamma_2 \ddot{\phi}_0(\xi) \Theta_0 + (\gamma_1' \dot{\phi}_0(\xi) \Theta_0) - \kappa^2 \ddot{\phi}_0(\xi) \Theta_0 \\
- \kappa^2 \gamma_1 \dot{\phi}_0(\xi) \Theta_0 \right] \dot{\phi}_0(\xi) d\xi \\
= \frac{1}{\dot{\phi}_0(0)} \left[ \lambda_2 \Theta_0 \int_{-\infty}^{+\infty} (\dot{\phi}_0(\xi))^2 d\xi - \partial_{ss} \Theta_0 \int_{-\infty}^{+\infty} (\dot{\phi}_0(\xi))^2 d\xi \\
- \Theta_0 \left\{ \int_{-\infty}^{+\infty} \tilde{F}_u^2(\dot{\phi}_0(\xi))^2 d\xi - (\gamma_1')^2 \int_{-\infty}^{+\infty} \ddot{\phi}_0(\xi)^2 d\xi \\
+ (\kappa(s))^2 \int_{-\infty}^{+\infty} \xi \dot{\phi}_0(\xi) \ddot{\phi}_0(\xi) d\xi \right\} \right] \equiv \mathcal{F}(\Theta_0). \]
Hence $(MC)_2$ is equivalent to
\[ \mathcal{F}(\Theta_0) + \mathcal{G}(\Theta_0) = 0, \]
and this yields (3.19).

**Remark 3.6.** (3.19) is a Sturm-Liouville eigenvalue problem with periodic boundary condition, and the existence and asymptotic behavior of the eigenvalues and their eigenfunctions are well-studied. We denote those eigenpairs by \((\lambda_2^{(n)}, \Theta_0^{(n)})\) for \(n = 1\).

By Proposition 3.5, \(\lambda_2\) and \(\Theta_0\) are determined. Generally \(\lambda_k\) and \(\Theta_{k-2}\) ($k \geq 3$) are determined by \((MC)_k\), that is

**Proposition 3.7.** For each $n \geq 1$, $(MC)_k$ ($k \geq 3$) is equivalent to the following problem
\[ \frac{d^2}{ds^2} \Theta_{k-2} + P(s) \Theta_{k-2} - \lambda_2^{(n)} \Theta_{k-2} = \Psi_k(s), \quad 0 \leq s \leq \ell \]
with Periodic boundary condition, where \( P(s) \) is the same one appeared in Proposition 3.5 and \( \Psi_k(s) \) is a smooth bounded function of \( s \) depending on \( \lambda_k, \lambda_{k-1}, \ldots, \hat{\lambda}^{(n)}_2, \lambda_1, \Theta_{k-3}, \ldots, \hat{\Theta}^{(n)}_0, \) and \( s \). Then \( \lambda_k \) and \( \Theta_{k-2} \) are determined once \( \lambda_{k-1}, \ldots, \lambda_1 \) and \( \Theta_{k-3}, \ldots, \hat{\Theta}^{(n)}_0 \) are known. In fact, \( \lambda_k \) is uniquely determined by the solvability condition
\[
\langle \Psi_k, \hat{\Theta}^{(n)}_0 \rangle_s = 0,
\]
where \( \langle \cdot, \cdot \rangle_s \) denotes \( L^2 \)-inner product with respect to \( s \), and then, \( \Theta_{k-2} \) is also uniquely determined by (3.21) with \( \langle \Theta_{k-2}, \hat{\Theta}^{(n)}_0 \rangle_s = 0 \).

**Proof.** We prove by using induction arguments. Assume that \( \lambda_j \) and \( \Theta_{j-2} \) \((2 \leq j \leq k)\) are known, and we determine \( \lambda_{k+1} \) and \( \Theta_{k-1} \). \((MC)_{k+1}\) is rewritten as
\[
0 = \left\langle \sum_{i=1}^{k+1} R_i(s, \xi) \zeta_{k+1-i}, \dot{\phi}_0(\xi) \right\rangle_{\xi}
\]
(3.23)
\[
= \left\langle R_1 \zeta_k, \dot{\phi}_0 \right\rangle_{\xi} + \left\langle R_2 \zeta_{k-1}, \dot{\phi}_0 \right\rangle_{\xi} + \left\langle \sum_{i=3}^{k+1} R_i \zeta_{k+1-i}, \dot{\phi}_0 \right\rangle_{\xi}.
\]
Then note that the third term of (3.23) does not depend on \( \Theta_k \) and \( \Theta_{k-1} \), and \( \lambda_{k+1} \) is only involved in the third term and the coefficient of \( \lambda_{k+1} \) is given by \( \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)} \Theta_0(s) \). Noting that the coefficients of \( \Theta_k \) and \( \Theta_{k-1} \) in the representation of \( \zeta_k \) are \( \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)} \) and \( \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)} \int_0^\xi (\dot{\phi}_0(t))^{-2} \int_{-\infty}^{t} (\dot{\phi}_0(\tau)(R_1(s, \tau) \dot{\phi}_0(t))d\tau dt, \) respectively, the first term of (3.23) is computed as follows:
\[
\left\langle R_1 \zeta_k, \dot{\phi}_0 \right\rangle_{\xi} = \frac{1}{\dot{\phi}_0(0)} \left[ \Theta_k (R_1 \phi_0, \dot{\phi}_0)_{\xi} + \Theta_{k-1} \int_{-\infty}^{+\infty} (\dot{\phi}_0(\xi))^2 \int_0^\xi (\dot{\phi}_0(t))^{-2} \right.

(3.24)
\[
\times \int_{-\infty}^{t} \dot{\phi}_0(\tau)(R_1(s, \tau) \dot{\phi}_0(\tau))d\tau dt d\xi \bigg] + \Sigma_{k-1}(s) = \mathcal{G}(\Theta_{k-1}) + \Sigma_{k-1}(s).
\]
Here, \( \mathcal{G}(\cdot) \) is the same one defined in Proposition 3.5 and \( \Sigma_{k-1}(s) \) is a known function depending on \( \lambda_j \) \((1 \leq j \leq k)\), \( \Theta_i \) \((0 \leq i \leq k-2)\), its derivatives, and so on. On the other hand, noting that \( \zeta_{k-1} \) is can be represented by
\[
\zeta_{k-1}(s, \xi) = \frac{\dot{\phi}_0(\xi)}{\dot{\phi}_0(0)} \Theta_{k-1}(s) + Z_{k-1}(s, \xi),
\]
\((Z_{k-1}(s, \xi) \) is a known function independent of \( \Theta_{k-1} \) and \( \lambda_{k+1} \) we have
\[
\left\langle R_2 \zeta_{k-1}, \dot{\phi}_0 \right\rangle_{\xi} = \mathcal{F}(\Theta_{k-1}) + \left\langle R_2 Z_{k-1}, \dot{\phi}_0 \right\rangle_{\xi},
\]
(3.25)
Using (3.24) and (3.25), we see that (3.23) is equivalent to (3.21). The remainder of the statement is obvious from Riesz-Schauder theory.

4 Over-determined reduced elliptic problem and the proof of Main Theorem

Formal asymptotic analysis in the previous sections tells us that the normal derivative of the $C^1$-matched solution $v_0$ at $\Gamma_0$ is constant along the interface (Prop. 3.4). On the other hand, $v_0$ satisfies the reduced problem (2.26) subject $v_0 = v^*$ on $\Gamma_0$. This is apparently an over-determined problem and only special $\Gamma_0$ and $v_0$ are allowable. In fact, the first part of the Main Theorem in Section 1 is clear from Theorem 2 of J. Serrin [11]. To show the second part, the following hypothesis is necessary.

**Hypothesis 4.1.** The nonlinear elliptic problem

$$D\Delta v_0 + g(h_-(v_0), v_0) = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Omega_0^+,\]

$$v_0 = v^* \quad \text{and} \quad (v_0)_y \equiv \text{constant} \quad \text{on} \quad \Gamma_0 = \partial \Omega_0^+$$

has only axisymmetric solutions.

Under this hypothesis it is clear that the Neumann boundary condition on $\partial \Omega$ of the reduced problem (2.26) cannot be satisfied for generic domains. This completes the proof of the second part of the Main Theorem.

References


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