1 Introduction

Some days ago, Stefan Luding asked one of the authors (Hisao Hayakawa) how to derive the transport coefficients of two-dimenisonal smooth but inelastic disks used in our previous paper[1]. Since we have already lost the technical note for the calculation, we have reexamined the validity of our calculation starting from the paper by Jenkins and Richman.[2] The calculation has been performed by Kuniyasu Saitoh, and Hisao Hayakawa has checked its validity.

2 Comments

In this note, Eq. (a)* represents the equation (a) in the paper of Jenkins and Richman [2]. For example, if we write Eq. (100)*, which means the equation (100) in Ref.[2]. On the other hand, "our paper" in this note means the Ref. [1].

It should be noted that the tangential restitution coefficient β is equal to -1 because we have mapped the system onto a system of smooth disks. Thus, from Eq. (9)* and (10)*, a is equal to zero and r is equal to $(1+e)/2$, where e is the normal restitution constant.

3 Energy loss rate

From Eqs. (102)* and (103)*, the energy sources $\chi_{\alpha\alpha}$ and χ_3 are respectively given by

$$\chi_{\alpha\alpha} = -\frac{\xi(1-e)}{2\sigma^2}\left[8T - 3\pi^{1/2}\sigma T^{1/2}(\nabla\cdot\mathbf{v})\right] , \tag{1}$$

and

$$\chi_{33} = 0 , \tag{2}$$

where σ, T and $\mathbf{v}$ are the diameter of a disk, the granular temperature which is $T \equiv \langle(\mathbf{c}-\mathbf{v})^2\rangle/2$ and the velocity field. Here, the bulk viscosity ξ appears in Eq.(1) which is defined by Eq. (99)*[*1]:

$$\xi \equiv \frac{8m}{\sigma\pi^{3/2}}\nu^2 g_0 r T^{1/2} , \tag{3}$$

where m is the mass of a disk. We should note that we adopt the different definition of the granular temperature from Ref.[2], where we have used $T \equiv m\langle(\mathbf{c}-\mathbf{v})^2\rangle/2$ with the mass of a disk m. This is because the granular temperature should have the dimension of the energy. From Eq. (3) the prefactor of Eq. (1) can be rewritten as

$$-\frac{\xi(1-e)}{2\sigma^2} = -\frac{4m\nu^2(1-e^2)}{\sigma^3\pi^{3/2}} g_0 T^{1/2} , \tag{4}$$

where ν and g_0 are respectively the area fraction and the radial distribution function at contact. Subsituting this into Eq.(1), we rewrite $\chi_{\alpha\alpha}$ as

$$\chi_{\alpha\alpha} = -\frac{4m\nu^2(1-e^2)}{\sigma^3\pi^{3/2}} g_0 T^{1/2}\left[8T - 3\pi^{1/2}\sigma T^{1/2}(\nabla\cdot\mathbf{v})\right] . \tag{5}$$

[*1] They used α in their paper[2]

Introducing the mass density $\rho = nm = 4m\nu/(\pi\sigma^2) = \rho_p \nu$ and mass density of each disk $\rho_p = 4m/(\pi\sigma^2)$, $\chi_{\alpha\alpha}$ further can be rewritten as

$$\chi_{\alpha\alpha} = -\frac{1-e^2}{\sigma\rho_p \pi^{1/2}}\rho^2 g_0 T^{1/2}\left[8T - 3\pi^{1/2}\sigma T^{1/2}(\nabla\cdot\mathbf{v})\right] . \tag{6}$$

Now, let us replace $\chi_{\alpha\alpha}$ by the energy loss rate χ by collisions by using the relation $\chi = -\chi_{\alpha\alpha}/2$. (Compare Eq. (60)* with Eq. (15) in our paper [1].) Thus, the energy loss rate due to collisions is given by

$$\chi = \frac{1-e^2}{2\sigma\rho_p \pi^{1/2}}\rho^2 g_0 T^{1/2}\left[8T - 3\pi^{1/2}\sigma T^{1/2}(\nabla\cdot\mathbf{v})\right] . \tag{7}$$

4 Pressure tensor

The pressure tensor given by Eq. (59)* is

$$P_{\alpha\beta} = \rho T\delta_{\alpha\beta} + \rho a_{\alpha\beta} + \Theta_{\alpha\beta} , \tag{8}$$

where, thanks to Eq. (98)*, $\Theta_{\alpha\beta}$ is given by

$$\Theta_{\alpha\beta} = (2\rho T\nu g_0 r - \xi\nabla\cdot\mathbf{v})\delta_{\alpha\beta} - \alpha\hat{D}_{\alpha\beta} + \nu g_0 r\rho a_{\alpha\beta} \tag{9}$$

with deviatoric strain rate $\hat{D}_{\alpha\beta} = (\partial_{x_\alpha} v_\beta + \partial_{x_\beta} v_\alpha)/2 - \delta_{\alpha\beta}\nabla\cdot\mathbf{v}/2$. Therefore the pressure tensor $P_{\alpha\beta}$ is written as

$$P_{\alpha\beta} = \left[\rho T(1+2r\nu g_0) - \alpha\nabla\cdot\mathbf{v}\right]\delta_{\alpha\beta} - \alpha\hat{D}_{\alpha\beta} + (1+r\nu g_0)\rho a_{\alpha\beta} . \tag{10}$$

Since $\rho a_{\alpha\beta} \propto -\hat{D}_{\alpha\beta}$ as in Eq. (67)* in Ref.[2], the pressure tensor can be written as the following form

$$P_{\alpha\beta} = \left[p - \xi\nabla\cdot\mathbf{v}\right]\delta_{\alpha\beta} - \eta\hat{D}_{\alpha\beta}, \tag{11}$$

where η is the viscosity. The pressure p and the bulk viscosity ξ in Eq,(11) is given by

$$p \equiv \rho T\left[1+2r\nu g_0\right] = \rho T\left[1+(1+e)\nu g_0\right] , \tag{12}$$

and

$$\xi = \frac{8m}{\sigma\pi^{3/2}}\nu^2 g_0 r T^{1/2} = \frac{4m}{\sigma\pi^{3/2}}(1+e)\nu^2 g_0 T^{1/2} . \tag{13}$$

where we have used Eq. (99)* or Eq. (3). Thus, what we should do is to determine the viscosity η.

Using Eq. (67)*, (68)* and (69)*, $\rho a_{\alpha\beta}$ in Eq. (10) is given by

$$\rho a_{\alpha\beta} = -\frac{2mT^{1/2}}{\sigma\pi^{1/2}g_0(5-3r)}\left[1+\nu g_0(3r-2)r\right]\hat{D}_{\alpha\beta} . \tag{14}$$

It should be noted that Eq. (70)* should be multiplied by rm. Now, the sum of the second term and the third term on the right hand side of Eq. (10) becomes

$$\begin{aligned}\eta\hat{D}_{\alpha\beta} &\equiv \xi\hat{D}_{\alpha\beta} - (1+r\nu g_0)\rho a_{\alpha\beta} \\ &= \left[\xi + \frac{2mT^{1/2}}{\sigma\pi^{1/2}g_0(5-3r)}\left[1+\nu g_0(3r-2)r\right](1+r\nu g_0)\right]\hat{D}_{\alpha\beta} \\ &= \frac{m}{\sigma\pi^{1/2}}\left[\frac{8}{\pi}\nu^2 g_0 r + \frac{2}{(5-3r)g_0}\left[1+(3r^2-r)\nu g_0 + r^2(3r-2)\nu^2 g_0^2\right]\right]T^{1/2}\hat{D}_{\alpha\beta}.\end{aligned} \tag{15}$$

Substituting $r = (1+e)/2$ into Eq. (15) we obtain

$$\begin{aligned}\eta\hat{D}_{\alpha\beta} &= \frac{m}{\sigma\pi^{1/2}}\left[\frac{4}{\pi}(1+e)\nu^2 g_0 + \frac{4}{(7-3e)g_0}\left[1+\frac{1}{4}(1+e)(3e+1)\nu g_0 + \frac{1}{8}(1+e)^2(3e-1)\nu^2 g_0^2\right]\right]T^{1/2}\hat{D}_{\alpha\beta} \\ &= \frac{4m}{\sigma\pi^{1/2}}\left[\frac{1}{7-3e}g_0^{-1} + \frac{(1+e)(3e+1)}{4(7-3e)}\nu + \left[\frac{(1+e)(3e-1)}{8(7-3e)} + \frac{1}{\pi}\right](1+e)\nu^2 g_0\right]T^{1/2}\hat{D}_{\alpha\beta}. \end{aligned} \tag{16}$$

Thus, the shear viscosity is given by

$$\eta \equiv \frac{4m}{\sigma\pi^{1/2}}\left[\frac{1}{7-3e}g_0^{-1} + \frac{(1+e)(3e+1)}{4(7-3e)}\nu + \left[\frac{(1+e)(3e-1)}{8(7-3e)} + \frac{1}{\pi}\right](1+e)\nu^2 g_0\right]T^{1/2}, \tag{17}$$

where we have used the relations

$$3r^2 - r = \frac{1}{4}(1+e)(3e+1) \tag{18}$$

$$r^2(3r-2) = \frac{1}{8}(1+e)^2(3e-1) \tag{19}$$

$$5 - 3r = \frac{1}{2}(7-3e). \tag{20}$$

5 Transport coefficients associate with the density gradient and the heat conductivity

The (translational) energy flux is given by Eq. (61)*:

$$q_\alpha = \frac{1}{2}\rho a_{\alpha\beta\beta} + \frac{1}{2}\Theta_{\alpha\beta\beta}\ . \tag{21}$$

Here, $\rho a_{\alpha\beta\beta}/2$ and $\Theta_{\alpha\beta\beta}/2$ are given by Eq. (89)* and (100)*, respectively. First, with the help of Eq. (100)*, we rewrite the energy flux q_α as

$$\begin{aligned} q_\alpha &= \frac{1}{2}\rho a_{\alpha\beta\beta} - \xi\nabla T + \frac{3}{2}r\nu g_0 \cdot \frac{1}{2}\rho a_{\alpha\beta\beta} \\ &= \left(1+\frac{3}{2}r\nu g_0\right)\frac{1}{2}\rho a_{\alpha\beta\beta} - \xi\nabla T\ . \end{aligned} \tag{22}$$

Now, we introduce κ_ρ and λ_ρ as

$$\frac{1}{2}\rho a_{\alpha\beta\beta} \equiv -\kappa_\rho\nabla T - \lambda_\rho\nabla\rho\ , \tag{23}$$

where κ_ρ and λ_ρ are respectively given by

$$\kappa_\rho = \frac{4mT^{1/2}}{\sigma g_0 r(17-15r)\pi^{1/2}}\left[1+\frac{3}{2}\nu g_0 r^2(4r-3)\right]\ , \tag{24}$$

and

$$\lambda_\rho = -\frac{3\sigma\pi^{1/2}(2r-1)(1-r)}{2\nu g_0(17-15r)}T^{3/2}\frac{d(\nu^2 g_0)}{d\nu}\ . \tag{25}$$

To derive Eqs. (24) and (25) we have used Eq.(89)* in Ref.[2]. Thus, the energy flux q_α becomes

$$q_\alpha = -\left[\kappa_\rho\left(1+\frac{3}{2}r\nu g_0\right)+\xi\right]\nabla T - \lambda_\rho\left(1+\frac{3}{2}r\nu g_0\right)\nabla\rho$$
$$\equiv -\kappa\nabla T - \lambda\nabla\rho\,. \tag{26}$$

Here, the heat conductivity κ and the transport coefficient λ associated with the density gradient are given by

$$\kappa = \kappa_\rho\left(1+\frac{3}{2}r\nu g_0\right)+\xi, \tag{27}$$

and

$$\lambda = \lambda_\rho\left(1+\frac{3}{2}r\nu g_0\right)\,, \tag{28}$$

respectively.

5.1 Heat conductivity κ

In this subsection, let us write the explicit expression of the heat conductivity κ,

From Eqs.(3), (26) and (24), we obtain the heat conductivity κ as

$$\kappa = \frac{4mT^{1/2}}{\sigma\pi^{1/2}g_0 r(17-15r)}\left[1+\frac{3}{2}r^2(4r-3)\nu g_0\right]\left(1+\frac{3}{2}r\nu g_0\right)+\frac{8m}{\sigma\pi^{3/2}}r\nu^2 g_0 T^{1/2} \tag{29}$$

$$= \frac{4mT^{1/2}}{\sigma\pi^{1/2}g_0 r(17-15r)}\left[1+\frac{3}{2}r(4r^2-3r+1)\nu g_0+\frac{9}{4}r^3(4r-3)\nu^2 g_0^2\right]+\frac{8m}{\sigma\pi^{3/2}}r\nu^2 g_0 T^{1/2} \tag{30}$$

$$= \frac{4m}{\sigma\pi^{1/2}}\left[\frac{1}{r(17-15r)}g_0^{-1}+\frac{3(4r^2-3r+1)}{2(17-15r)}\nu+\frac{9r^2(4r-3)}{4(17-15r)}\nu^2 g_0+\frac{2}{\pi}r\nu^2 g_0\right]T^{1/2}. \tag{31}$$

Substituting $r = (1+e)/2$ into the above equation, κ is rewritten as

$$\kappa = \frac{16m}{\sigma\pi^{1/2}}\left[\frac{1}{(1+e)(19-15e)}g_0^{-1}+\frac{3(2e^2+e+1)}{8(19-15e)}\nu+\left\{\frac{9(1+e)^2(2e-1)}{64(19-15e)}+\frac{1}{4\pi}\right\}(1+e)\nu^2 g_0\right]T^{1/2}\,. \tag{32}$$

Here, we have used the following relations

$$17-15r = \frac{1}{2}(19-15e) \tag{33}$$

$$r(17-15r) = \frac{1}{4}(1+e)(19-15e) \tag{34}$$

$$4r^2-3r+1 = \frac{1}{2}(2e^2+e+1) \tag{35}$$

$$r^2(4r-3) = \frac{1}{4}(1+e)^2(2e-1). \tag{36}$$

We should note that the third term on the right hand side of Eq. (32) differs from our paper[1]. Indeed, one of the coefficients used $r(4r-2)$ instead of the correct form $r^2(4r-2)$, and the coefficient $1/4\pi$ in the last term on the right hand side of Eq. (32) is different from $1/2\pi$.

5.2 Transport coefficient λ

In this subsection, let us write the explicit form of λ.
Substituting (25) into Eq.(26) we obtain

$$\lambda = -\frac{3\sigma\pi^{1/2}(2r-1)(1-r)}{2\nu g_0(17-15r)} T^{3/2} \frac{d(\nu^2 g_0)}{d\nu} \left(1 + \frac{3}{2} r\nu g_0\right) . \tag{37}$$

With the aid of $r = (1+e)/2$, λ is rewritten as

$$\lambda = -\frac{3e(1-e)}{8(19-15e)} \sigma\pi^{1/2} \left[4g_0^{-1} + 3(1+e)\nu\right] \frac{1}{\nu} \frac{d(\nu^2 g_0)}{d\nu} T^{3/2} , \tag{38}$$

where we have used the relations

$$1 + \frac{3}{2} r\nu g_0 = \frac{1}{4} g_0 \left[4g_0^{-1} + 3(1+e)\nu\right], \tag{39}$$

$$(2r-1)(1-r) = \frac{1}{2} e(1-e), \tag{40}$$

$$2(17-15r) = 19 - 15e. \tag{41}$$

5.3 Summary in the dimensional form

In this subsection, we explicitly list the collisional energy loss rate and all the transport coefficients.

$$\chi = \frac{1-e^2}{2\sigma\rho_p\pi^{1/2}} \rho^2 g_0 T^{1/2} \left[8T - 3\pi^{1/2}\sigma T^{1/2}(\nabla\cdot\mathbf{v})\right] \tag{42}$$

$$p = \rho T \left[1 + (1+e)\nu g_0\right] \tag{43}$$

$$\xi = \frac{4m}{\sigma\pi^{3/2}} (1+e)\nu^2 g_0 T^{1/2} \tag{44}$$

$$\eta = \frac{4m}{\sigma\pi^{1/2}} \left[\frac{1}{7-3e} g_0^{-1} + \frac{(1+e)(3e+1)}{4(7-3e)}\nu + \left[\frac{(1+e)(3e-1)}{8(7-3e)} + \frac{1}{\pi}\right](1+e)\nu^2 g_0\right] T^{1/2} \tag{45}$$

$$\kappa = \frac{16m}{\sigma\pi^{1/2}} \left[\frac{1}{(1+e)(19-15e)} g_0^{-1} + \frac{3(2e^2+e+1)}{8(19-15e)}\nu + \left[\frac{9(1+e)^2(2e-1)}{64(19-15e)} + \frac{1}{4\pi}\right](1+e)\nu^2 g_0\right] T^{1/2} \tag{46}$$

$$\lambda = -\frac{3e(1-e)}{8(19-15e)} \sigma\pi^{1/2} \left[4g_0^{-1} + 3(1+e)\nu\right] \frac{1}{\nu} \frac{d(\nu^2 g_0)}{d\nu} T^{3/2}. \tag{47}$$

6 Non-dimensionalization

Now, we non-dimensionalize the expressions for the transport coefficients listed in the previous section.
Using the dimensionless temperature θ, dimensionless pressure and the transport coefficients are given by

$$p^* = p(\nu)\theta \tag{48}$$

$$\xi^* = \xi(\nu)\theta^{1/2} \tag{49}$$

$$\eta^* = \eta(\nu)\theta^{1/2} \tag{50}$$

$$\kappa^* = \kappa(\nu)\theta^{1/2} \tag{51}$$

$$\lambda^* = \lambda(\nu)\theta^{3/2}. \tag{52}$$

Finally, the functions $p(\nu)$, $\xi(\nu)$, $\eta(\nu)$, $\kappa(\nu)$ and $\lambda(\nu)$ in the dimensionless forms are given by

$$p(\nu) = \frac{1}{2}\nu\left[1 + (1+e)\nu g_0\right] \tag{53}$$

$$\xi(\nu) = \frac{1}{\sqrt{2\pi}}(1+e)\nu^2 g_0 \tag{54}$$

$$\eta(\nu) = \sqrt{\frac{\pi}{2}}\left[\frac{1}{7-3e}g_0^{-1} + \frac{(1+e)(3e+1)}{4(7-3e)}\nu + \left[\frac{(1+e)(3e-1)}{8(7-3e)} + \frac{1}{\pi}\right](1+e)\nu^2 g_0\right] \tag{55}$$

$$\kappa(\nu) = \sqrt{2\pi}\left[\frac{1}{(1+e)(19-15e)}g_0^{-1} + \frac{3(2e^2+e+1)}{8(19-15e)}\nu + \left[\frac{9(1+e)^2(2e-1)}{64(19-15e)} + \frac{1}{4\pi}\right](1+e)\nu^2 g_0\right] \tag{56}$$

$$\lambda(\nu) = -\sqrt{\frac{\pi}{2}}\left[4g_0^{-1} + 3(1+e)\nu\right]\frac{1}{\nu}\frac{d(\nu^2 g_0)}{d\nu}. \tag{57}$$

In the table I of our paper[1], the radial distribution function g_0 is represented by $g(\nu)$. Note that there are two mistakes for the third term of κ in the table I, in which the first term of the third term misused $r(4r-3)$ instead of $r^2(4r-3)$, and $1/2\pi$ in the last term on the right hand side of $\kappa(\nu)$ in the table I should be replaced by $1/4\pi$.

参考文献

[1] K. Saitoh and H. Hayakawa, Phys. Rev. E **75**, 021302 (2007).
[2] J. T. Jenkins and M. W. Richman, Phys. Fluids **28**, 3485 (1985) .