Weights, Kovalevskaya exponents and the Painlevé property

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Abstract

Weighted degrees of quasihomogeneous Hamiltonian functions of the Painlevé equations are investigated. A tuple of positive integers, called a regular weight, satisfying certain conditions related to singularity theory is classified. For each polynomial Painlevé equation a regular weight associates. Conversely, for 2 and 4-dim cases, it is shown that there exists a differential equation satisfying the Painlevé property associated with each regular weight. Kovalevskaya exponents of quasihomogeneous Hamiltonian systems are also investigated by means of regular weights, singularity theory and dynamical systems theory. It is shown that there is a one-to-one correspondence between Laurent series solutions and stable manifolds of the associated vector field obtained by the blow-up of the system. For 4-dim autonomous Painlevé equations, the level surface of Hamiltonian functions can be decomposed into a disjoint union of stable manifolds.

Keywords: Painlevé equation; quasihomogeneous vector field; regular weight; Kovalevskaya exponent

1 Introduction

A differential equation defined on a complex region is said to have the Painlevé property if any movable singularity (a singularity of a solution which depends on an initial condition) of any solution is a pole. Painlevé and his group classified second order ODEs having the Painlevé property and found new six differential equations called the Painlevé equations $\mathrm{P_I}, \cdots, \mathrm{P_{VI}}$. Nowadays, it is known that they are written in Hamiltonian forms

$$(\mathrm{P_J}) : \frac{dq}{dz} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H_J}{\partial q}, \quad J = \mathrm{I}, \cdots, \mathrm{VI}.$$

Among six Painlevé equations, the Hamiltonian functions of the first, second and fourth Painlevé equations are polynomials in both of the independent variable z and

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the dependent variables (q, p). They are given by

$$\begin{aligned} H_{\rm I} &= \frac{1}{2}p^2 - 2q^3 - zq, \\ H_{\rm II} &= \frac{1}{2}p^2 - \frac{1}{2}q^4 - \frac{1}{2}zq^2 - \alpha q, \\ H_{\rm IV} &= -pq^2 + p^2q - 2pqz - \alpha p + \beta q, \end{aligned} \tag{1.1}$$

respectively, where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters.

In general, a polynomial $H(x_1, \cdots, x_n)$ is called a quasihomogeneous polynomial if there are positive integers $a_1, \cdots, a_n$ and h such that

$$H(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_n) = \lambda^h H(x_1, \cdots, x_n) \tag{1.2}$$

for any $\lambda \in \mathbb{C}$. A polynomial H is called a semi-quasihomogeneous if H is decomposed into two polynomials as $H = H^P + H^N$, where H^P satisfies (1.2) and H^N satisfies

$$H^N(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_n) \sim o(\lambda^h), \quad |\lambda| \to \infty.$$

The integer $\deg(H) := h$ is called the weighted degree of H with respect to the weight $\deg(x_1, \cdots, x_n) := (a_1, \cdots, a_n)$. H^P and H^N are called the principal part and the non-principal part of H, respectively. The weight of H, H^P and H^N are also calculated by the Newton diagram as follows. Plot all exponents $(r_1, \cdots, r_n)$ of monomials $x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$ included in H^P on the integer lattice in $\mathbb{R}^n$. If they lie on a unique hyperplane $a_1x_1 + \cdots + a_nx_n = h$, then $\deg(H^P) = h$ with respect to the weight $(a_1, \cdots, a_n)$. Exponents of monomials included in H^N should be on the lower side of the hyperplane. See [3] for the detail.

The above Hamiltonian functions for $\mathrm{P_I}, \mathrm{P_{II}}$ and $\mathrm{P_{IV}}$ are semi-quasihomogeneous. If we define degrees of variables by $\deg(q, p, z) = (2, 3, 4)$ for $H_{\rm I}$, $\deg(q, p, z) = (1, 2, 2)$ for $H_{\rm II}$ and $\deg(q, p, z) = (1, 1, 1)$ for $H_{\rm IV}$, then Hamiltonian functions have the weighted degrees $6, 4$ and 3, respectively, (Table 1) with $H_{\rm I}^N = 0, H_{\rm II}^N = -\alpha q$ and $H_{\rm IV}^N = -\alpha p + \beta q$.

The Hamiltonian functions of the third, fifth and sixth Painlevé equations are not polynomials in z, and their weights include nonpositive integers (Table 1). They are not treated in this paper, while the analysis of them using weighted projective spaces is given in [7].

Higher dimensional Painlevé equations have not been classified yet, however, a lot of such equations have been reported in the literature. A list of four dimensional Painlevé equations derived from the monodromy preserving deformation is given in [21, 20]. Lie-algebraic approach is often employed to find new Painlevé equations [13, 15, 12, 27]. Several Painlevé hierarchies, which are hierarchies of $2n$-dimensional Painlevé equations, are obtained by the similarity reductions of soliton equations such as the KdV equation. Among them, it is known that Hamiltonian functions of the the first Painlevé hierarchy $(\mathrm{P_I})_n$[22, 23, 31], the second-first Painlevé hierarchy $(\mathrm{P_{II\text{-}1}})_n$[9, 10, 22, 23], the second-second Painlevé hierarchy $(\mathrm{P_{II\text{-}2}})_n$ and the

fourth Painlevé hierarchy $(\mathrm{P_{IV}})_n$[16, 22] can be expressed as polynomials with respect to both of the dependent variables and the independent variables. They are Hamiltonian PDEs of the form

$$\begin{cases} \dfrac{\partial q_j}{\partial z_i} = \dfrac{\partial H_i}{\partial p_j},\ \dfrac{\partial p_j}{\partial z_i} = -\dfrac{\partial H_i}{\partial q_j}, \quad j = 1, \cdots, n;\ i = 1, \cdots, n \\ H_i = H_i(q_1, \cdots, q_n, p_1, \cdots, p_n, z_1, \cdots, z_n) \end{cases} \tag{1.3}$$

consisting of n Hamiltonians $H_1, \cdots, H_n$ with n independent variables $z_1, \cdots, z_n$. When $n = 1$, $(\mathrm{P_I})_1$ and $(\mathrm{P_{IV}})_1$ are reduced to the first and fourth Painlevé equations, respectively. Both of $(\mathrm{P_{II\text{-}1}})_1$ and $(\mathrm{P_{II\text{-}2}})_1$ coincide with the second Painlevé equation, while they are different systems for $n \geq 2$. When $n = 2$, Hamiltonians of $(\mathrm{P_I})_2$, $(\mathrm{P_{II\text{-}1}})_2$, $(\mathrm{P_{II\text{-}2}})_2$ and $(\mathrm{P_{IV}})_2$ are given by

$$(\mathrm{P_I})_2 \begin{cases} H_1^{9/2} = 2p_2p_1 + 3p_2^2q_1 + q_1^4 - q_1^2q_2 - q_2^2 - z_1q_1 + z_2(q_1^2 - q_2), \\ H_2^{9/2} = p_1^2 + 2p_2p_1q_1 - q_1^5 + p_2^2q_2 + 3q_1^3q_2 - 2q_1q_2^2 \\ \qquad\qquad +z_1(q_1^2 - q_2) + z_2(z_2q_1 + q_1q_2 - p_2^2), \end{cases} \tag{1.4}$$

$$(\mathrm{P_{II\text{-}1}})_2 \begin{cases} H_1^{7/2+1} = 2p_1p_2 - p_2^3 - p_1q_1^2 + q_2^2 - z_1p_2 + z_2p_1 + 2\alpha q_1, \\ H_2^{7/2+1} = -p_1^2 + p_1p_2^2 + p_1p_2q_1^2 + 2p_1q_1q_2 \\ \qquad +z_1p_1 + z_2(z_2p_1 - p_1q_1^2 + p_1p_2) - \alpha(2p_2q_1 + 2q_2 + 2z_2q_1), \end{cases} \tag{1.5}$$

$$(\mathrm{P_{II\text{-}2}})_2 \begin{cases} H_1^5 = p_1p_2 - p_1q_1^2 - 2p_1q_2 + p_2q_1q_2 + q_1q_2^2 + q_2z_1 + z_2(q_1q_2 - p_1) + \alpha q_1, \\ H_2^5 = p_1^2 - p_1p_2q_1 + p_2^2q_2 - 2p_1q_1q_2 - p_2q_2^2 + q_1^2q_2^2 \\ \qquad +z_1(q_1q_2 - p_1) - z_2(p_1q_1 + q_2^2 + q_2z_2) + \alpha p_2, \end{cases} \tag{1.6}$$

$$(\mathrm{P_{IV}})_2 \begin{cases} H_1^{4+1} = p_1^2 + p_1p_2 - p_1q_1^2 + p_2q_1q_2 - p_2q_2^2 - z_1p_1 + z_2p_2q_2 + \alpha q_2 + \beta q_1, \\ H_2^{4+1} = p_1p_2q_1 - 2p_1p_2q_2 - p_2^2q_2 + p_2q_1q_2^2 \\ \qquad +p_2q_2z_1 + z_2(p_1p_2 - p_2q_2^2 + p_2q_2z_2) + (p_1 - q_1q_2 + q_2z_2)\alpha - \beta p_2, \end{cases} \tag{1.7}$$

respectively, with arbitrary parameters $\alpha, \beta \in \mathbb{C}$ (these notations for Hamiltonian functions are related to the spectral type of a monodromy preserving deformation [21]). The weighted degrees of these hierarchies determined by the Newton diagrams are shown in Table 2 (see also Table 3). From Table 1, 2 and the equations, we deduce the following properties.

- $\deg(q_i) + \deg(p_i) = \deg(H_1) - 1$.
- $\deg(z_1) = \deg(H_1) - 2$.
- $\deg(z_i) + \deg(H_i)$ is independent of $i = 1, \cdots, n$.
- $\min\limits_{1 \leq i \leq n} \{\deg(q_i), \deg(p_i)\} = 1$ or 2.
- The equation (1.3) is invariant under the $\mathbb{Z}_s$-action

$$(q_i, p_i, z_i) \mapsto (\omega^{\deg(q_i)}q_i,\ \omega^{\deg(p_i)}p_i,\ \omega^{\deg(z_i)}z_i),$$

  where $s := \deg(H_1) - 1$ and $\omega := e^{2\pi i/s}$.

- The symplectic form $\sum_{i=1}^{n} dq_i \wedge dp_i + \sum_{i=1}^{n} dz_i \wedge dH_i$ is also invariant under the same $\mathbb{Z}_s$-action, for which $H_i \mapsto \omega^{\deg(H_i)} H_i$.

We decompose the Hamiltonian function H_i into the principal part H_i^P and the non-principal part H_i^N. Then, we further deduce

- The non-principal part H_i^N consists of monomials including arbitrary parameters.
- $\deg(H_i^N) = \deg(H_i) - \deg(H_1) + 1$. In particular $\deg(H_1^N) = 1$.
- The variety defined by

$$H_1^P(q_1, \cdots, q_n, p_1, \cdots, p_n, 0, \cdots, 0) = 0$$

in $\mathbb{C}^{2n}$ has a unique singularity at the origin.

In section 2.4, several above properties will be proved from the others. For ($\mathrm{P_I}$), ($\mathrm{P_{II}}$) and ($\mathrm{P_{IV}}$), we have

$$\begin{aligned} H_{\mathrm{I}}^P(q,p,0) &= \frac{1}{2}p^2 - 2q^3, \\ H_{\mathrm{II}}^P(q,p,0) &= \frac{1}{2}p^2 - \frac{1}{2}q^4, \\ H_{\mathrm{IV}}^P(q,p,0) &= -pq^2 + p^2 q. \end{aligned}$$

They define A_2, A_3 and D_4 singularities at the origin, respectively. In singularity theory, it is known that if a singularity defined by a quasihomogeneous polynomial $H(x_1, \cdots, x_n) = 0$ is isolated, then the rational function

$$\chi(T) := \frac{(T^{h-a_1} - 1) \cdots (T^{h-a_n} - 1)}{(T^{a_1} - 1) \cdots (T^{a_n} - 1)} \tag{1.8}$$

becomes a polynomial (Poincaré polynomial), where $\deg(x_i) = a_i$ and $\deg(H) = h$.

Motivated by these observation, we classify regular weights $(a_1, \cdots, a_n, b_1, \cdots, b_n; h)$ satisfying certain conditions in Section 2. In particular, for $n = 1$ and 2, we will show that there is a corresponding Painlevé equation for each weight such that $\deg(q_i) = a_i, \deg(p_i) = b_i$ and $\deg(H) = h$. In Section 2.4, a Hamiltonian system, whose Hamiltonian function satisfies certain assumptions on the quasihomogeneity, will be considered. Then, some of the above properties of weights will be proved.

In Section 3, a brief review of Kovalevskaya exponents of quasihomogeneous vector fields, which seems to be closely related to regular weights, is given. A list of Kovalevskaya exponents of 4-dim Painlevé equations are shown in Table 4. From the table, it is expected that Painlevé equations defined by semi-quasihomogeneous Hamiltonian functions can be classified by their weights and Kovalevskaya exponents.

In Section 4, Kovalevskaya exponents of quasihomogeneous systems are further studied by means of singularity theory and dynamical systems theory. In general, the level surface of quasihomogeneous Hamiltonian functions has a singularity at the origin. The weighted blow-up of the singularity at the origin induces a vector field on the exceptional divisor. Then, Laurent series solutions, Kovalevskaya exponents and the level surface are investigated via the vector field. In particular, it is shown that there is a one-to-one correspondence between Laurent series solutions and fixed points of the vector field, and the eigenvalues of the Jacobi matrix of the vector field at the fixed point precisely coincide with Kovalevskaya exponents. With the aid of these results, it is shown for several 4-dim Painlevé equations that the level surface of Hamiltonian functions can be decomposed into a disjoint sum of stable manifolds of the fixed points.

	$\deg(q,p,z)$	$\deg(H)$	κ
$\mathrm{P_I}$	$(2,3,4)$	6	6
$\mathrm{P_{II}}$	$(1,2,2)$	4	4
$\mathrm{P_{IV}}$	$(1,1,1)$	3	3
$\mathrm{P_{III}}(D_8)$	$(-1,2,4)$	2	2
$\mathrm{P_{III}}(D_7)$	$(-1,2,3)$	2	2
$\mathrm{P_{III}}(D_6)$	$(0,1,2)$	2	2
$\mathrm{P_V}$	$(1,0,1)$	2	2
$\mathrm{P_{VI}}$	$(1,0,0)$	2	2

Table 1: $\deg(H)$ denotes the weighted degree of the Hamiltonian function with respect to the weight $\deg(q,p,z)$. κ denotes the Kovalevskaya exponent defined in Section 3.

	$\deg(q_j,p_j)$	$\deg(z_i)$	$\deg(H_i)$
$(\mathrm{P_I})_n$	$(2j, 2n+3-2j)$	$2n-2i+4$	$2n+2i+2$
$(\mathrm{P_{II\text{-}1}})_n$	$(2j-1, 2n+2-2j)$	$2n-2i+2$	$2n+2i$
$(\mathrm{P_{II\text{-}2}})_n$	$(j, n+2-j)$	$n-i+2$	$n+i+2$
$(\mathrm{P_{IV}})_n$	$(j, n+1-j)$	$n-i+1$	$n+i+1$

Table 2: Weights for four classes of the Painlevé hierarchies.

2 Classification of regular weights

Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ and h be positive integers such that $1 \leq a_i, b_i < h$. Motivated by the observation in Section 1, we suppose the following.

(W1) $\min_{1\leq i\leq n}\{a_i, b_i\} = 1$ or 2.

	$\{\deg(q_j), \deg(p_j)\}$	$\deg(z_i)$	$\deg(H_i)$
$(\mathrm{P_I})_2$	$(2,3,4,5)$	$6,4$	$8,10$
$(\mathrm{P_I})_3$	$(2,3,4,5,6,7)$	$8,6,4$	$10,12,14$
$(\mathrm{P_{II\text{-}1}})_2$	$(1,2,3,4)$	$4,2$	$6,8$
$(\mathrm{P_{II\text{-}1}})_3$	$(1,2,3,4,5,6)$	$6,4,2$	$8,10,12$
$(\mathrm{P_{II\text{-}2}})_2$	$(1,2,2,3)$	$3,2$	$5,6$
$(\mathrm{P_{II\text{-}2}})_3$	$(1,2,2,3,3,4)$	$4,3,2$	$6,7,8$
$(\mathrm{P_{IV}})_2$	$(1,1,2,2)$	$2,1$	$4,5$
$(\mathrm{P_{IV}})_3$	$(1,1,2,2,3,3)$	$3,2,1$	$5,6,7$

Table 3: Weights for four classes of the Painlevé hierarchies when $n = 2, 3$, where $\deg(q_j), \deg(p_j)$'s are shown in ascending order.

(W2) $a_i + b_i = h - 1$ for $i = 1, \cdots, n$.
(W3) A function

$$\chi(T) = \frac{(T^{h-a_1}-1)(T^{h-b_1}-1)\cdots(T^{h-a_n}-1)(T^{h-b_n}-1)}{(T^{a_1}-1)(T^{b_1}-1)\cdots(T^{a_n}-1)(T^{b_n}-1)} \tag{2.1}$$

is polynomial.

In Saito[29], a tuple of integers $(a_1, \cdots, a_n, b_1, \cdots, b_n; h)$ satisfying (W3) is called a regular weight. In this paper, a tuple is called a regular weight if it satisfies (W1) to (W3). In this section, we will classify all regular weights for $n = 1, 2, 3$. In particular, for $n = 1$ and $n = 2$, we will show that there are Hamiltonians of Painlevé equations associated with regular weights such that $\deg(q_i) = a_i, \deg(p_i) = b_i$ and $\deg(H) = h$.

2.1 $n = 1$

Proposition 2.1. When $n = 1$, regular weights satisfying (W1) to (W3) are only

$$(a, b; h) = (2, 3; 6), \quad (1, 2; 4), \quad (1, 1; 3).$$

They coincide with the weights $(\deg(q), \deg(p); \deg(H))$ of $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV} for 2-dim Painlevé equations, respectively, given in Sec.1.

Hence, there is a one to one correspondence between regular weights and the 2-dim Painlevé equations written in polynomial Hamiltonians. Note that $\deg(z)$ is recovered by the rule $\deg(z) = \deg(H) - 2$ (see also Prop.2.5). Now we show that $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV} can be reconstructed from the regular weights with the aid of singularity theory.

Step 1. Consider generic polynomials $H(q, p)$ whose weighted degrees are $\deg(q, p; H) =$

$(2,3;6), (1,2;4)$ and $(1,1;3)$. They are given by

$$\begin{aligned} H &= c_1p^2 + c_2q^3, \\ H &= c_1p^2 + c_2q^2p + c_3q^4, \\ H &= c_1q^3 + c_2pq^2 + c_3p^2q + c_4p^3, \end{aligned}$$

respectively, with arbitrary constants $c_1, \cdots, c_4$.
Step 2. Simplify by symplectic transformations. One of the results are

$$\begin{aligned} H &= \frac{1}{2}p^2 - 2q^3, \\ H &= \frac{1}{2}p^2 - \frac{1}{2}q^4, \\ H &= -pq^2 + p^2q, \end{aligned}$$

respectively.
Step 3. Consider the versal deformations of them[3]. We obtain

$$\begin{aligned} H &= \frac{1}{2}p^2 - 2q^3 + \alpha_4 q + \alpha_6, \\ H &= \frac{1}{2}p^2 - \frac{1}{2}q^4 + \alpha_2 q^2 + \alpha_3 q + \alpha_4, \\ H &= -pq^2 + p^2q + \alpha_1 pq + \alpha_2 p + \beta_2 q + \alpha_3, \end{aligned}$$

respectively, where $\alpha_i, \beta_i \in \mathbb{C}$ are deformation parameters. The subscripts i of α_i, β_i denote the weighted degrees of α_i, β_i so that H becomes a quasihomogeneous.
Step 4. Now we use the ansatz $\deg(z) = \deg(H) - 2$ observed in Sec.1. If there is a parameter α_i such that $i = \deg(H) - 2$, then replace it by z. The results are

$$\begin{aligned} H &= \frac{1}{2}p^2 - 2q^3 + zq + \alpha_6, \\ H &= \frac{1}{2}p^2 - \frac{1}{2}q^4 + zq^2 + \alpha_3 q + \alpha_4, \\ H &= -pq^2 + p^2q + zpq + \alpha_2 p + \beta_2 q + \alpha_3, \end{aligned}$$

respectively. They are equivalent to $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV} up to the scaling of z (constant terms in Hamiltonians such as α_6 do not play a role).

Hence, when $n = 1$, there is a one to one correspondence between the regular weights and 2-dim polynomial Painlevé equations, and we can recover one of them from the other.

2.2 $n = 2$

Proposition 2.2. When $n = 2$, regular weights satisfying (W1) to (W3) are only

$$\begin{aligned}(a_1, a_2, b_2, b_1; h) \;=\; & (2,3,4,5;8),\\ & (1,2,3,4;6),\\ & (2,2,3,3;6),\\ & (1,2,2,3;5),\\ & (1,1,2,2;4),\\ & (1,1,1,1;3),\end{aligned}$$

where we assume without loss of generality that $a_1 \le a_2 \le b_2 \le b_1$. For each weight, there exists a polynomial Hamiltonian of a 4-dim Painlevé equation (not unique). Explicit forms of Hamiltonian functions are given as follows.

(2,3,4,5;8). The first Hamiltonian $H_1^{9/2}$ of $(\mathrm{P_I})_2$ shown in Eq.(1.4) has this weight with $\deg(q_1, q_2, p_1, p_2) = (2, 4, 5, 3)$. Another example is

$$H_{\text{Cosgrove}} = -4p_1p_2 - 2p_2^2q_1 - \frac{73}{128}q_1^4 + \frac{11}{8}q_1^2q_2 - \frac{1}{2}q_2^2 - q_1z - \frac{1}{48}\left(q_1 + \frac{\alpha}{6}\right)q_1^2\alpha. \quad (2.2)$$

This Hamiltonian system is derived by a Lie-algebraic method of type B_2 and can be written in Lax form [8], thus it enjoys the Painlevé property. It seems that it does not appear in the list of 4-dim Painlevé equations in [20, 21, 26]. If we rewrite the system as the fourth order single equation of $q_1 = y$, we obtain

$$y'''' = 18yy'' + 9(y')^2 - 24y^3 + 16z + \alpha y(y + \frac{1}{9}\alpha). \quad (2.3)$$

This equation was first given in Cosgrove [11], denoted by F-VI. He conjectured that this equation defines a new Painlevé transcendents (i.e. it is not reduced to known equations).

(1,2,3,4;6). The first Hamiltonian $H_1^{7/2+1}$ of $(\mathrm{P_{II\text{-}1}})_2$ shown in Eq.(1.5) has this weight $\deg(q_1, q_2, p_1, p_2) = (1, 3, 4, 2)$. Another example is the matrix Painlevé equation of the first type $H_{\mathrm{I}}^{\mathrm{Mat}}$[20, 21] defined by

$$H_{\mathrm{I}}^{\mathrm{Mat}} = \frac{1}{2}p_1^2 - 2q_1^3 - 2p_2^2q_2 + 6q_1q_2 - 2q_1z + 2\alpha p_2, \quad (2.4)$$

with $\deg(q_1, q_2, p_1, p_2) = (2, 4, 3, 1)$.

(2,2,3,3;6). For $H_1^{7/2+1}$ and $H_2^{7/2+1}$ of $(\mathrm{P_{II\text{-}1}})_2$ shown in Eq.(1.5), perform the symplectic transformation

$$q_1 = -\frac{y_1}{2x_1},\ p_1 = -x_1^2,\ q_2 = \frac{y_2}{2},\ p_2 = 2x_2. \quad (2.5)$$

Then we obtain the Hamiltonians

$$\begin{cases} H_1^{(2,3,2,3)} = -4x_1^2x_2 - 8x_2^3 + \dfrac{y_1^2}{4} + \dfrac{y_2^2}{4} - 2z_1x_2 - z_2x_1^2 - \dfrac{\alpha y_1}{x_1}, \\ H_2^{(2,3,2,3)} = -x_1^4 - 4x_1^2x_2^2 - \dfrac{x_2y_1^2}{2} + \dfrac{x_1y_1y_2}{2} \\ \qquad -z_1x_1^2 - z_2^2x_1^2 - 2z_2x_1^2x_2 + \dfrac{z_2y_1^2}{4} - \dfrac{\alpha z_2y_1}{x_1} + \dfrac{2\alpha x_2y_1}{x_1} - \alpha y_2. \end{cases} \tag{2.6}$$

Thus, putting $\alpha = 0$ yields semi-quasihomogeneous Hamiltonians of $\deg(H_1^{(2,3,2,3)}, H_2^{(2,3,2,3)}) = (6, 8)$ with respect to $\deg(x_1, y_1, x_2, y_2) = (2, 3, 2, 3)$ and $\deg(z_1, z_2) = (4, 2)$. Although this is equivalent to $(\mathrm{P_{II\text{-}1}})_2$ for $\alpha = 0$, they should be distinguished from each other from a viewpoint of a geometric classification of Painlevé equations (i.e. a classification based on the spaces of initial conditions) because the above symplectic transformation is not a one-to-one mapping. The direct product of two $(\mathrm{P_I})$ also has this weight, see Example 4.14.

(1,2,2,3;5). The first Hamiltonian H_1^5 of $(\mathrm{P_{II\text{-}2}})_2$ shown in Eq.(1.6) has this weight with $\deg(q_1, q_2, p_1, p_2) = (1, 2, 3, 2)$.

(1,1,2,2;4). The first Hamiltonian H_1^{4+1} of $(\mathrm{P_{IV}})_2$ shown in Eq.(1.7) has this weight with $\deg(q_1, q_2, p_1, p_2) = (1, 1, 2, 2)$. Another example is the matrix Painlevé equation of the second type $H_{\mathrm{II}}^{\mathrm{Mat}}$[20, 21] defined by

$$H_{\mathrm{II}}^{\mathrm{Mat}} = \frac{1}{2}p_1^2 - p_1q_1^2 + p_1q_2 - 2p_2^2q_2 - 4p_2q_1q_2 - p_1z + 2\alpha p_2 + 2\beta(p_2 + q_1), \tag{2.7}$$

with $\deg(q_1, q_2, p_1, p_2) = (1, 2, 2, 1)$. The direct product of two $(\mathrm{P_{II}})$ also has this weight.

(1,1,1,1;3). The Noumi-Yamada system of type A_4 [21, 27] defined by

$$H_{\mathrm{NY}}^{A_4} = 2p_1p_2q_1 + p_1q_1(p_1 - q_1 - z) + p_2q_2(p_2 - q_2 - z) + \alpha p_1 + \beta q_1 + \gamma p_2 + \delta q_2 \tag{2.8}$$

has the weight $\deg(q_1, q_2, p_1, p_2) = (1, 1, 1, 1)$, where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters. The direct product of two $(\mathrm{P_{IV}})$ also has this weight.

2.3 $n = 3$

To determine all regular weights satisfying (W1) to (W3), the following lemma is useful. Without loss of generality, we assume $a_1 \le a_2 \le \cdots \le a_n \le b_n \le \cdots \le b_2 \le b_1$. There exist integers N and $j(1), \cdots, j(N)$ such that

$$\begin{aligned} a_1 = \cdots = a_{j(1)} < a_{j(1)+1} = \cdots = a_{j(2)} < \cdots < a_{j(N)+1} = \cdots = a_n \\ \le b_n = \cdots = b_{j(N)+1} < \cdots < b_{j(2)} = \cdots = b_{j(1)+1} < b_{j(1)} = \cdots = b_1. \end{aligned}$$

We put $J_l = j(l) - j(l-1)$ $(l = 1, \cdots, N+1)$, where $j(0) = 0$ and $j(N+1) = n$.

Lemma 2.3.
(i) When $N = 0$ (i.e. $a_1 = a_n$), then

$$\begin{aligned}(a_1, \cdots, a_n, b_n, \cdots, b_1; h) &= (1, \cdots, 1, 1, \cdots, 1; 3) \\ &= (1, \cdots, 1, 2, \cdots, 2; 4) \\ &= (2, \cdots, 2, 3, \cdots, 3; 6).\end{aligned}$$

(ii) When $N \geq 1$, the equality $b_{j(i)} = b_{j(i+1)} + 1$ holds for $i = 1, \cdots, N$ and $J_{i+1} \geq J_i$ holds for $i = 1, \cdots, N-1$. If $a_n \neq b_n$, further $b_n = a_n + 1$ and $J_{N+1} \geq J_N$ hold.
(iii) If $a_i < a_{i+1}$ for any $i = 1, \cdots, n-1$, then

$$\begin{aligned}(a_1, \cdots, a_n, b_n, \cdots, b_1; h) &= (1, \cdots, n, n, \cdots, 2n-1; 2n+1) \\ &= (1, \cdots, n, n+1, \cdots, 2n; 2n+2) \\ &= (2, \cdots, n+1, n+2, \cdots, 2n+1; 2n+4).\end{aligned}$$

Proof. Because of (W2), Eq.(2.1) is rewritten as

$$\chi(T) = \frac{(T^{a_1+1}-1)\cdots(T^{a_n+1}-1)(T^{b_n+1}-1)\cdots(T^{b_1+1}-1)}{(T^{a_1}-1)\cdots(T^{a_n}-1)(T^{b_n}-1)\cdots(T^{b_1}-1)}. \tag{2.9}$$

(i) In this case, $a_1 = a_n \leq b_n = b_1$ due to (W2), which implies

$$\chi(T) = \frac{(T^{a_1+1}-1)^n(T^{b_1+1}-1)^n}{(T^{a_1}-1)^n(T^{b_1}-1)^n}.$$

Since it is polynomial, either $b_1 + 1$ or $a_1 + 1$ is a multiple of b_1. If $b_1 m = b_1 + 1$, then $(m, b_1) = (2, 1)$ and we obtain $(a_1, \cdots, a_n, b_n, \cdots, b_1) = (1, \cdots 1, 1, \cdots, 1)$. If $a_1 = b_1$, the same result is obtained. Now suppose that $b_1 m = a_1 + 1 < b_1 + 1$. It is easy to verify that $m = 1$ and $b_1 = a_1 + 1$. Then,

$$\chi(T) = \frac{(T^{a_1+2}-1)^n}{(T^{a_1}-1)^n}.$$

Since $a_1 + 2$ is a multiple of a_1, we have $a_1 m = a_1 + 2$. This provides $a_1 = 1$ or 2 (we need not use (W1)).

(ii) In what follows, we suppose that $b_1 > 1$. In this case, $b_j > 1$ for any $j = 1, \cdots, n$ due to the assumption $1 \leq a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$ and (W2).

Step 1. Since $\chi(T)$ is polynomial, there is a multiple of $b_{j(1)}$ among exponents $b_{j(l)} + 1$ in the numerator. If $b_{j(1)} m = b_{j(1)} + 1$, then $(m, b_{j(1)}) = (2, 1)$ and it contradicts the assumption $b_{j(1)} = b_1 > 1$.

If $b_{j(1)} m = b_{j(l)} + 1 < b_{j(1)} + 1$ for some $l > 1$, it is easy to verify $m = 1, l = 2$ and $b_{j(1)} = b_{j(2)} + 1$. There are J_1 factors $T^{b_{j(1)}} - 1$ in the denominator. This implies that $2J_2 \geq J_1$ when $N = 1$ and $a_n = b_n$, and $J_2 \geq J_1$ otherwise.

Step 2. Now we assume that for some $r \leq N$, $b_{j(i)} = b_{j(i+1)} + 1$ holds for $i = 1, \cdots, r-1$. There exists a multiple of $b_{j(r)}$ among $b_{j(l)} + 1$. If $l \leq r$, we have

$$b_{j(r)}m = b_{j(l)} + 1 = b_{j(l+1)} + 2 = \cdots = b_{j(r)} + r - l + 1,$$

which yields

$$1 < b_{j(r)} \leq r - l + 1 \leq r.$$

This proves $b_{j(r)} = b_n = a_n = r$ (otherwise, a_1 becomes nonpositive). Hence, $r = N + 1$, which contradicts the assumption $r \leq N$.

If $b_{j(r)}m = b_{j(l)} + 1$ for some $l > r$, then $m = 1, l = r + 1$ and $b_{j(r)} = b_{j(r+1)} + 1$. There are J_r factors $T^{b_{j(r)}} - 1$ in the denominator. This implies that $2J_{r+1} \geq J_r$ when $r = N$ and $a_n = b_n$, and $J_{r+1} \geq J_r$ otherwise.

Step 3. By induction, we obtain $b_{j(i)} = b_{j(i+1)} + 1$ for $i = 1, \cdots, N$, and $J_{i+1} \geq J_i$ for $i = 1, \cdots, N-1$. In particular, if $a_n \neq b_n$, $J_{N+1} \geq J_N$ also holds.

Step 4. There exists a multiple of $b_{j(N+1)} = b_n$ among exponents of the numerator. Suppose $b_{j(N+1)}m = b_{j(l)} + 1$ for some $l = 1, \cdots, N+1$. The same argument as Step 2 shows that $a_n = b_n$. Suppose $b_{j(N+1)}m = a_{j(l)} + 1 < b_{j(N+1)} + 1$ for some $l = 1, \cdots, N+1$. Then, we obtain $m = 1, l = N + 1$ and $b_{j(N+1)} = b_n = a_n + 1$. This completes the proof of (ii).

(iii) This is verified by a direct calculation with the aid of (ii). □

Proposition 2.4. When $n = 3$, regular weights satisfying (W1) to (W3) are only

$$\begin{aligned}(a_1, a_2, a_3, b_3, b_2, b_1; h) \;=\; & (2,3,4,5,6,7;10),\\ & (2,3,3,4,4,5;8),\\ & (1,2,3,4,5,6;8),\\ & (1,2,3,3,4,5;7),\\ & (2,2,2,3,3,3;6),\\ & (1,2,2,3,3,4;6),\\ & (1,1,2,2,3,3;5),\\ & (1,1,1,2,2,2;4),\\ & (1,1,1,1,1,1;3),\end{aligned}$$

where we assume without loss of generality that $a_1 \leq a_2 \leq a_3 \leq b_3 \leq b_2 \leq b_1$.

This proposition is easily obtained with the aid of Lemma 2.3. To find corresponding Painlevé equations is a future work. The weights of 6-dim Painlevé equations $(\mathrm{P_I})_3, (\mathrm{P_{II\text{-}1}})_3, (\mathrm{P_{II\text{-}2}})_3$ and $(\mathrm{P_{IV}})_3$ shown in Table 3 are included in Prop.2.4. The author does not know a Painlevé equation whose Hamiltonian function is semi-quasihomogeneous but its degree does not satisfy (W1) to (W3).

2.4 Properties of weights for semi-quasihomogeneous Hamiltonian systems

We gave the definition of a regular weight which is independent of differential equations so far. Now let us consider the $2n$-dimensional Hamiltonian system

$$\frac{dq_i}{dz} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dz} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \cdots, n, \tag{2.10}$$

with the Hamiltonian function $H(q_1, \cdots, q_n, p_1, \cdots, p_n, z)$ with a single time variable z for simplicity. We suppose the following.

(A1) H is semi-quasihomogeneous; it is decomposed into two polynomials as $H = H^P + H^N$. For the principal part H^P, there exist integers $1 \leq a_i, b_i, r < h$ such that

$$H^P(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^h H^P(q, p, z), \tag{2.11}$$

for any $\lambda \in \mathbb{C}$, where $\lambda^a q = (\lambda^{a_1} q_1, \cdots, \lambda^{a_n} q_n)$ and $\lambda^b p = (\lambda^{b_1} p_1, \cdots, \lambda^{b_n} p_n)$.
(A2) The Hamiltonian vector field of H^P satisfies

$$\frac{\partial H^P}{\partial p_i}(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^{1+a_i} \frac{\partial H^P}{\partial p_i}(q, p, z), \quad \frac{\partial H^P}{\partial q_i}(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^{1+b_i} \frac{\partial H^P}{\partial q_i}(q, p, z).$$

(A3) The non-principal part satisfies $H^N(\lambda^a q, \lambda^b p, \lambda^r z) \sim o(\lambda^h)$ as $|\lambda| \to \infty$.
(A4) The Hamiltonian vector field of $H = H^P + H^N$ is invariant under the $\mathbb{Z}_s$ action

$$(q_j, p_j, z) \mapsto (\omega^{a_j} q_j, \omega^{b_j} p_j, \omega^r z), \tag{2.12}$$

where $s = h - 1$ and $\omega := e^{2\pi i/s}$.
(A5) The symplectic form $\sum_{j=1}^{n} dq_j \wedge dp_j + dz \wedge dH$ is also invariant under the same $\mathbb{Z}_s$-action, for which $H \mapsto \omega^h H$.

From these assumptions, we will explain some of the properties of weights shown in Section 1.

Remark. The assumption (A2) is used to define the Kovalevskaya exponents in the next section. In this case, we can construct Laurent series solutions of (2.10) systematically. Due to the assumptions (A1), (A2) and (A5), it is easy to show that the Hamiltonian vector field of H^P is invariant under the action (2.12). The assumption (A4) requires that the vector field of H^N is also invariant under the action. Then, Eq.(2.10) induces a rational differential equation on the weighted projective space $\mathbb{C}P^{2n+1}(a, b, r, s)$ [6, 5].

In what follows, we assume $h \geq 3$ (if $h \leq 2$, Eq.(2.10) is linear).

Proposition 2.5. Suppose that Eq.(2.10) satisfies (A1) to (A5) and $h \geq 3$. Then,
(i) $a_i + b_i = h - 1$ for $i = 1, \cdots, n$,
(ii) $r = h - 2$,

(iii) $\deg(H^N) = 1$,
(iv) if Eq.(2.10) is non-autonomous, $\min\limits_{1\leq i\leq n}\{a_i, b_i\} = 1$ or 2.

Proof. The first statement (i) immediately follows from (A1) and (A2).

(ii) Because of (A5), there exists an integer N such that $r+h = N(h-1)$. Since $r < h$, we obtain $0 < r = N(h-1) - h < h$. This yields $h < N/(N-2)$ if $N \neq 2$. This contradicts the assumption $h \geq 3$. Therefore, $N = 2$, which proves $r = h-2$.

(iii) Let $q_1^{\mu_1}\cdots q_n^{\mu_n}p_1^{\nu_1}\cdots p_n^{\nu_n}z^{\eta}$ be a monomial included in H^N. Due to (A3), the exponents satisfy

$$0 \leq \sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta \leq h-1.$$

Further, (A4) implies that there exists an integer N such that

$$\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta - a_j - b_j + r = N(h-1).$$

This and (i),(ii) give

$$\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta = N(h-1) + 1.$$

Hence, we obtain $0 \leq N(h-1)+1 \leq h-1$. This proves $N = 0$ and $\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta = 1$.

(iv) Suppose that H includes z. Since $\deg(H) = h$ and $\deg(z) = h-2$, z is multiplied by a function whose weighted degree is 2. It exists only when $\min\limits_{1\leq i\leq n}\{a_i, b_i\} = 1$ or 2. □

3 Kovalevskaya exponents

Kovalevskaya exponents are the most important invariants of a quasihomogeneous vector field related to the Painlevé test. Here, we give a brief review of properties of them according to [5]. Let us consider the system of differential equations on $\mathbb{C}^m$

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_m, z) + g_i(x_1, \cdots, x_m, z), \quad i = 1, \cdots, m, \tag{3.1}$$

where f_i and g_i are polynomials in $(x_1, \cdots, x_m, z) \in \mathbb{C}^{m+1}$. We suppose that

(K1) $(f_1, \cdots, f_m)$ is a quasihomogeneous vector field satisfying

$$f_i(\lambda^{a_1}x_1, \cdots, \lambda^{a_m}x_m, \lambda^r z) = \lambda^{1+a_i}f_i(x_1, \cdots, x_m, z) \tag{3.2}$$

for any $\lambda \in \mathbb{C}$ and $i = 1, \cdots, m$, where $(a_1, \cdots, a_m, r) \in \mathbb{Z}_{>0}^{m+1}$.

(K2) $(g_1, \cdots, g_m)$ satisfies

$$g_i(\lambda^{a_1}x_1, \cdots, \lambda^{a_m}x_m, \lambda^r z) = o(\lambda^{a_i+1}), \quad |\lambda| \to \infty.$$

Put $f_i^A(x_1, \cdots, x_m) := f_i(x_1, \cdots, x_m, 0)$ and $f_i^{NA} := f_i - f_i^A$ (i.e. f_i^A and f_i^{NA} are autonomous and nonautonomous parts, respectively). We also consider the truncated system

$$\frac{dx_i}{dz} = f_i^A(x_1, \cdots, x_m), \quad i = 1, \cdots, m. \tag{3.3}$$

By substituting $x_i(z) = c_i(z-z_0)^{-a_i}$ into the truncated system, we find the following definition.

Definition 3.1. A root $c = (c_1, \cdots, c_m) \in \mathbb{C}^m$ of the equation

$$-a_i c_i = f_i^A(c_1, \cdots, c_m), \quad i = 1, \cdots, m \tag{3.4}$$

is called the indicial locus.

For each indicial locus, $x_i(z) = c_i(z-z_0)^{-a_i}$ is an exact solution of the truncated system for any $z_0 \in \mathbb{C}$. Due to the assumption (K1), $c = 0$ is always an indicial locus, which corresponds to the fixed point at the origin. Usually, we assume $c \neq 0$ for an indicial locus. Considering the variational equation along the exact solution $x_i(z) = c_i(z-z_0)^{-a_i}$ suggests the following definition.

Definition 3.2. For an indicial locus $c = (c_1, \cdots, c_m) \neq 0$, the matrix

$$K = K(c) := \left\{ \frac{\partial f_i^A}{\partial x_j}(c_1, \cdots, c_m) + a_i\delta_{ij} \right\}_{i,j=1}^m \tag{3.5}$$

and its eigenvalues are called the Kovalevskaya matrix and the Kovalevskaya exponents, respectively, of the system (3.1) associated with c.

Proposition 3.3 (see [2, 5, 17] for the detail.) Suppose (K1) and (K2).
(i) -1 is always a Kovalevskaya exponent with the eigenvector $(a_1c_1, \cdots, a_mc_m)^T$.
(ii) $\lambda = 0$ is a Kovalevskaya exponent associated with c if and only if c is not an isolated root of the equation $-a_ic_i = f_i^A(c_1, \cdots, c_m)$.
(iii) The Kovalevskaya exponents are invariant under weight preserving diffeomorphisms.

Consider a formal power series solution of Eq.(3.1) of the form

$$x_i = c_i(z-z_0)^{-a_i} + b_{i,1}(z-z_0)^{-a_i+1} + b_{i,2}(z-z_0)^{-a_i+2} + \cdots \tag{3.6}$$

Coefficients $b_{i,j}$ are determined by substituting it into Eq.(3.1). The column vector $b_j = (b_{1,j}, \cdots, b_{m,j})^T$ satisfies

$$(K - jI)b_j = (\text{a function of } c_i \text{ and } b_{i,k} \text{ with } k < j). \tag{3.7}$$

If a positive integer j is not an eigenvalue of K, b_j is uniquely determined. If a positive integer j is an eigenvalue of K and (3.7) has no solutions, we have to introduce a logarithmic term $\log(z-z_0)$ into the coefficient b_j. In this case, the system (3.1) has no Laurent series solution of the form (3.6) with a given indicial locus c. If a positive integer j is an eigenvalue of K and (3.7) has a solution b_j, then $b_j + v$ is also a solution for any eigenvectors v. This implies that the power series solution (3.6) includes a free parameter in $(b_{1,j}, \cdots, b_{m,j})$. Therefore, if (3.6) represents a k-parameter family of formal Laurent series solutions which includes $k-1$ free parameters other than z_0, at least $k-1$ Kovalevskaya exponents have to be nonnegative integers. Hence, the classical Painlevé test [1, 17, 33] for the necessary condition for the Painlevé property is stated as follows;

Classical Painlevé test. If the system (3.1) satisfying (K1) and (K2) has the Painlevé property in a sense that any solutions are meromorphic, then there exists an indicial locus $c = (c_1, \cdots, c_m)$ such that all Kovalevskaya exponents except for one -1 are nonnegative integers (such an indicial locus is called principal), and the Kovalevskaya matrix is semisimple. In this case, (3.6) represents an m-parameter family of formal Laurent series solutions.

Due to (K1), the system $dx_i/dt = f_i(x_1, \cdots, x_m, z)$ is invariant under the $\mathbb{Z}_s$ action

$$(x_1, \cdots, x_m, z) \mapsto (\omega^{a_1} x_1, \cdots, \omega^{a_m} x_m, \omega^r z), \quad \omega := e^{2\pi i/s}, \tag{3.8}$$

where $s = r + 1$. We assume that the full system (3.1) is also invariant under the same action (i.e. the perturbation term g_i admit the same $\mathbb{Z}_s$ action as f_i);

(K3) The system (3.1) is invariant under the above $\mathbb{Z}_s$ action.

Proposition 3.4. Suppose (K1) to (K3). If the system (3.1) has a formal power series solution (3.6), then it is a convergent power series on $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$. In particular, when $g_i = 0$ $(i = 1, \cdots, m)$ this is true without the assumption (K3).

This proposition is shown in [18] for autonomous systems and extended to nonautonomous systems (3.1) in [5] by using the weighted projective space $\mathbb{C}P^{m+1}(a_1, \cdots, a_m, r, s)$. The assumption (K1) and (K3) are used to confirm that the system (3.1) induces a rational vector field on $\mathbb{C}P^{m+1}(a_1, \cdots, a_m, r, s)$. The classical Painlevé test gives the *necessary condition* that (3.1) has an m-parameter family of *formal* Laurent series solutions. Prop.3.4 means that *if* a formal power series solution of the form (3.6) exists, it is convergent. In Prop.3.5 of Chiba [5], the *necessary and sufficient condition* that (3.1) has a k-parameter family of *convergent* Laurent series solution (3.6) is given under the assumption (K1) to (K3) with the aid of the weighted projective space, Kovalevskaya exponents and the normal form theory of dynamical systems.

For the next theorem, we further assume that

(S) The origin is the only fixed point of the truncated system (3.3), i.e,

$$f_i^A(x_1, \cdots, x_m) = 0 \quad (i = 1, \cdots, m) \Rightarrow (x_1, \cdots, x_m) = (0, \cdots, 0). \tag{3.9}$$

Theorem 3.5. [5] If the system (3.1) satisfies (K1), (K2) and (S), any formal Laurent series solutions with a pole at $z = z_0$ are of the form (3.6) such that $(c_1, \cdots, c_m) \neq (0, \cdots, 0)$. If we further assume (K3), they are convergent (due to Prop.3.4).

This theorem means that there are no Laurent series solution $(x_1(z), \cdots, x_m(z))$ of (3.1) such that the order of a pole of x_i is larger than a_i for some i (For the proof, (S) is essentially used). Furthermore, if $(c_1, \cdots, c_m) = 0$ (i.e. the orders of a pole of $x_1, \cdots, x_m$ are smaller than $a_1, \cdots, a_m$), it should be a local analytic solution. Therefore, the leading term of a Laurent series solution is strictly given by $c_i(z - z_0)^{-a_i}$ with a given weight $(a_1, \cdots, a_m)$ and an indicial locus $(c_1, \cdots, c_m) \neq 0$.

In the rest of this section, we consider the semi-quasihomogeneous Hamiltonian system (2.10). If it satisfies (A1) to (A5), then it also satisfies (K1) to (K3) and the above results are applicable. Further, the assumption (S) implies that a singularity of the algebraic variety defined by $\{H = 0\}$ is isolated. This fact is used to study a relationship between the Painlevé equations and singularity theory (see Eq.(1.8)). The next lemma is well known [4, 17, 19].

Lemma 3.6. For a semi-quasihomogeneous Hamiltonian system (2.10) of $\deg(H) = h$ satisfying (A1) and (A2), if κ is a Kovalevskaya exponent, so is μ given by $\kappa + \mu = h - 1$. In particular, h is always a Kovalevskaya exponent for any indicial loci.

Example 3.7. The first Painlevé equation in Hamiltonian form is given by

$$(\mathrm{P_I}) \begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\ \dfrac{dy}{dz} = x, \end{cases}$$

It satisfies the assumptions (A1) to (A5) as is mentioned with $(a_1, a_2; h) = (3, 2; 6)$ (Table 1). The indicial locus is uniquely given by $(c_1, c_2) = (-2, 1)$. The associated Laurent series solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} T^{-3} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} T^{-2} - \begin{pmatrix} z_0/5 \\ 0 \end{pmatrix} T - \begin{pmatrix} 1/2 \\ z_0/10 \end{pmatrix} T^2 + \begin{pmatrix} A_6 \\ -1/6 \end{pmatrix} T^3 + \cdots,$$

where $T = z - z_0$ and A_6 is an arbitrary constant. Lemma 3.6 shows that $h = 6$ is a Kovalevskaya exponent. As a result, an arbitrary constant appears in the sixth place from the beginning (i.e. in the coefficient of $T^{-3+h} = T^3$).

We give a list of Kovalevskaya exponents of 4-dim Painlevé equations shown in Section 2.2. In Table 4, $H_1^{9/2}, H_1^{7/2+1}, H_1^5$ and H_1^{4+1} denote the first Hamiltonians of $(\mathrm{P_I})_2, (\mathrm{P_{II\text{-}1}})_2, (\mathrm{P_{II\text{-}2}})_2$ and $(\mathrm{P_{IV}})_2$, respectively, given in Section 1 (this notation is related to the spectral type of a monodromy preserving deformation [21]). For

example, $(-1,2,3,6)\times 2$ in Table 4 implies that there are two indicial loci c, for which the associated Kovalevskaya exponents are $\kappa = -1, 2, 3$ and 6. Since Kovalevskaya exponents are invariant under weight preserving diffeomorphisms, we can conclude that two Hamiltonian systems having the same weights are actually different systems if their Kovalevskaya exponents are different from each other.

For a differential equation $dx/dz = f(x,z)$ on $(x,z) \in \mathbb{C}^{m+1}$, an m-dim manifold $\mathcal{M}(z)$ parameterized by z is called the space of initial conditions if any solutions of the system give global holomorphic sections of the fiber bundle $\mathcal{P} = \{(x,z) \,|\, x \in \mathcal{M}(z), z \in \mathbb{C}\}$ over $\mathbb{C}$ [28]. In particular, the space of initial conditions exists for a system having the Painlevé property in the sense that any solutions are meromorphic. Many experts believe that the Painlevé equations can be classified by the geometry of the space of initial conditions, which was confirmed for two dimensional Painlevé equations by Sakai [30] and Takano et al. [32, 24, 25]. In Chiba [5], an algorithm to construct the space of initial conditions for semi-quasihomogeneous systems is obtained by the weighted blow-up of the weighted projective space. The weight for the weighted projective space is just the weight of the variables, and the weight for the blow-up is given by Kovalevskaya exponents. This suggests the conjecture that polynomial systems having the Painlevé property can be classified by their weights and Kovalevskaya exponents.

For 2-dim Painlevé equations, we have constructed the Painlevé equations $P_{\rm I}, P_{\rm II}$ and $P_{\rm IV}$ from the weights (Prop.2.1). In this case, the Kovalevskaya exponent is given by h (Lemma 3.6), which is included in the information of the weight $(a,b;h)$. For 4-dim Painlevé equations listed in Table 4, they are classified by the weights with Kovalevskaya exponents. Thus, the above conjecture looks true at least up to four dimensional quasihomogeneous systems.

As a convenience for readers, we provide a few 4-dim Painlevé equations whose Hamiltonian functions are polynomial, but the weights are not positive integers. Thus, they do not satisfy the assumption (W3).

$$\begin{aligned} H_{\rm IV}^{\rm Mat} &= \frac{p_1^2 q_1}{2} - p_1 q_1^2 + p_1 q_2 + 2p_1 p_2 q_2 - 4p_2 q_1 q_2 - 2p_2^2 q_1 q_2 - p_1 q_1 z - 2p_2 q_2 z \\ &\quad + 2p_2 q_1 \theta_0 - p_1 \theta_1 + 2p_2 q_1 \theta_1 - p_1 \theta_2 + 2q_1 \theta_2 + 2p_2 q_1 \theta_2, \qquad (3.10) \\ H^{(1,2,1,0)} &= -p_1^2 q_1 - 2p_1 q_1^2 + 2p_1 q_2 - 2p_1 p_2 q_2 - 2p_2 q_1 q_2 \\ &\quad + (2p_1 q_1 + 2p_2 q_2) z + (2\alpha_2 + 2\beta_2) q_1 + 2\beta_2 p_1 + 2\beta_3 p_2, \qquad (3.11) \\ H^{(-1,1,4,2)} &= p_1 - p_2^2 - 2p_1 q_1 q_2 - p_2 q_2^2 + 2\beta_3 q_2 + 2\beta_5 q_1 + p_2 z. \qquad (3.12) \end{aligned}$$

The first one $H_{\rm IV}^{\rm Mat}$, whose degree is $\deg(q_1, q_2, p_1, p_2; h) = (1,2,1,0;3)$, is the matrix Painlevé equation of the fourth type $H_{\rm IV}^{\rm Mat}$ [20, 21]. $H^{(1,2,1,0)}$ $(h=3)$ and $H^{(-1,1,4,2)}$ $(h=4)$ are obtained in [8] by a Lie algebraic method as well as $H_{\rm Cosgrove}$. Although the weights are nonpositive, they still satisfy (W2) and (A1) to (A5). See also Table 4.

	$(a_1, a_2, b_2, b_1; h)$	κ	
$H_1^{9/2}$	$(2, 3, 4, 5; 8)$	$(-1, 2, 5, 8)$	$\times 1$
Eq.(1.4)		$(-3, -1, 8, 10)$	$\times 1$
H_{Cosgrove}	$(2, 3, 4, 5; 8)$	$(-1, 3, 4, 8)$	$\times 1$
Eq.(2.2)		$(-5, -1, 8, 12)$	$\times 1$
$H_1^{7/2+1}$	$(1, 2, 3, 4; 6)$	$(-1, 2, 3, 6)$	$\times 2$
Eq.(1.5)		$(-3, -1, 6, 8)$	$\times 2$
$H_{\text{I}}^{\text{Mat}}$	$(1, 2, 3, 4; 6)$	$(-1, 2, 3, 6)$	$\times 2$
Eq.(2.4)		$(-2, -1, 6, 7)$	$\times 1$
		$(-7, -1, 6, 12)$	$\times 1$
$H_1^{(2,3,2,3)}$	$(2, 2, 3, 3; 6)$	$(-1, 1, 4, 6)$	$\times 1$
Eq.(2.6)		$(-3, -1, 6, 8)$	$\times 2$
H_1^5	$(1, 2, 2, 3; 5)$	$(-1, 1, 3, 5)$	$\times 2$
Eq.(1.6)		$(-2, -1, 5, 6)$	$\times 3$
H_1^{4+1}	$(1, 1, 2, 2; 4)$	$(-1, 1, 2, 4)$	$\times 3$
Eq.(1.7)		$(-2, -1, 4, 5)$	$\times 5$
$H_{\text{II}}^{\text{Mat}}$	$(1, 1, 2, 2; 4)$	$(-1, 1, 2, 4)$	$\times 3$
Eq.(2.7)		$(-2, -1, 4, 5)$	$\times 2$
		$(-5, -1, 4, 8)$	$\times 2$
		$(-1, -1, 4, 4)$	$\times 1$
$H_{\text{NY}}^{A_4}$	$(1, 1, 1, 1; 3)$	$(-1, 1, 1, 3)$	$\times 5$
Eq.(2.8)		$(-1, -1, 3, 3)$	$\times 5$
		$(-3, -1, 3, 5)$	$\times 5$
$H_{\text{IV}}^{\text{Mat}}$	$(0, 1, 1, 2; 3)$	$(-1, 1, 1, 3)$	$\times 2$
Eq.(3.10)		$(-1, -1, 3, 3)$	$\times 3$
		$(-2, -1, 3, 4)$	$\times 2$
		$(-4, -1, 3, 6)$	$\times 3$
$H^{(1,2,1,0)}$	$(0, 1, 1, 2; 3)$	$(-1, 1, 1, 3)$	$\times 2$
Eq.(3.11)		$(-2, -1, 3, 4)$	$\times 4$
$H^{(-1,1,4,2)}$	$(-1, 1, 2, 4; 4)$	$(-1, 1, 2, 4)$	$\times 2$
Eq.(3.12)		$(-3, -1, 4, 6)$	$\times 2$

Table 4: Weights and Kovalevskaya exponents κ of 4-dim Painlevé equations. The weights for dependent variables q_1, q_2, p_1, p_2 are shown in ascending order. For example, $(-1, 2, 3, 6) \times 2$ means that there are two indicial loci whose Kovalevskaya exponents are given by $\kappa = -1, 2, 3, 6$.

4 Blow-up of quasihomogeneous systems

Let us investigate the role of Kovalevskaya exponents for quasihomogeneous systems from a view point of dynamical systems theory. Since the Kovalevskaya exponents are defined by the autonomous part of a quasihomogeneous system, we consider the following autonomous system

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_m), \quad i = 1, \cdots, m \tag{4.1}$$

satisfying the assumptions (K1) and (S) for the weight $(a_1, \cdots, a_m) \in \mathbb{Z}^m_{>0}$. For an indicial locus $c = (c_1, \cdots, c_m) \neq 0 \in \mathbb{C}^m$ given as a root of $-a_i c_i = f_i(c_1, \cdots, c_m)$, $x_i(z) = c_i z^{-a_i}$ is an exact solution.

We introduce the weighted blow-up $\pi : B \to \mathbb{C}^m$ of the system (4.1) at the origin by the coordinates transformations

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} r_1^{a_1} \\ r_1^{a_2} X_2^{(1)} \\ \vdots \\ r_1^{a_m} X_m^{(1)} \end{pmatrix} = \begin{pmatrix} r_2^{a_1} X_1^{(2)} \\ r_2^{a_2} \\ \vdots \\ r_2^{a_m} X_m^{(2)} \end{pmatrix} = \cdots = \begin{pmatrix} r_m^{a_1} X_1^{(m)} \\ r_m^{a_2} X_2^{(m)} \\ \vdots \\ r_m^{a_m} \end{pmatrix}, \tag{4.2}$$

and the blow-up space B by

$$B = B_1 \cup B_2 \cup \cdots \cup B_m, \quad B_j \simeq \mathbb{C}^m / \mathbb{Z}_{a_j}.$$

Here, the space $\mathbb{C}^m/\mathbb{Z}_{a_j}$ is defined as follows: Let $(r_1, X_2^{(1)}, \cdots, X_m^{(1)})$ be the coordinates of $\mathbb{C}^m$. Then, B_1 is defined as a quotient space by the $\mathbb{Z}_{a_1}$ action

$$(r_1, X_2^{(1)}, \cdots, X_m^{(1)}) \mapsto (e^{2\pi i/a_1} r_1, e^{-2\pi i a_2/a_1} X_2^{(1)}, \cdots, e^{-2\pi i a_m/a_1} X_m^{(1)}), \tag{4.3}$$

and similar for $B_2, \cdots, B_m$. Let $\pi : B \to \mathbb{C}^m$ be the surjection defined through (4.2). The exceptional divisor

$$D := \pi^{-1}(\{0\}) = \{r_1 = 0\} \cup \{r_2 = 0\} \cup \cdots \cup \{r_m = 0\} \subset B \tag{4.4}$$

is isomorphic to the $m-1$ dimensional weighted projective space $\mathbb{C}P^{m-1}(a_1, \cdots, a_m)$, and $\pi|_{B\setminus D} : B\setminus D \to \mathbb{C}^m\setminus\{0\}$ is a diffeomorphism. Since $(c_1, \cdots, c_m) \neq (0, \cdots, 0)$, we assume $c_1 \neq 0$ and denote the first local coordinates $(r_1, X_2^{(1)}, \cdots, X_m^{(1)})$ on the chart B_1 as $(r, X_2, \cdots, X_m)$ for simplicity. In this coordinates, Eq.(4.1) is written as

$$\begin{cases} \dfrac{dr}{dz} = \dfrac{1}{a_1} r^2 f_1(1, X_2, \cdots, X_m) \\ \dfrac{dX_i}{dz} = r f_i(1, X_2, \cdots, X_m) - \dfrac{a_i}{a_1} r X_i f_1(1, X_2, \cdots, X_m), \quad i = 2, \cdots, m. \end{cases}$$

A new independent variable t satisfying the relation $d/dz = r \cdot d/dt$ is introduced, that results in

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{1}{a_1} r f_1(1, X_2, \cdots, X_m) \\ \dfrac{dX_i}{dt} = f_i(1, X_2, \cdots, X_m) - \dfrac{a_i}{a_1} X_i f_1(1, X_2, \cdots, X_m), \quad i = 2, \cdots, m. \end{cases} \tag{4.5}$$

We regard it as a vector field on B_1. The set $\{(0, X_2, \cdots, X_m)\} \subset D$ is an invariant manifold.

Lemma 4.1. (i) For an indicial locus $(c_1, \cdots, c_m)$ of (4.1) with $c_1 \neq 0$,

$$(r, X_2, \cdots, X_m) = (0, c_1^{-a_2/a_1} c_2, \cdots, c_1^{-a_m/a_1} c_m) \tag{4.6}$$

is a fixed point of the vector field (4.5). Conversely, for any fixed point $(0, X_2, \cdots, X_m)$ of (4.5) on the divisor, there exists an indicial locus $(c_1, \cdots, c_m)$ satisfying (4.6).
(ii) For an indicial locus c, the exact solution $x_i(z) = c_i z^{-a_i}$, $(i = 1, \cdots, m)$ on the blow-up space converges to the fixed point (4.6) as $z \to \infty$.

Proof. (i) A fixed point satisfying $r = 0$ is given by a root of the equation

$$a_1 f_i(1, X_2, \cdots, X_m) - a_i X_i f_1(1, X_2, \cdots, X_m) = 0, \ (i = 2, \cdots, m). \tag{4.7}$$

If there is a root $(X_2, \cdots, X_m)$ satisfying $f_1(1, X_2, \cdots, X_m) = 0$, then $f_i(1, X_2, \cdots, X_m) = 0$ for $i = 2, \cdots, m$. This contradicts the assumption (S). Thus, there is a number $\lambda \neq 0$ such that (4.7) is equivalent to

$$\begin{cases} f_i(1, X_2, \cdots, X_m) = -a_i X_i \lambda^{-1} \\ f_1(1, X_2, \cdots, X_m) = -a_1 \lambda^{-1}. \end{cases} \tag{4.8}$$

By using the assumption (K1), we rewrite the above equation as

$$\begin{aligned} & \lambda^{a_i+1} f_i(1, X_2, \cdots, X_m) \\ = \ & f_i(\lambda^{a_1}, \lambda^{a_2} X_2, \cdots, \lambda^{a_m} X_n) \\ = \ & -\lambda^{a_i+1} a_i X_i \lambda^{-1} = -a_i \lambda^{a_i} X_i, \end{aligned}$$

for $i = 2, \cdots, m$, and

$$\begin{aligned} & \lambda^{a_1+1} f_1(1, X_2, \cdots, X_m) \\ = \ & f_1(\lambda^{a_1}, \lambda^{a_2} X_2, \cdots, \lambda^{a_m} X_n) \\ = \ & -\lambda^{a_1+1} a_1 \lambda^{-1} = -a_1 \lambda^{a_1}. \end{aligned}$$

By putting $\lambda^{a_1} = c_1$ and $\lambda^{a_i} X_i = c_i$, it turns out that (4.8) is equivalent to the equation $-a_i c_i = f_i(c_1, \cdots, c_m)$ to determine an indicial locus. A proof of (ii) is straightforward. □

Note that the choice of a branch of $c_1^{-a_j/a_1}$ does not matter because of the $\mathbb{Z}_{a_1}$ action (4.3). When $c_1 = 0$, there are no fixed points in the chart B_1 but exists in B_j when $c_j \neq 0$. In this manner, there is a one-to-one correspondence between indicial loci c and fixed points of the vector field induced on the divisor, denoted by $\mathrm{P}(c)$. If we do not assume (S), there is a fixed point of (4.5) on the divisor, which results not from an indicial locus but from a fixed point of (4.1) other than the origin. The next proposition associates the Kovalevskaya exponents with the local dynamics around a fixed point of the vector field.

Proposition 4.2. Let $\kappa_1 = -1, \kappa_2, \cdots, \kappa_m$ be Kovalevskaya exponents of the system (4.1) associated with an indicial locus $c = (c_1, \cdots, c_m)$. The eigenvalues of the Jacobi matrix of the vector field (4.5) at the fixed point $\mathrm{P}(c)$ are given by

$$\lambda_1 = -c_1^{-1/a_1},\ \lambda_2 = c_1^{-1/a_1}\kappa_2,\ \cdots,\ \lambda_m = c_1^{-1/a_1}\kappa_m. \tag{4.9}$$

Hence, the ratio of eigenvalues is the same as that of the Kovalevskaya exponents.

Proof. Let K be the Kovalevskaya matrix for an indicial locus c. Set $v_1 = a_1c_1$ and $v_2 = (a_2c_2, \cdots, a_mc_m)$. Then, $(v_1, v_2)^T$ is an eigenvector of K associated with $\kappa_1 = -1$ (Prop.3.3). Define

$$P = \begin{pmatrix} v_1 & 0 \\ v_2^T & \mathrm{id} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} v_1^{-1} & 0 \\ -v_1^{-1}v_2^T & \mathrm{id} \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

We obtain

$$P^{-1}KP = \begin{pmatrix} -1 & v_1^{-1}K_2 \\ 0 & K_4 - v_1^{-1}v_2^TK_2 \end{pmatrix} =: \begin{pmatrix} -1 & v_1^{-1}K_2 \\ 0 & \tilde{K} \end{pmatrix},$$

where an $(m-1)\times(m-1)$ matrix $\tilde{K} = (\tilde{K}_{ij})_{i,j=2}^m$ is given by

$$\tilde{K}_{ij} = \frac{\partial f_i}{\partial x_j}(c) - \frac{a_ic_i}{a_1c_1}\frac{\partial f_1}{\partial x_j}(c) + a_i\delta_{ij}. \tag{4.10}$$

By the definition, eigenvalues of $\tilde{K}$ are $\kappa_2, \cdots, \kappa_m$.

On the other hand, the Jacobi matrix of (4.5) at the fixed point $\mathrm{P}(c)$ is given by

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \tilde{J} \end{pmatrix},$$

where

$$\lambda_1 = \frac{1}{a_1}f_1(1, c_1^{-a_2/a_1}c_2, \cdots, c_1^{-a_m/a_1}c_m) = \frac{1}{a_1}c_1^{-(a_1+1)/a_1}f_1(c_1, \cdots, c_m) = -c_1^{-1/a_1},$$

and

$$\begin{aligned}\tilde{J}_{ij} &= \frac{\partial f_i}{\partial x_j}(1, c_1^{-a_2/a_1}c_2, \cdots, c_1^{-a_m/a_1}c_m) - \frac{a_i}{a_1}X_i\frac{\partial f_1}{\partial x_j}(1, c_1^{-a_2/a_1}c_2, \cdots, c_1^{-a_m/a_1}c_m) \\ &\quad -\frac{a_i}{a_1}f_1(1, c_1^{-a_2/a_1}c_2, \cdots, c_1^{-a_m/a_1}c_m)\delta_{ij} \\ &= c_1^{-(a_i+1-a_j)/a_1}\frac{\partial f_i}{\partial x_j}(c) - \frac{a_i}{a_1}c_1^{-a_i/a_1}c_i c_1^{-(a_1+1-a_j)/a_1}\frac{\partial f_1}{\partial x_j}(c) - \frac{a_i}{a_1}c_1^{-(a_1+1)/a_1}f_1(c)\delta_{ij} \\ &= c_1^{-1/a_1}c_1^{-(a_i-a_j)/a_1}\left(\frac{\partial f_i}{\partial x_j}(c) - \frac{a_i c_i}{a_1 c_1}\frac{\partial f_1}{\partial x_j}(c) + a_i\delta_{ij}\right).\end{aligned}$$

This shows

$$c_1^{1/a_1}\tilde{J}_{ij} = c_1^{-(a_i-a_j)/a_1}\tilde{K}_{ij}.$$

Let κ be an eigenvalue of $\tilde{K}$ with the eigenvector $u = (u_2, \cdots, u_m)^T$ satisfying $\sum \tilde{K}_{ij}u_j = \kappa u_i$. Putting $u_j = c_1^{a_j/a_1}\tilde{u}_j$ yields

$$\sum c_1^{-(a_i-a_j)/a_1}\tilde{K}_{ij}\tilde{u}_j = \kappa\tilde{u}_i.$$

This proves that κ is an eigenvalue of the matrix $c_1^{1/a_1}\tilde{J}$. □

We turn to the quasihomogeneous Hamiltonian system of degree m

$$\frac{dq_i}{dz} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dz} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \cdots, m. \tag{4.11}$$

We assume stronger conditions than (A1) and (A2) as follows.

(H0) There exist polynomials $H = H_1, H_2, \cdots, H_k$ $(1 \le k \le m)$ that commute with respect to the canonical Poisson structure; $\{H_i, H_j\} = 0$ for $i, j = 1, \cdots, k$.
(H1) H_i is quasihomogeneous; there exist positive integers a_j, b_j and h_i such that

$$H_i(\lambda^a q, \lambda^b p) = \lambda^{h_i}H_i(q, p), \quad i = 1, \cdots, k, \tag{4.12}$$

for any $\lambda \in \mathbb{C}$, where $\lambda^a q = (\lambda^{a_1}q_1, \cdots, \lambda^{a_m}q_m)$ and $\lambda^b p = (\lambda^{b_1}p_1, \cdots, \lambda^{b_m}p_m)$.
(H2) $h_1 = a_j + b_j + 1$ for $j = 1, \cdots, m$ and $h_1 \le h_i$ for $i = 1, \cdots, k$.
(S) The origin is the only fixed point;

$$\frac{\partial H_1}{\partial p_i} = \frac{\partial H_1}{\partial q_i} = 0 \ (i = 1, \cdots, m) \Rightarrow (q_1, \cdots, q_m, p_1, \cdots, p_m) = 0.$$

Note that (H1) and (H2) for $k = 1$ is equivalent to (A1) and (A2), see Prop.2.5 (i). Let $c = (c_1, \cdots, c_{2m})$ be an indicial locus determined by H_1 as

$$\frac{\partial H_1}{\partial p_i}(c) = -a_i c_i, \quad \frac{\partial H_1}{\partial q_i}(c) = b_i c_{m+i}, \quad i = 1, \cdots, m. \tag{4.13}$$

The Kovalevskaya matrix at c is

$$K(c) = \begin{pmatrix} \dfrac{\partial^2 H_1}{\partial p \partial q}(c) & \dfrac{\partial^2 H_1}{\partial p \partial p}(c) \\ -\dfrac{\partial^2 H_1}{\partial q \partial q}(c) & -\dfrac{\partial^2 H_1}{\partial q \partial p}(c) \end{pmatrix} + \begin{pmatrix} \mathrm{diag}(a_1, \cdots, a_m) & 0 \\ 0 & \mathrm{diag}(b_1, \cdots, b_m) \end{pmatrix},$$

where $\partial^2/\partial p \partial q = (\partial^2/\partial p_i \partial q_j)_{i,j}$.

Lemma 4.3.

$$\sum_{j=1}^{m} \left(a_j q_j \frac{\partial H_i}{\partial q_j} + b_j p_j \frac{\partial H_i}{\partial p_j} \right) = h_i H_i(q,p). \tag{4.14}$$

Proof. This is obtained by the derivative of (4.12) at $\lambda = 1$.

Lemma 4.4. For any $i = 1, \cdots, k$ and indicial loci c, we have $H_i(c) = 0$.

Proof. Use the relations (4.13), (4.14) and $\{H_1, H_j\} = 0$ at $(q,p) = c$.

In what follows, the gradient of a function H is denoted by a row vector

$$dH := \left(\frac{\partial H}{\partial q_1}, \cdots, \frac{\partial H}{\partial q_m}, \frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_m} \right). \tag{4.15}$$

The assumption (S) implies that $dH_1(q,p) = 0$ if and only if $(q,p) = 0$; i.e. the origin is a unique singularity of the variety $\{H_1 = 0\}$. The following result was first obtained by Yoshida [34]. Here, we give a simple proof.

Theorem 4.5. For an indicial locus c, the following equality

$$dH_j(c)(K(c) - h_j \cdot \mathrm{Id}_{2m \times 2m}) = 0, \quad j = 1, \cdots, k \tag{4.16}$$

holds. In particular, if $dH_j(c) \neq 0$, then h_j is a Kovalevskaya exponent.

Proof. $\{H_1, H_j\} = 0$ gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial q_l}\{H_1, H_j\} \\ &= \sum_{i=1}^{m} \left(\frac{\partial^2 H_1}{\partial q_i \partial q_l}\frac{\partial H_j}{\partial p_i} + \frac{\partial H_1}{\partial q_i}\frac{\partial^2 H_j}{\partial p_i \partial q_l} - \frac{\partial^2 H_j}{\partial q_i \partial q_l}\frac{\partial H_1}{\partial p_i} - \frac{\partial H_j}{\partial q_i}\frac{\partial^2 H_1}{\partial p_i \partial q_l} \right). \end{aligned}$$

Substituting $(q,p) = c$ yields

$$0 = \sum_{i=1}^{m} \left(a_i c_i \frac{\partial^2 H_j}{\partial q_i \partial q_l} + b_i c_{m+i} \frac{\partial^2 H_j}{\partial p_i \partial q_l} + \frac{\partial^2 H_1}{\partial q_i \partial q_l}\frac{\partial H_j}{\partial p_i} - \frac{\partial H_j}{\partial q_i}\frac{\partial^2 H_1}{\partial p_i \partial q_l} \right).$$

By the derivative of (4.14) with respect to q_l, we obtain

$$\sum_{i=1}^{m}\left(a_i q_i \frac{\partial^2 H_j}{\partial q_i \partial q_l} + b_i p_i \frac{\partial^2 H_j}{\partial p_i \partial q_l}\right) + a_l \frac{\partial H_j}{\partial q_l} = h_j \frac{\partial H_j}{\partial q_l}.$$

Thus, we obtain

$$0 = \sum_{i=1}^{m}\left(\frac{\partial^2 H_1}{\partial q_i \partial q_l}\frac{\partial H_j}{\partial p_i} - \frac{\partial^2 H_1}{\partial p_i \partial q_l}\frac{\partial H_j}{\partial q_i}\right) + (h_j - a_l)\frac{\partial H_j}{\partial q_l} \tag{4.17}$$

for $l = 1, \cdots, m$. The derivative of (4.14) with respect to p_l gives similar m relations. The resultant $2m$ relations are equivalent to (4.16). □

Example 4.6. We consider the Hamiltonians of degree 2

$$\begin{cases} H_1 = 2p_2p_1 + 3p_2^2q_1 + q_1^4 - q_1^2q_2 - q_2^2, \\ H_2 = p_1^2 + 2p_2p_1q_1 - q_1^5 + p_2^2q_2 + 3q_1^3q_2 - 2q_1q_2^2. \end{cases} \tag{4.18}$$

They are autonomous parts of $(\mathrm{P_I})_2$ given in (1.4). They satisfy (H0), (H1), (H2) and (S) for the weight $(a_1, a_2, b_1, b_2) = (2, 4, 5, 3)$ and $h_1 = 8, h_2 = 10$ shown in Table 2. There are two indicial loci $c_1 = (1, 1, 1, -1)$ and $c_2 = (3, 0, 27, -3)$. For the former c_1, the Kovalevskaya exponents are $\kappa = -1, 2, 5, 8$. Thus, the corresponding Laurent series solution (3.6) represents a general solution including four free parameters (Painlevé test). Since $\kappa \neq -10$, Thm.4.5 implies that $dH_2(c_1) = 0$. For the indicial locus c_2, we can verify that $dH_1(c_2) \neq 0$, $dH_2(c_2) \neq 0$. Therefore, Thm.4.5 and Lemma 3.6 show that the Kovalevskaya exponents are given by $\kappa = -3, -1, 8, 10$ (see $H_1^{9/2}$ of Table 4).

Let V be a variety defined by the level set

$$V = \{(q, p) \in \mathbb{C}^{2m} \,|\, H_j(q, p) = 0,\ j = 1, \cdots, k\} \ni 0. \tag{4.19}$$

Lemma 4.4 shows $c \in V$ for an indicial locus c. Because of (H1), the orbit

$$\{(\lambda^{a_1}c_1, \cdots, \lambda^{a_m}c_m, \lambda^{b_1}c_{m+1}, \cdots, \lambda^{b_m}c_{2m}) \,|\, \lambda \in \mathbb{C}\}$$

is also included in V. Let us consider the weighted blow-up $\pi : B \to \mathbb{C}^{2m}$ at the origin

$$B = B_1 \cup \cdots \cup B_{2m}, \quad B_i = \mathbb{C}^{2m}/\mathbb{Z}_{a_i},\ B_{m+i} = \mathbb{C}^{2m}/\mathbb{Z}_{b_i} \tag{4.20}$$

for $i = 1, \cdots, m$. The exceptional divisor D is a $2m-1$ dimensional weighted projective space

$$D = \pi^{-1}(\{0\}) = \mathbb{C}P^{2m-1}(a_1, \cdots, a_m, b_1, \cdots, b_m). \tag{4.21}$$

For an indicial locus c, we assume $c_1 \neq 0$ as before. The local coordinates $(r, Q_2, \cdots, Q_m, P_1, \cdots, P_m)$ on B_1 is defined by

$$\begin{aligned} &q_1 = r^{a_1},\ q_i = r^{a_i}Q_i \quad (i = 2, \cdots, m), \\ &p_i = r^{b_i}P_i \quad (i = 1, \cdots, m). \end{aligned}$$

In particular, $D \cap B_1$ is given by the set $\{r = 0\}$. The set $\pi^{-1}(V) \subset B$ is a disjoint union of D and $\pi^{-1}(V \backslash \{0\})$. Let $\overline{\pi^{-1}(V \backslash \{0\})}$ be the closure with respect to the usual topology and

$$V_0 := D \cap \overline{\pi^{-1}(V \backslash \{0\})} \subset D,$$

see Fig.1. On the chart B_1, we have $H_j(q,p) = r^{h_j} H_j(1, Q_2, \cdots, P_m)$. Hence, define

$$V_{01} := \{(0, Q_2, \cdots, P_m) \in D \,|\, H_j(1, Q_2, \cdots, P_m) = 0,\ j = 1, \cdots, k\} \subset D \cap B_1.$$

The sets V_{0i} on the chart B_i are also defined in the same way for $i = 2, \cdots, 2m$. Then, we have

$$\begin{aligned} V_0 &= V_{01} \cup V_{02} \cup \cdots \cup V_{0,2m} \subset D \\ \pi^{-1}(V) &= D \cup \pi^{-1}(V \backslash \{0\}) \simeq D \cup (V_0 \times \mathbb{C}). \end{aligned}$$

V and V_0 are $2m-k$ and $2m-k-1$ dimensional manifolds, respectively, with singularities. As in Eq.(4.5), the system (4.11) induces the vector field $\mathcal{X}$ on B after a suitable change of the independent variable. On B_1, $\mathcal{X}$ is expressed as

$$\left\{ \begin{array}{l} \dfrac{dr}{dt} = \dfrac{1}{a_1} r \dfrac{\partial H_1}{\partial p_1}(1, Q_2, \cdots, P_m) \\ \dfrac{dQ_i}{dt} = \dfrac{\partial H_1}{\partial p_i}(1, Q_2, \cdots, P_m) - \dfrac{a_i}{a_1} Q_i \dfrac{\partial H_1}{\partial p_1}(1, Q_2, \cdots, P_m) \\ \dfrac{dP_i}{dt} = -\dfrac{\partial H_1}{\partial q_i}(1, Q_2, \cdots, P_m) - \dfrac{b_i}{a_1} P_i \dfrac{\partial H_1}{\partial p_1}(1, Q_2, \cdots, P_m). \end{array} \right. \tag{4.22}$$

Proposition 4.7.
(i) D is an invariant manifold of $\mathcal{X}$.
(ii) $V_0 \subset D$ is an invariant manifold of $\mathcal{X}$.
(iii) All fixed points of $\mathcal{X}$ are included in V_0.
(iv) For an indicial locus $c = (c_1, \cdots, c_{2m})$, the orbit of the exact solution $q_i(z) = c_i z^{-a_1}$, $p_i(z) = c_{m+i} z^{-b_i}$ of (4.11) is included in V. On the blow-up space B, it tends to a fixed point $\mathrm{P}(c)$ on V_0 as $z \to \infty$.

Proof. It is sufficient to prove the statements on the chart B_1. Since $D \cap B_1 = \{r = 0\}$, (i) immediately follows from Eq.(4.22). By the assumption (S), all fixed points of $\mathcal{X}$ lie on the divisor D. Due to Lemma 4.1(i), a fixed point $\mathrm{P}(c)$ on $D \cap B_1$ is of the form $(0, c_1^{-a_2/a_1} c_2, \cdots, c_1^{-b_m/a_1} c_{2m})$ for an indicial locus c. Then, Lemma 4.4 implies

$$H_j(1, c_1^{-a_2/a_1} c_2, \cdots, c_1^{-b_m/a_1} c_{2m}) = c_1^{-h_j/a_1} H_j(c) = 0,$$

which proves (iii): $\mathrm{P}(c) \in V_{01} \subset V_0$. Part (iv) is shown by Lemma 4.1 (ii) and Lemma 4.4. Finally, let us show the statement (ii). Along an integral curve of (4.22), we have

$$\frac{d}{dt} H_j(1, Q_2, \cdots, P_m) = \sum_{i=2}^{m} \frac{\partial H_j}{\partial q_i} \left(\frac{\partial H_1}{\partial p_i} - \frac{a_i}{a_1} Q_i \frac{\partial H_1}{\partial p_1} \right) + \sum_{i=1}^{m} \frac{\partial H_j}{\partial p_i} \left(-\frac{\partial H_1}{\partial q_i} - \frac{b_i}{a_1} P_i \frac{\partial H_1}{\partial p_1} \right).$$

By introducing a dummy parameter $Q_1 = 1$, it is rewritten as

$$\begin{aligned}\frac{d}{dt}H_j(1, Q_2, \cdots, P_m) &= \sum_{i=1}^{m} \frac{\partial H_j}{\partial q_i}\left(\frac{\partial H_1}{\partial p_i} - \frac{a_i}{a_1}Q_i\frac{\partial H_1}{\partial p_1}\right) + \sum_{i=1}^{m}\frac{\partial H_j}{\partial p_i}\left(-\frac{\partial H_1}{\partial q_i} - \frac{b_i}{a_1}P_i\frac{\partial H_1}{\partial p_1}\right) \\ &= \sum_{i=1}^{m}\left(\frac{\partial H_j}{\partial q_i}\frac{\partial H_1}{\partial p_i} - \frac{\partial H_1}{\partial q_i}\frac{\partial H_j}{\partial p_i}\right) - \frac{1}{a_1}\frac{\partial H_1}{\partial p_1}\sum_{i=1}^{m}\left(a_iQ_i\frac{\partial H_j}{\partial q_i} + b_iP_i\frac{\partial H_j}{\partial p_i}\right).\end{aligned}$$

Lemma 4.3 and $\{H_1, H_j\} = 0$ show

$$\frac{d}{dt}H_j(1, Q_2, \cdots, P_m) = -\frac{1}{a_1}\frac{\partial H_1}{\partial p_1}h_jH_j(1, Q_2, \cdots, P_m).$$

This is a linear equation of $H_j(1, Q_2, \cdots, P_m)$ solved as

$$H_j(1, Q_2(t), \cdots, P_m(t)) = H_j(1, Q_2(0), \cdots, P_m(0)) \cdot \exp\left[-\frac{h_j}{a_1}\int_0^t \frac{\partial H_1}{\partial p_1}ds\right].$$

This proves that if $(0, Q_2, \cdots, P_m) \in V_{01}$ at the initial time $t = 0$, so that $H_j(1, Q_2(0), \cdots, P_m(0)) = 0$, then $(0, Q_2, \cdots, P_m) \in V_{01}$ for any $t \in \mathbb{R}$. $\square$

Fix an indicial locus $c = (c_1, \cdots, c_{2m}) \neq 0$ with $c_1 \neq 0$. Without loss of generality we assume that $c_1 = 1$ by a suitable scaling of the independent variable. By Lemma 4.1, the indicial locus associates the fixed point $\mathrm{P}(c) : (r, Q_2, \cdots, P_m) = (0, c_2, \cdots, c_{2m})$ of the vector field (4.22) on the chart B_1. Prop.4.2 shows that the Jacobi matrix at the fixed point written in $(r, Q_2, \cdots, P_m)$-coordinates is of the form

$$J = \begin{pmatrix} -1 & 0 \\ 0 & \tilde{J} \end{pmatrix}, \tag{4.23}$$

and its eigenvalues coincide with the Kovalevskaya exponents $\kappa_1 = -1, \kappa_2, \cdots, \kappa_{2m}$. Thus, eigenvectors of $\kappa_2, \cdots, \kappa_{2m}$ are tangent to the divisor $D = \{r = 0\}$. Let $\mathbf{E}_s, \mathbf{E}_u$ and $\mathbf{E}_c$ be the stable, unstable and center subspace at the point $\mathrm{P} = \mathrm{P}(c)$, which are eigenspaces of eigenvalues with negative real parts, positive real parts, and zero real parts, respectively. Let $W_s(\mathrm{P}), W_u(\mathrm{P})$ and $W_c(\mathrm{P})$ be a local stable manifold, unstable manifold and center manifold, respectively, which are tangent to $\mathbf{E}_s, \mathbf{E}_u$ and $\mathbf{E}_c$ at P. Because of Lemma 3.6 ($\kappa + \mu = h_1 - 1 > 0$), $\dim\mathbf{E}_u \geq m$ and $1 \leq \dim\mathbf{E}_s \leq m$.

Proposition 4.8. Under the above situation, the unstable manifold $W_u(\mathrm{P})$ is included in D and the stable manifold $W_s(\mathrm{P})$ is included in $\overline{\pi^{-1}(V\backslash\{0\})} \simeq V_0 \times \mathbb{C}$. If there are no purely imaginary eigenvalues ($\neq 0$), $W_c(\mathrm{P})$ is included in V_0.

Proof. Since $\mathbf{E}_u$ is tangent to the divisor D and D is an invariant manifold, $W_u(\mathrm{P}) \subset D$. Let x be a point on $W_s(\mathrm{P})$ and suppose $x \notin D \cup V$. Since $x \in W_s(\mathrm{P})$, a solution of (4.22) with an initial value at x tends to the fixed point P as $t \to \infty$. Since $x \notin V$, $H_j(x) \neq 0$ for some j. This is a contradiction because $H_j = 0$ at P and

H_j is a constant along a solution. Let x' be a point on $W_s(\mathrm{P})$ such that $x' \in D \backslash V_0$. Then, there is a point x on $W_s(\mathrm{P})$ and $x \notin D \cup V$ because the eigenvector of $\kappa_1 = -1$ is transverse to D, that is again a contradiction. If the Kovalevskaya matrix at c has zero eigenvalues, then an indicial locus c is not isolated (Prop.3.3). Thus, the fixed point $\mathrm{P}(c)$ is not isolated and there exists a neighborhood U of $\mathrm{P}(c)$ such that $U \cap V_0$ consists of fixed points of the vector field. If there are no purely imaginary eigenvalues, $U \cap V_0$ gives a local center manifold. $\square$

Next, we consider the system (4.11) satisfying (H0), (H1), (H2) and (S) with $k = m$. In this case, V and $\overline{\pi^{-1}(V \backslash \{0\})}$ are m dimensional and V_0 is an $m-1$ dimensional variety with singularities. If the system satisfies the Painlevé property in a sense that any solution is meromorphic, there is an indicial locus c such that all the Kovalevskaya exponents but unique -1 are positive integers (Painlevé test). Thus, a stable manifold at $\mathrm{P}(c)$ is one dimensional, which is precisely given by the orbit of the special solution $q_i(z) = c_i z^{-a_i}$, $p_i(z) = c_{m+i} z^{-b_i}$. The next theorem consider the opposite situation.

Theorem 4.9. Suppose that the system (4.11) satisfies (H0), (H1), (H2) and (S) with $k = m$. Suppose that there exists an indicial locus c such that vectors $dH_1(c), \cdots, dH_m(c)$ are linearly independent. Then, there exists a neighborhood U of $\mathrm{P}(c)$ such that $\overline{\pi^{-1}(V \backslash \{0\})} \cup U = W_s(\mathrm{P}(c)) \cup U$.

Proof. Theorem 4.5 shows that $h_1, \cdots, h_m > 0$ are Kovalevskaya exponents. Due to Lemma 3.6, negative integers $\mu_j := h_1 - 1 - h_j$, $j = 1, \cdots, m$ are also Kovalevskaya exponents. Thus, a local stable manifold $W_s(\mathrm{P}(c))$ of the vector field (4.22) is an m-dimensional smooth manifold included in $\overline{\pi^{-1}(V \backslash \{0\})}$. Indeed, again Theorem 4.5 implies that the (right) eigenvectors of $K(c)$ associated with eigenvalues $\mu_1, \cdots, \mu_m$ are orthogonal to $dH_1(c), \cdots, dH_m(c)$. Hence, the stable subspace $\mathbf{E}_s$ coincides with the tangent space of $\overline{\pi^{-1}(V \backslash \{0\})}$ at $\mathrm{P}(c)$. $\square$

An indicial locus c satisfying the assumption of the theorem (i.e. $h_1, \cdots, h_m > 0$ are Kovalevskaya exponents), for which $\dim \mathbf{E}_u = \dim \mathbf{E}_s = m$, is called the lowest indicial locus. The existence of a lowest indicial locus is proved by [14] for a certain class of integrable systems called the hyperelliptically separable systems, while the existence for more general systems is not known. Let us demonstrate our results for several 4-dim systems obtained from the autonomous parts of Painlevé equations. See also Table 5. They have lowest indicial loci and $\overline{\pi^{-1}(V \backslash \{0\})}$ is decomposed into the disjoint union of stable manifolds at the fixed points on the divisor.

Example 4.10. We consider the autonomous part of $(\mathrm{P_I})_2$ given in Example 4.6. Since the Kovalevskaya exponents of the indicial locus $c_1 = (1, 1, 1, -1)$, which corresponds to the principal Laurent series solution, are $-1, 2, 5, 8$, the stable manifold $W_s(\mathrm{P}_1)$ at the fixed point $\mathrm{P}(c_1)$ is given by the orbit of the special solution $q_1 = z^{-2}, q_2 = z^{-4}, p_1 = z^{-5}, p_2 = -z^{-3}$. The Kovalevskaya exponents of the indicial locus $c_2 = (3, 0, 27, -3)$, which satisfies the assumptions for Theorem 4.9, are $-3, -1, 8, 10$. The 2-dim stable manifold $W_s(\mathrm{P}_2)$ locally coincides with $\overline{\pi^{-1}(V \backslash \{0\})}$.

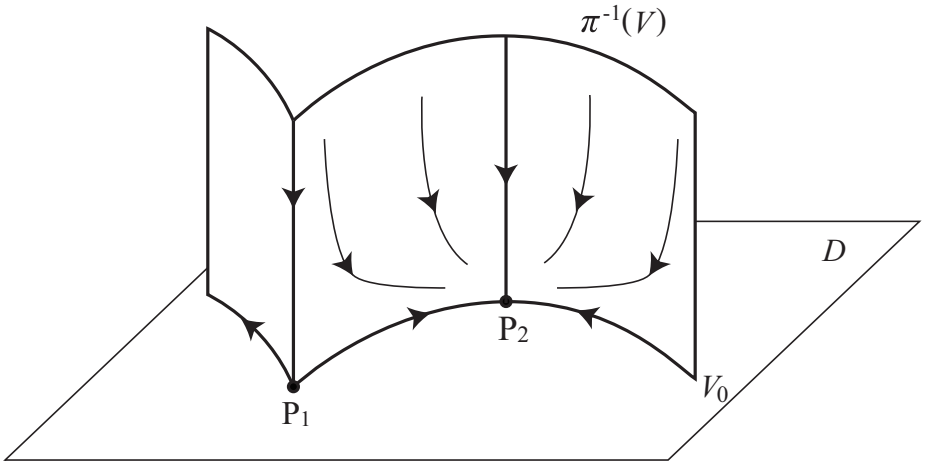

Fig. 1: A schematic view of $\pi^{-1}(V), V_0$ and the dynamics on it for Example 4.10. The singularity P_1 of V_0 is of type A_4.

In this case, $\overline{\pi^{-1}(V\backslash\{0\})}$ is decomposed into the disjoint union of $W_s(P_1)$ and $W_s(P_2)$, see Fig.1.

On the (r, Q_2, P_1, P_2)-coordinates, H_1 and H_2 are written as

$$\begin{aligned} H_1 &= r^8(1 + 2P_1P_2 + 3P_2^2 - Q_2 - Q_2^2), \\ H_2 &= r^{10}(-1 + P_1^2 + 2P_1P_2 + 3Q_2 + P_2^2Q_2 - 2Q_2^2). \end{aligned}$$

Thus, V_{01} is defined by

$$V_{01} = \{1 + 2P_1P_2 + 3P_2^2 - Q_2 - Q_2^2 = 0,\ -1 + P_1^2 + 2P_1P_2 + 3Q_2 + P_2^2Q_2 - 2Q_2^2 = 0\}.$$

Since $dH_2(c_1) = 0$, V_0 is singular at $P_1 : (Q_2, P_1, P_2) = (1, 1, -1)$. We can verify that it is a A_4-singularity, for example, by using blow-up of a singularity or direct suitable coordinate transformations (a singularity whose normal form of defining equation is $y^2 + x^5 = 0$).

Example 4.11. The autonomous, quasihomogeneous part of $(P_{\text{II-1}})_2$ given in (1.5) is defined by the Hamiltonians

$$\begin{cases} H_1 = 2p_1p_2 - p_2^3 - p_1q_1^2 + q_2^2, \\ H_2 = -p_1^2 + p_1p_2^2 + p_1p_2q_1^2 + 2p_1q_1q_2. \end{cases} \tag{4.24}$$

The weights are $(a_1, a_2, b_1, b_2) = (1, 3, 4, 2)$ and $h_1 = 6, h_2 = 8$. Its four indicial loci and the Kovalevskaya exponents are given by

$$\begin{aligned} c_1 &= (1, 0, 0, 0), \quad \kappa = -1, 2, 3, 6 \\ c_2 &= (-1, 1, 0, 1), \quad \kappa = -1, 2, 3, 6 \\ c_3 &= (2, 1, 0, 1), \quad \kappa = -1, -3, 6, 8 \\ c_4 &= (-2, 3, 9, 3), \quad \kappa = -1, -3, 6, 8. \end{aligned}$$

Among them, c_3 and c_4 satisfy the assumptions for Theorem 4.9. Thus, the fixed points $\mathrm{P}(c_3)$ and $\mathrm{P}(c_4)$ have 2-dim stable manifolds that locally coincide with $\overline{\pi^{-1}(V\backslash\{0\})}$. The fixed points $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$ have 1-dim stable manifolds that are given by the orbit of special solutions.

On the (r, Q_2, P_1, P_2)-coordinates, H_1 and H_2 are written as

$$\begin{aligned} H_1 &= r^6(2P_1P_2 - P_2^3 - P_1 + Q_2^2) \\ H_2 &= r^8(-P_1^2 + P_1P_2^2 + P_1P_2 + 2P_1Q_2). \end{aligned}$$

Thus, V_{01} is defined by

$$V_{01} = \{2P_1P_2 - P_2^3 - P_1 + Q_2^2 = 0,\ -P_1^2 + P_1P_2^2 + P_1P_2 + 2P_1Q_2 = 0\}.$$

Theorem 4.5 shows that $dH_2(c_1) = dH_2(c_2) = 0$. Hence, V_0 is singular at $\mathrm{P}(c_1) : (Q_2, P_1, P_2) = (0, 0, 0)$ and $\mathrm{P}(c_2) : (-1, 0, 1)$. We can verify that both $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$ are D_5-singularities (the normal form is $y(x^2 + y^3) = 0$).

Example 4.12. The autonomous, quasihomogeneous part of $(\mathrm{P}_{\text{II-2}})_2$ given in (1.6) is defined by the Hamiltonians

$$\begin{cases} H_1 = p_1p_2 - p_1q_1^2 - 2p_1q_2 + p_2q_1q_2 + q_1q_2^2, \\ H_2 = p_1^2 - p_1p_2q_1 + p_2^2q_2 - 2p_1q_1q_2 - p_2q_2^2 + q_1^2q_2^2. \end{cases} \tag{4.25}$$

The weights are $(a_1, a_2, b_1, b_2) = (1, 2, 3, 2)$ and $h_1 = 5, h_2 = 6$. It has five indicial loci given by

$$\begin{aligned} c_1 &= (1, 0, 0, 0), \quad \kappa = -1, 1, 3, 5 \\ c_2 &= (-1, -1, 1, 0), \quad \kappa = -1, 1, 3, 5 \\ c_3 &= (0, -2, 4, -4), \quad \kappa = -1, -2, 5, 6 \\ c_4 &= (-2, -2, 0, 2), \quad \kappa = -1, -2, 5, 6 \\ c_5 &= (2, 0, 0, 2), \quad \kappa = -1, -2, 5, 6. \end{aligned}$$

Among them, c_3, c_4 and c_5 satisfy the assumptions for Theorem 4.9. Thus, the fixed points $\mathrm{P}(c_3), \mathrm{P}(c_4)$ and $\mathrm{P}(c_5)$ have 2-dim stable manifolds that locally coincide with $\overline{\pi^{-1}(V\backslash\{0\})}$. The fixed points $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$ have 1-dim stable manifolds that are given by the orbit of special solutions.

On the (r, Q_2, P_1, P_2)-coordinates, H_1 and H_2 are written as

$$\begin{aligned} H_1 &= r^5(P_1P_2 - P_1 - 2P_1Q_2 + P_2Q_2 + Q_2^2) \\ H_2 &= r^6(P_1^2 - P_1P_2 + P_2^2Q_2 - 2P_1Q_2 - P_2Q_2^2 + Q_2^2). \end{aligned}$$

V_{01} is defined by

$$\begin{aligned} V_{01} = \{&P_1P_2 - P_1 - 2P_1Q_2 + P_2Q_2 + Q_2^2 = 0, \\ &P_1^2 - P_1P_2 + P_2^2Q_2 - 2P_1Q_2 - P_2Q_2^2 + Q_2^2 = 0\}. \end{aligned}$$

Since $dH_2(c_1) = dH_2(c_2) = 0$, V_0 is singular at $\mathrm{P}(c_1) : (Q_2, P_1, P_2) = (0, 0, 0)$ and $\mathrm{P}(c_2) : (-1, -1, 0)$. We can verify that both $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$ are A_5-singularities (the normal form is $y^2 + x^6 = 0$).

Example 4.13. The autonomous, quasihomogeneous part of $(\mathrm{P_{IV}})_2$ given in (1.7) is defined by the Hamiltonians

$$\begin{cases} H_1 = p_1^2 + p_1 p_2 - p_1 q_1^2 + p_2 q_1 q_2 - p_2 q_2^2, \\ H_2 = p_1 p_2 q_1 - 2 p_1 p_2 q_2 - p_2^2 q_2 + p_2 q_1 q_2^2. \end{cases} \tag{4.26}$$

The weights are $(a_1, a_2, b_1, b_2) = (1, 1, 2, 2)$ and $h_1 = 4, h_2 = 5$. It has eight indicial loci given by

$$\begin{aligned} c_1 &= (-1, -1, 1, 0), \quad \kappa = -1, 1, 2, 4 \\ c_2 &= (1, 0, 0, 0), \quad \kappa = -1, 1, 2, 4 \\ c_3 &= (0, 1, 0, 0), \quad \kappa = -1, 1, 2, 4 \\ c_4 &= (0, -1, 2, -4), \quad \kappa = -1, -2, 4, 5 \\ c_5 &= (2, 0, 0, 2), \quad \kappa = -1, -2, 4, 5 \\ c_6 &= (-1, 1, 1, 0), \quad \kappa = -1, -2, 4, 5 \\ c_7 &= (1, 2, 0, 0), \quad \kappa = -1, -2, 4, 5 \\ c_8 &= (-2, -2, 2, 2), \quad \kappa = -1, -2, 4, 5 \end{aligned}$$

Among them, c_4 to c_8 satisfy the assumptions for Theorem 4.9. To investigate the fixed points $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$, we move to B_1 chart with the (r, Q_2, P_1, P_2)-coordinates, on which H_1 and H_2 are written as

$$\begin{aligned} H_1 &= r^4(P_1^2 + P_1 P_2 - P_1 + P_2 Q_2 - P_2 Q_2^2) \\ H_2 &= r^5(P_1 P_2 - 2 P_1 P_2 Q_2 - P_2^2 Q_2 + P_2 Q_2^2). \end{aligned}$$

V_{01} is defined by

$$\begin{aligned} V_{01} = \{ & P_1^2 + P_1 P_2 - P_1 + P_2 Q_2 - P_2 Q_2^2 = 0, \\ & P_1 P_2 - 2 P_1 P_2 Q_2 - P_2^2 Q_2 + P_2 Q_2^2 = 0 \}. \end{aligned}$$

Since $dH_2(c_1) = dH_2(c_2) = 0$, V_{01} is singular at $\mathrm{P}(c_1) : (Q_2, P_1, P_2) = (1, 1, 0)$ and $\mathrm{P}(c_2) : (0, 0, 0)$. We can verify that both $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$ are D_6-singularities (the normal form is $y(x^2 + y^4) = 0$).

Note that $\mathrm{P}(c_3)$ is not included in B_1 chart because the first component of c_3 is zero. To study $\mathrm{P}(c_3)$ we use to B_2 chart with the coordinates (Q_1, r, P_1, P_2) defined as (4.2). In this coordinates, H_1 and H_2 are written as

$$\begin{aligned} H_1 &= r^4(P_1^2 + P_1 P_2 - P_1 Q_1^2 + P_2 Q_1 - P_2) \\ H_2 &= r^5(P_1 P_2 Q_1 - 2 P_1 P_2 - P_2^2 + P_2 Q_1). \end{aligned}$$

V_{02} is defined by

$$\begin{aligned} V_{02} &= \{P_1^2 + P_1P_2 - P_1Q_1^2 + P_2Q_1 - P_2 = 0, \\ &\qquad P_1P_2Q_1 - 2P_1P_2 - P_2^2 + P_2Q_1 = 0\}. \end{aligned}$$

Since $dH_2(c_3) = 0$, V_{02} is singular at $\mathrm{P}(c_3) : (Q_1, P_1, P_2) = (0, 0, 0)$, which is also a D_6-singularity.

Example 4.14. Let us consider the following Hamiltonians

$$\begin{cases} H_1 = (p_1^2/2 - 2q_1^3) + (p_2^2/2 - 2q_2^3), \\ H_2 = p_1^2/2 - 2q_1^3. \end{cases} \tag{4.27}$$

The Hamiltonian equation of H_1 is a direct product of the autonomous part of the first Painlevé equation. The weights are $(a_1, a_2, b_1, b_2) = (2, 2, 3, 3)$ and $h_1 = 6, h_2 = 6$. It has three indicial loci c_1, c_2, c_3, whose Kovalevskaya exponents are $\kappa = -1, 2, 3, 6$ for c_1, c_2 and $\kappa = -1, -1, 6, 6$ for c_3. Since $dH_2(c_1) = dH_2(c_2) = 0$, V_0 is singular at $\mathrm{P}(c_1)$ and $\mathrm{P}(c_2)$. We can show that both singularities are A_2-singularity.

Similarly, consider the direct product of the autonomous part of the second Painlevé equation

$$\begin{cases} H_1 = (p_1^2/2 - q_1^4/2) + (p_2^2/2 - q_2^4/2), \\ H_2 = p_1^2/2 - q_1^4/2. \end{cases} \tag{4.28}$$

The weights are $(a_1, a_2, b_1, b_2) = (1, 1, 2, 2)$ and $h_1 = 4, h_2 = 4$. It has eight indicial loci $c_1, \cdots, c_8$, whose Kovalevskaya exponents are $\kappa = -1, 1, 2, 4$ for c_1, c_2, c_3 and $\kappa = -1, -1, 4, 4$ for the others. Since $dH_2(c_1) = dH_2(c_2) = dH_2(c_3) = 0$, V_0 is singular at $\mathrm{P}(c_1)$ to $\mathrm{P}(c_3)$. We can show that singularities of them are A_3- singularity.

Finally, consider the direct product of the autonomous part of the fourth Painlevé equation

$$\begin{cases} H_1 = (-p_1q_1^2 + p_1^2q_1) + (-p_2q_2^2 + p_2^2q_2), \\ H_2 = -p_1q_1^2 + p_1^2q_1. \end{cases} \tag{4.29}$$

The weights are $(a_1, a_2, b_1, b_2) = (1, 1, 1, 1)$ and $h_1 = 3, h_2 = 3$. It has fifteen indicial loci $c_1, \cdots, c_{15}$, whose Kovalevskaya exponents are $\kappa = -1, 1, 1, 3$ for c_1 to c_5 and $\kappa = -1, -1, 3, 3$ for the others. Since $dH_2(c_1) \cdots = dH_2(c_5) = 0$, V_0 is singular at $\mathrm{P}(c_1)$ to $\mathrm{P}(c_5)$. They are D_4- singularities.

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	$(a_1, a_2, b_1, b_2; h_1, h_2)$	κ		sing.
$H_{\mathrm{I}} \times H_{\mathrm{I}}$	$(2, 2, 3, 3; 6, 6)$	$(-1, 2, 3, 6)$	$\times 2$	A_2
Eq.(4.27)		$(-1, -1, 6, 6)$	$\times 1$	
$H_{\mathrm{II}} \times H_{\mathrm{II}}$	$(1, 1, 2, 2; 4, 4)$	$(-1, 1, 2, 4)$	$\times 3$	A_3
Eq.(4.28)		$(-1, -1, 4, 4)$	$\times 5$	
$H_{\mathrm{IV}} \times H_{\mathrm{IV}}$	$(1, 1, 1, 1; 3, 3)$	$(-1, 1, 1, 3)$	$\times 5$	D_4
Eq.(4.29)		$(-1, -1, 3, 3)$	$\times 10$	
$H_1^{9/2}$	$(2, 4, 5, 3; 8, 10)$	$(-1, 2, 5, 8)$	$\times 1$	A_4
Eq.(1.4)		$(-3, -1, 8, 10)$	$\times 1$	
$H_1^{7/2+1}$	$(1, 3, 4, 2; 6, 8)$	$(-1, 2, 3, 6)$	$\times 2$	D_5
Eq.(1.5)		$(-3, -1, 6, 8)$	$\times 2$	
H_1^5	$(1, 2, 3, 2; 5, 6)$	$(-1, 1, 3, 5)$	$\times 2$	A_5
Eq.(1.6)		$(-2, -1, 5, 6)$	$\times 3$	
H_1^{4+1}	$(1, 1, 2, 2; 4, 5)$	$(-1, 1, 2, 4)$	$\times 3$	D_6
Eq.(1.7)		$(-2, -1, 4, 5)$	$\times 5$	

Table 5: Weights, Kovalevskaya exponents and singularity-types of 4-dim autonomous Painlevé equations.

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