The generalized spectral theory and its application to the Kuramoto conjecture

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Abstract

A spectral theory of linear operators based on a Gelfand triplet (rigged Hilbert space) is developed under the assumptions that a linear operator T on a Hilbert space $\mathcal{H}$ is a perturbation of a self-adjoint operator, and the spectral measure of the self-adjoint operator has an analytic continuation near the real axis. It is shown that for a suitable dense subspace X of $\mathcal{H}$ and its dual space X', for any $\phi \in X$, the resolvent $(\lambda - T)^{-1}\phi$ of the operator T has an analytic continuation from the lower half plane to the upper half plane as an X'-valued holomorphic function even when T has a continuous spectrum on $\mathbf{R}$. The Gelfand triplet consists of three topological vector spaces $X \subset \mathcal{H} \subset X'$. Basic tools of the usual spectral theory, such as spectra, resolvents and Riesz projections are extended to those defined on a Gelfand triplet. They prove to have the same properties as those of the usual spectral theory. The results are applied to estimate exponential decays of the semigroups of linear operators and bifurcations of nonlinear dynamical systems. In particular, a conjecture on a bifurcation of the Kuramoto model (Kuramoto conjecture) will be solved.

1 Kuramoto model

In the last few decades, the study of large / infinite-dimensional dynamical systems becomes more and more important. Let us consider the system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, N,$$

where $x_i = x_i(t)$ moves on some phase space. This dynamical system assigns a directed graph in the following way:

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Let $V = \{v_i\}_{i=1}^N$ be the set of vortices. If the function f_i depends on x_j, we define the edge e_{ji} from the vortex v_j to v_i, that means that x_j affects the dynamics of x_i. It is a challenging problem to understand how the graph structure is related to the dynamics. For simplicity, we consider the system of the form

$$\frac{d\theta_i}{dt} = f_i(\theta_1, \cdots, \theta_n), \quad i = 1, \cdots, N, \tag{1.1}$$

where $\theta_i \in S^1$ rotates on a circle. Thus, the phase space is an N-torus. A dynamical system of the form is often called a **system of coupled oscillators** [21, 23]. For example, if a given dynamical system on a phase space X has an N-torus as an invariant manifold, we obtain a coupled oscillators by restricting the dynamics on it.

The **Kuramoto model** is one of the most famous coupled oscillators given by

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \tag{1.2}$$

which is well-known as a typical mathematical model for synchronization phenomena [20, 21, 26]. Here, ω_i and K are constants called natural frequencies and the coupling strength, respectively. When the coupling strength is zero, there are no interactions between oscillators and they rotate with their own velocity ω_i. Hence, if $\omega_j > \omega_i$ then θ_j overtakes θ_i many times. However, if K is positive, there are interactions between oscillators through the term $\sin(\theta_j - \theta_i)$ and we expect that if K is large enough, such an overtaking does not occur. Indeed, it is easy to observe by numerics that there exists a threshold K_c such that when $K > K_c$, a synchronized state appears; a subset of oscillators forms a cluster on a circle and it behaves like a big oscillator without overtaking. The cluster consists of oscillators whose natural frequency ω_i is close to the average Ω of all natural frequencies. As K increases, the number of oscillators that are entrained into the cluster gets larger (Fig.1).

Figure 1: (left) synchronization. (right) de-synchronization.

In order to observe that whether a synchronization occurs or not, it is convenient to introduce the **order parameter** defined by

$$\eta := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}. \tag{1.3}$$

This gives the center of mass of oscillators on a unit circle. Hence, when its absolute value $r := |\eta|$ is positive (resp. zero), a synchronization occurs (resp. does not occur). Kuramoto performed a certain formal and technical calculation using the order parameter, and reached the following result, though there are no mathematical proofs.

The Kuramoto conjecture [21, 26].

Suppose $N \to \infty$ and the natural frequencies are independent and identically distributed according to a probability density function $g(\omega)$. If $g(\omega)$ is an even and unimodal function, a bifurcation diagram of the order parameter $r = |\eta|$ is given as Fig. 2. This means that when K is smaller than $K_c := 2/(\pi g(0))$, the desynchronized state $r = 0$ is asymptotically stable. At $K = K_c$, a bifurcation (phase transition) occurs and a stable synchronized state ($r > 0$) exists for $K > K_c$. Near the bifurcation point, r is approximately given by $r \sim O(\sqrt{K - K_c})$.

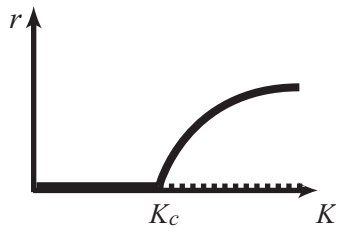

Figure 2: A bifurcation diagram of the order parameter.

Note that by a translation of the coordinate $\theta_i \mapsto \theta_i + \Omega t$, we can assume without loss of generality that the average value of $g(\omega)$ is zero. Then, " $g(\omega)$ is unimodal" means that when $\omega > 0$ (resp. $\omega < 0$), it is strictly monotonically decreasing (resp. increasing). The bifurcation point $K_c := 2/(\pi g(0))$ is often called Kuramoto's transition point. See [26] for Kuramoto's formal calculation.

As explained in later sections, the difficulty of a mathematical approach to the Kuramoto conjecture is that a certain linear operator obtained by the linearization of the model has a continuous spectrum. Recently, the author developed the generalized spectrum theory of linear operators based on Gelfand triplets and proposed an effective method to investigate linear operators having continuous spectra [5].

By applying this theory to the infinite dimensional Kuramoto model, he proved the Kuramoto conjecture under a suitable condition [3, 4]. In what follows, $h(\theta)$ denotes a distribution of the initial values $\{\theta_j(0)\}_{j=1}^{\infty}$ of oscillators.

Theorem 1.1 *Suppose that $g(\omega)$ is the Gaussian distribution. When $0 < K < K_c$, there exists $\delta > 0$ such that if $h(\theta)$ satisfies*

$$\left| \int_0^{2\pi} e^{ij\theta} h(\theta) d\theta \right| < \delta, \quad j = 1, 2, \cdots,$$

then the order parameter $\eta(t)$ tends to zero as $t \to \infty$ with an exponential rate.

Theorem 1.2 *Suppose that $g(\omega)$ is the Gaussian distribution. There exist numbers $\varepsilon_0, \delta > 0$ such that if $h(\theta)$ satisfies*

$$\left| \int_0^{2\pi} e^{ij\theta} h(\theta) d\theta \right| < \delta, \quad j = 1, 2, \cdots,$$

then for $K_c < K < K_c + \varepsilon_0$, the absolute value of the order parameter converges to the following value as $t \to \infty$

$$|\eta(t)| = \sqrt{\frac{-16}{\pi K_c^4 g''(0)}} \sqrt{K - K_c} + O(K - K_c).$$

In particular, a bifurcation diagram of the order parameter is given as Fig. 2.

This result holds even if $g(\omega)$ is not Gaussian. The most essential assumption is that $g(\omega)$ is analytic on $\mathbf{R}$ and it has an analytic continuation around the real axis. On the other hand, the Kuramoto conjecture was proved when $g(\omega)$ is a C^n function in [13] by a different way. In this case, the decay rate (Thm.1.1) is not exponential but algebraic $O(t^{-n})$.

More general form of the Kuramoto model is given as

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i + \alpha), \quad i = 1, \cdots, N, \tag{1.4}$$

and $g(\omega)$ is not unimodal, where α is a phase lag and a_{ij} denotes the adjacency matrix that determines the graph structure (we have an edge connecting θ_j and θ_i only when $a_{ij} \neq 0$). We can also consider other periodic function as the interaction term such as $\sin 2(\theta_j - \theta_i)$. Even in these cases, we can obtain similar results as above, see [4, 7, 8, 9, 10] for the details.

The generalized spectral theory used to prove the Kuramoto conjecture is also applicable to any problems related to continuous spectra, such as Schrödinger equations [6], chaos in symbolic dynamical systems [11], the onset of the human brain wave [12], and so on.

The purpose of this article is to illustrate the generalized spectral theory. For it, we begin to investigate the Kuramoto model within the usual spectral theory in Sec. 2 and 3.

2 Continuous limit

Since we assume $N \to \infty$ for the Kuramoto conjecture, we first start to define the infinite-dimensional version of the Kuramoto model.

By substituting the definition of the order parameter η into the Kuramoto model (1.2), we obtain

$$\frac{d\theta_i}{dt} = \omega_i + Kr\sin(\psi - \theta_i),$$

where we put $\eta = re^{i\psi}$. With this in mind, the continuous limit of the Kuramoto model is defined by the following system

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} + \dfrac{\partial}{\partial \theta}(v\rho_t) = 0, \quad \rho_t = \rho_t(\theta, \omega), \\ v := \omega + Kr\sin(\psi - \theta), \\ \eta := re^{i\psi} = \displaystyle\int_{\mathbf{R}} g(\omega)d\omega \int_0^{2\pi} e^{i\theta}\rho_t(\theta, \omega)d\theta. \end{cases} \tag{2.1}$$

Now infinitely many oscillators rotate on a circle like a fluid, and ρ_t denotes its distribution. More precisely, $\rho_t(\theta, \omega)$ implies a probability density function of θ for each time t and natural frequency ω. The first line of the system is the equation of continuity (conservation law) of ρ_t. The velocity field v is given by the second line, that comes from the right hand side of the finite-dimensional one by removing the subscript i. The third line is the definition of the infinite-dimensional version of the order parameter, that is obtained by replacing the summation in the finite-dim model by the integral with respect to the measure $g(\omega)\rho_t(\theta, \omega)d\omega d\theta$, where $g(\omega)$ is a given density function. In this article, we assume that it is the Gaussian distribution for simplicity. It is easy to show that this system has a unique weak solution for a given initial condition $\rho_0(\theta, \omega) = h(\theta, \omega)$ for any $t > 0$.

Since the unknown function ρ_t is periodic in θ, we consider the Fourier series of it. The Fourier coefficients are given by

$$Z_j(t, \omega) := \int_0^{2\pi} e^{ij\theta}\rho_t(\theta, \omega)d\theta.$$

Rewriting the equation of ρ_t by Z_j yields the system of equations of Z_j as

$$\frac{dZ_1}{dt} = i\omega Z_1 + \frac{K}{2}\eta(t) - \frac{K}{2}\overline{\eta(t)}Z_2,$$

and

$$\frac{dZ_j}{dt} = ij\omega Z_j + \frac{jK}{2}(\eta(t)Z_{j-1} - \overline{\eta(t)}Z_{j+1}),$$

for $j = 2, 3, \cdots$. Note that $Z_0 = 1$ because of the normalization of a probability density. This system has the trivial solution $Z_j = 0$, $j = 1, 2, \cdots$. In this case $\rho_t = 1/(2\pi)$ is the uniform distribution on a circle which corresponds to the de-synchronization state.

Let us investigate the stability of the trivial solution. Since η is written as $\eta = \int_{\mathbf{R}} Z_1 g(\omega) d\omega$, the terms such as $\eta(t) Z_j$ in the system are nonlinear terms. Therefore, the linearization of the system around the trivial solution is given by

$$\frac{dZ_1}{dt} = T_1 Z_1 := \left(i\mathcal{M} + \frac{K}{2}\mathcal{P} \right) Z_1,$$
$$\frac{dZ_j}{dt} = ij\mathcal{M} Z_j, \quad j = 2, 3, \cdots$$

where the linear operators $\mathcal{M}$ and $\mathcal{P}$ are defined by $\mathcal{M} : f(\omega) \mapsto \omega f(\omega)$ and

$$\mathcal{P} f(\omega) = \int_{\mathbf{R}} f(\omega) g(\omega) d\omega.$$

The order parameter $\eta = \int_{\mathbf{R}} Z_1 g(\omega) d\omega = \mathcal{P} Z_1$ depends only on Z_1 at least for the linearized system. Hence, let us investigate the spectrum of the operator $T_1 = i\mathcal{M} + K\mathcal{P}/2$, that defines the linearized system for Z_1, as an operator on the Hilbert space $L^2(\mathbf{R}, g(\omega)d\omega)$ (weighted Lebesgue space).

3 Spectrum of linear operators

For the comparison with the generalized spectrum, we give a brief review of the (usual) spectral theory on a Banach space.

The spectrum set $\sigma(T)$ of a linear operator T on a Banach space X is the set of the singularities of the resolvent operator $(\lambda - T)^{-1}$. More precisely, it consists of ;

point spectrum $\sigma_p(T)$**.** The set of point λ such that $\lambda - T$ is not injective on X.

residual spectrum $\sigma_r(T)$**.** The set of point λ such that $\lambda - T$ is injective on X but its range is not a dense subspace of X.

continuous spectrum $\sigma_c(T)$**.** The set of point λ such that $\lambda - T$ is injective and the range is dense but the inverse $(\lambda - T)^{-1}$ is not a continuous operator on X.

The **resolvent set** is defined by $\rho(T) = \mathbf{C} \backslash \sigma(T)$.

The point spectrum is just the set of eigenvalues ; $Tv = \lambda v$ has a solution $v \neq 0$ in X. If X is a finite dimensional space, $\lambda - T$ is surjective if and only if

it is injective, however, it is not true for infinite one. Unfortunately, $\lambda - T$ is not surjective for most problems. Hence, we consider a more mild condition that the range of $\lambda - T$ is dense or not dense in X. When it is not dense, the set of such λ is the residual spectrum. If λ is neither in the point spectrum nor the residual spectrum, the inverse $(\lambda - T)^{-1}$ exists and the domain is dense. If $(\lambda - T)^{-1}$ is a continuous operator, we can continuously extend its domain to the whole space X, and λ is a regular point. The continuous spectrum is the set of λ such that we cannot extend the domain of $(\lambda - T)^{-1}$ to the whole space (recall the closed graph theorem) [2].

Let us consider the linear differential equation $du/dt = Tu$, $u \in X$ defined on a Banach space X. The asymptotic behavior of a solution as $t \to \infty$ is almost characterized by the spectrum of T. Indeed, under a suitable assumption for T, it is known that a solution is expressed by the Laplace inversion formula

$$u(t) = e^{Tt}u(0) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}(\lambda - T)^{-1}u(0)d\lambda, \tag{3.1}$$

for $t > 0$, where the integral path is a vertical straight line such that the spectrum set of T is included in the left half plane $\mathrm{Re}(\lambda) < a$. The operator e^{Tt} is called the semigroup generated by T. For example when X is a finite dimensional space, the set of singularities of the integrant $(\lambda - T)^{-1}$ consists only of the eigenvalues of T. In this case, we can calculate the Laplace inversion formula by deforming the integral path and using the residue theorem as is shown in Fig. 3. Hence, the real parts of eigenvalues completely determine the asymptotic behavior of solutions because of the factor $e^{\lambda t}$. This is true even if X is an infinite-dim space as long as T is a bounded operator or a sectorial operator, that admit the deformation of the integral path as in Fig. 3 [3].

[2]The concept of the spectrum makes sense only when T is a closed operator. Let us consider the point $\lambda \in \rho(T)$. By the definition, the resolvent $(\lambda - T)^{-1}$ is a continuous operator on X, in particular it is a closed operator. It is known that the inverse of a closed operator is also closed. Thus, $((\lambda - T)^{-1})^{-1} = \lambda - T$ is also closed. Then, $T = -(\lambda - T) + \lambda$ is also closed. Consider the contraposition. If T is not a closed operator, there are no $\lambda \in \rho(T)$; the whole complex plane is the spectrum set.

[3]Roughly speaking, a sectorial operator is an operator such that its spectrum set is included in a small sector that is open toward the left direction (i.e. included in the > shape region). A bounded operator is always sectorial because its spectrum set is compact.

If T is not sectorial, solutions $u(t)$ may diverge as $t \to \infty$ even when the spectrum set is included in the left half plane. A typical situation is that the spectrum set is not bounded for imaginary direction (so we can not take > shape region). This means that the spectrum set does not determine the behavior of solutions [18].

This difficulty essentially comes from the fact that the spectral mapping theorem does not hold. Let $S(t) = e^{Tt}$ be a semi-group generated by T. If T is bounded or a self-adjoint operator on a Hilbert space, we have the spectral mapping theorem, that states $e^{\sigma(T)t} = \sigma(S(t))$. However, in general we only have $e^{\sigma(T)t} \subset \sigma(S(t))$. This implies that $S(t)$ has an information that is not

With this in mind, let us calculate the spectrum of the operator $T_1 = i\mathcal{M} + K\mathcal{P}/2$. We only give the sketch of proofs, see [3] for the detail.

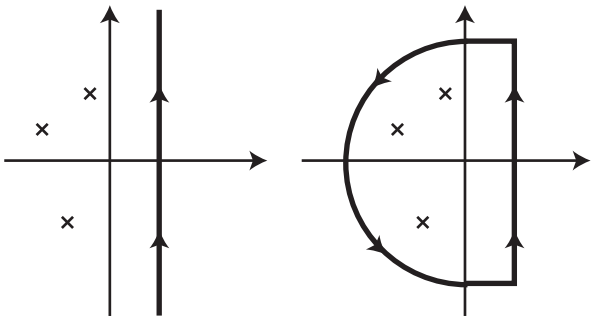

Figure 3: A deformation of the integral path. $\times$ denotes an eigenvalue.

Proposition 3.1 *T_1 is a densely defined closed operator on $L^2(\mathbf{R}, g(\omega)d\omega)$ satisfying*
(i) the continuous spectrum is $\sigma_c(T_1) = i \cdot \mathrm{supp}(g)$,
(ii) the residual spectrum is empty,
(iii) an eigenvalue is given as a root of the equation

$$\int_{\mathbf{R}} \frac{1}{\lambda - i\omega} g(\omega)d\omega = \frac{2}{K}, \quad \lambda \in \mathbf{C}\backslash\sigma_c(T_1), \tag{3.2}$$

if it exists.

Since $\mathcal{P}$ is a compact operator, (i) and (ii) immediately follow from the perturbation theory of linear operators. If g is the Gaussian distribution, $\sigma_c(T_1) = i\mathbf{R}$ is the whole imaginary axis. Let us derive the eigen-equation (3.2). Let $P_0(\omega) \equiv 1$ be a constant function. By using the inner product on $L^2(\mathbf{R}, g(\omega)d\omega)$, $\mathcal{P}$ is written as $\mathcal{P}f = (f, P_0)P_0$. This gives

$$\begin{aligned} \lambda v = T_1 v &= i\omega v + \frac{K}{2}(v, P_0)P_0 \\ \Longrightarrow \quad v &= \frac{K}{2}(v, P_0)(\lambda - i\omega)^{-1}P_0. \end{aligned}$$

obtained from an information of $\sigma(T)$.

The domain $\mathsf{D}(T)$ of an unbounded operator T is not the whole space X but its dense subspace. However, the domain of its semigroup $S(t)$ can be the whole space. Hence, if we take an initial condition $u(0)$ from $X\backslash\mathsf{D}(T)$, then the behavior of a solution $u(t)$ is not captured by $\sigma(T)$. If we choose $u(0)$ from $\mathsf{D}(T)$, we may obtain nice information about a solution from $\sigma(T)$, see [18] for the detail.

By taking the inner product with P_0 and dividing by (v, P_0), we obtain (3.2). From this calculation, it turns out that an eigenvector is given by

$$v(\omega) = \frac{1}{\lambda - i\omega} \tag{3.3}$$

if an eigenvalue λ exists.

Proposition 3.2 *Suppose that g is an even, unimodal and continuous function.*
(i) When $K > K_c := 2/(\pi g(0))$, there exists a unique eigenvalue on the positive real axis,
(ii) it converges to the origin as $K \to K_c + 0$ and
(iii) when $K \le K_c$, there are no eigenvalues.

Proof. By setting $\lambda = x + iy$ and decomposing (3.2) into the real and complex parts, we have

$$\begin{cases} \displaystyle\int_{\mathbf{R}} \frac{x}{x^2 + (\omega - y)^2} g(\omega) d\omega = \frac{2}{K}, \\ \displaystyle\int_{\mathbf{R}} \frac{\omega - y}{x^2 + (\omega - y)^2} g(\omega) d\omega = 0. \end{cases}$$

Further the second line is written as

$$\int_0^\infty \frac{\omega}{x^2 + \omega^2} \left(g(y + \omega) - g(y - \omega) \right) d\omega = 0.$$

If g is even, $y = 0$ satisfies it. By using that g is unimodal, we can verify that $y \neq 0$ does not satisfy. The first line shows that when $K > 0$, x is positive, which means that if an eigenvalue exists, it should be on the positive real line. It also follows from the first line that an eigenvalue is unique if it exists. When $|\lambda| = x$ is sufficiently large, the equation (3.2) is estimated as $1/\lambda + O(1/\lambda^2) = 2/K$, which proves that the eigenvalue exists and given by $\lambda \sim K/2$. On the other hand, since the left hand side of (3.2) is bounded on the right half plane, there are no eigenvalues when $K > 0$ is sufficiently small. This shows that there is a number $K_c > 0$ such that $x \to +0$ as $K \to K_c + 0$. The value K_c is obtained from the well-known formula

$$\lim_{x \to +0} \int_{\mathbf{R}} \frac{x}{x^2 + \omega^2} g(\omega) d\omega = \pi g(0).$$

This result proves that the trivial solution (de-synchro state) is unstable when $K > K_c$ because of the eigenvalue on the right half plane. The eigenvalue goes to the left side as K decreases, and finally it is absorbed into the continuous spectrum

on the imaginary axis at $K = K_c$. When $K \leq K_c$, there are no eigenvalues and the spectrum set consists only of the continuous spectrum on the imaginary axis. Since a point of the spectrum on the imaginary axis implies the neutral stability, we cannot prove the asymptotic stability from the spectrum $\sigma(T_1)$. This problem will be resolved by introducing the generalized spectrum.

4 Gelfand triplet

To handle the difficulty caused by the continuous spectrum on the imaginary axis, we develop the generalized spectral theory based on a Gelfand triplet. In this section, we will illustrate how the triplet naturally arises by a simple example.

Let us consider the multiplication operator $\mathcal{M} : f(x) \mapsto xf(x)$ on $L^2(\mathbf{R})$. The continuous spectrum is the whole real axis. Indeed, the resolvent is given by

$$(\lambda - \mathcal{M})^{-1} f(x) = \frac{1}{\lambda - x} f(x)$$

and it is not included in $L^2(\mathbf{R})$ when $\lambda \in \mathbf{R}$. Nevertheless, we will show that there exists a topological vector space larger than $L^2(\mathbf{R})$ on which the resolvent operator makes sense even if $\lambda \in \mathbf{R}$.

To this end, we consider the $L^2(\mathbf{R})$-inner product with some functions ϕ, ψ

$$((\lambda - \mathcal{M})^{-1}\phi, \psi^*) = \int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx,$$

where $\psi^*(x) := \overline{\psi(\overline{x})}$ is introduced to avoid the complex conjugate in the right hand side. The right hand side above is holomorphic in λ on the lower half plane $\{\mathrm{Im}(\lambda) < 0\}$.

Next, suppose λ approaches the real axis from below

$$\lim_{\mathrm{Im}(\lambda)\to 0} \int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx.$$

The factor $1/(\lambda - x)$ diverges at $x = \lambda \in \mathbf{R}$, however, it is known that as long as ϕ and ψ are continuous functions on $\mathbf{R}$, the above integral exists as an improper integral and is continuous in $\lambda \in \mathbf{R}$.

Further suppose that λ moves to the upper half plane. It is known that as long as ϕ and ψ are holomorphic on the region $\{\mathrm{Im}(\lambda) \geq 0\}$, the above function of λ has an analytic continuation to the upper half plane given by

$$\int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx + 2\pi i \phi(\lambda)\psi(\lambda), \quad \mathrm{Im}(\lambda) > 0.$$

Now we have shown that if ϕ and ψ are holomorphic on the real axis and the upper half plane, the function $((\lambda - \mathcal{M})^{-1}\phi, \psi^*)$ of λ has an analytic continuation

from the lower to the upper half plane across the continuous spectrum on the real axis. We denote it as

$$R(\lambda;\phi,\psi) := \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - x}\phi(x)\psi(x)dx, & \mathrm{Im}(\lambda) < 0 \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - x}\phi(x)\psi(x)dx + 2\pi i\phi(\lambda)\psi(\lambda), & \mathrm{Im}(\lambda) > 0. \end{cases}$$

Motivated by this observation, let X be a dense subspace of $L^2(\mathbf{R})$ consisting of some class of holomorphic functions and X' be its dual space, the vector space of continuous linear functionals on X. The mapping $\phi \mapsto R(\lambda;\phi,\psi)$ defines a linear functional on X, which is denoted by $R(\lambda;\bullet,\psi) \in X'$. The topology on X is defined so that this functional is continuous. Then, the mapping $\psi \mapsto R(\lambda;\bullet,\psi)$ gives a linear mapping from X to X', denoted by $\mathcal{R}_\lambda$, that is holomorphic in $\lambda \in \mathbf{C}$. By the definition, $\mathcal{R}_\lambda = (\lambda - \mathcal{M})^{-1}$ when $\mathrm{Im}(\lambda) < 0$. We call $\mathcal{R}_\lambda$ the generalized resolvent of $\mathcal{M}$.

This discussion is summarized as follows: As an operator from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$, the resolvent operator $(\lambda - \mathcal{M})^{-1}$ is singular on the real axis because of the continuous spectrum. Nevertheless, if we regard it as an operator from X into X', it has an analytic continuation $\mathcal{R}_\lambda$ from the lower to the upper half plane. For any $\psi \in X$, $\mathcal{R}_\lambda\psi$ is an X'-valued holomorphic function.

If X is a dense subspace of $L^2(\mathbf{R})$ and the embedding is continuous, $L^2(\mathbf{R})$ is continuously embedded to the dual space X'. In this manner, we obtain the triplet

$$X \subset L^2(\mathbf{R}) \subset X' \tag{4.1}$$

called the **Gelfand triplet** or **rigged Hilbert space**.

The spectrum set is also generalized as follows. Let $\mathcal{H}$ be a Hilbert space and T a linear operator on $\mathcal{H}$. Recall that the spectrum set of T is the set of singularities of the resolvent $(\lambda - T)^{-1}$. Suppose that T has a continuous spectrum. In a similar manner to the above, suppose that there exists a suitable subspace $X \subset \mathcal{H}$ such that if we regard the resolvent as an operator from X to X', then it has an analytic continuation $\mathcal{R}_\lambda$ across the continuous spectrum. In general, the Riemann surface of $\mathcal{R}_\lambda$ is nontrivial. If the analytic continuation $\mathcal{R}_\lambda$ has a new singularity on the Riemann surface different from the original complex plane, we call it a **generalized spectrum**. By the definition, it is not a true eigenvalue in $\mathcal{H}$-sense, however, it is expected that it plays a similar role to a usual eigenvalue and provides a new information that is not obtained from the framework of a Hilbert space.

It is applied to the dynamics of the Kuramoto model as follows. Recall that the semigroup e^{Tt} generated by T is given by the Laplace inversion formula (3.1). As explained, if T is a bounded operator, we can estimate the formula by deforming the integral path as shown in Fig. 3. However, for the operator T_1 obtained

from the Kuramoto model, any point on the imaginary axis is a singularity of the integrant $e^{\lambda t}(\lambda - T_1)^{-1}$ and we cannot deform the integral path toward the left half plane. Now we assume that there exists a subspace $X \subset L^2(\mathbf{R}, g(\omega)d\omega)$ such that the resolvent $(\lambda - T_1)^{-1}$ has an analytic continuation $\mathcal{R}_\lambda$ from the right to the left half plane as an operator from X into X'. Hence, we interpret (3.1) as

$$e^{Tt}\phi = \lim_{y\to\infty} \frac{1}{2\pi i} \int_{a-iy}^{a+iy} e^{\lambda t}\mathcal{R}_\lambda \phi \, d\lambda, \quad \phi \in X. \tag{4.2}$$

Then, we can deform the integral path toward the left half plane (more precisely, the second sheet of the Riemann surface), on which $\mathcal{R}_\lambda\phi \in X'$ [4]. A singularity of $\mathcal{R}_\lambda$ on the second Riemann sheet is called the generalized eigenvalue. By picking up the residue of the generalized eigenvalue, we can estimate the asymptotic behavior of the semigroup. Recall that there is the eigenvalue on the positive real axis when $K > K_c$, it moves to the left side as K decreases and is absorbed into the continuous spectrum as $K \to K_c + 0$, and disappears. Actually, the eigenvalue does not disappear. It moves to the second Riemann sheet when $K < K_c$ (now the imaginary axis is a branch cut of the Riemann surface). After getting across the branch cut, it is not an eigenvalue in the usual sense but becomes a generalized eigenvalue, that is not found in Hilbert space theory, see Fig. 4. We can deform the integral path to the second Riemann sheet and calculate the residue around the generalized eigenvalue, which prove the exponential decay of a solution of $dZ_1/dt = T_1 Z_1$ with respect to the topology of X' (asymptotic stability of the de-synchronization when $K < K_c$). Further, we can show that when the generalized eigenvalue crosses the imaginary axis at $K = K_c$ as K increases, a bifurcation from the de-synchro state to the synchronized state occurs. In this manner, the Kuramoto conjecture was proved [3].

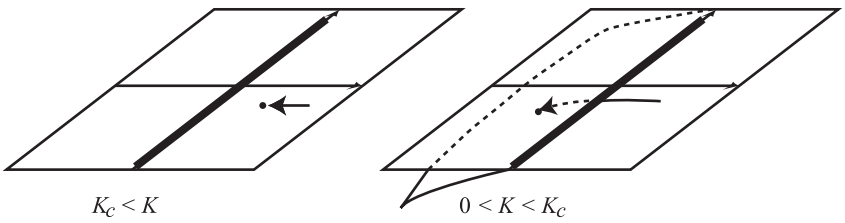

Figure 4: The motion of the (generalized) eigenvalue as K decreases. When $K > K_c$, it is a usual eigenvalue in L^2-sense. When $0 < K < K_c$, it is a generalized eigenvalue that lies on the second Riemann sheet different from the original complex plane.

[4] Thus, the limit $\lim_{y\to\infty}$ in (4.2) is considered in weak sense (weak dual topology on X').

From the next section, the generalized spectral theory will be formulated in a general setting. The reader can find all omitted proofs in [5, 6]. Throughout this article, $\mathsf{D}(\cdot)$ and $\mathsf{R}(\cdot)$ denote the domain and range of an operator, respectively. For terminologies of topological vector spaces, refer to [27].

5 The generalized spectral theory

Let X be a locally convex Hausdorff topological vector space over $\mathbf{C}$, X' be its dual space that is a vector space consisting of all continuous anti-linear functionals on X.

For $\mu \in X'$ and $\phi \in X$, $\mu(\phi)$ is denoted by $\langle \mu \,|\, \phi \rangle$. For any $a, b \in \mathbf{C}$, $\phi, \psi \in X$ and $\mu, \xi \in X'$, the equalities

$$\langle \mu \,|\, a\phi + b\psi \rangle = \overline{a}\langle \mu \,|\, \phi \rangle + \overline{b}\langle \mu \,|\, \psi \rangle,$$
$$\langle a\mu + b\xi \,|\, \phi \rangle = a\langle \mu \,|\, \phi \rangle + b\langle \xi \,|\, \phi \rangle,$$

hold. On the dual space X', there are several ways to introduce a topology. The most commonly used are the weak dual topology and the strong dual topology. A sequence $\{\mu_j\} \subset X'$ is said to be weakly convergent to $\mu \in X'$ if we have $\langle \mu_j \,|\, \phi \rangle \to \langle \mu \,|\, \phi \rangle$ for each $\phi \in X$. If we have $\langle \mu_j \,|\, \phi \rangle \to \langle \mu \,|\, \phi \rangle$ uniformly on arbitrary bounded subset in X, then $\{\mu_j\} \subset X'$ is said to be strongly convergent to $\mu \in X'$.

Let $\mathcal{H}$ be a Hilbert space with a Hermitian inner product $(\cdot\,,\,\cdot)$. Suppose X is a dense subspace of $\mathcal{H}$ and the embedding into $\mathcal{H}$ is continuous (i.e. the topology of X is stronger than that of $\mathcal{H}$). By considering their duals, it turns out that $\mathcal{H}'$ is continuously embedded into X'. Since a Hilbert space is isomorphic to itself, we have $\mathcal{H}' \simeq \mathcal{H} \subset X'$.

Definition 5.1 *Assume that a locally convex Hausdorff topological vector space X is a dense subspace of a Hilbert space $\mathcal{H}$ and the topology of X is stronger than that of $\mathcal{H}$. The triplet*

$$X \subset \mathcal{H} \subset X'$$

is called the ***Gelfand triplet*** *or the* ***rigged Hilbert space****.*

The embedding $i : \mathcal{H} \to X'$ is defined as follows: For $\psi \in \mathcal{H}$, $i(\psi)$ is denoted by $\langle \psi |$ and defined as

$$i(\psi)(\phi) = \langle \psi \,|\, \phi \rangle = (\psi, \phi), \quad \phi \in X.$$

In other words, the isomorphism $\mathcal{H} \simeq \mathcal{H}'$ is defined so that $\langle \psi \,|\, \phi \rangle$ is compatible with the inner product when $\psi \in \mathcal{H}$. The embedding is injective and continuous by the assumption in Def. 5.1. A Gelfand triplet was proposed to generalize the theory of Schwartz distribution [15], for which $X = C_0^\infty(\mathbf{R}^m)$ and , $\mathcal{H} = L^2(\mathbf{R}^m)$.

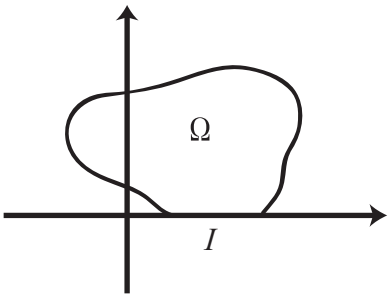

Figure 5: The region Ω and the interval I.

5.1 Generalized eigenvalue

Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$, H be a self-adjoint operator densely defined on $\mathcal{H}$ and $\{E(B)\}_{B\in\mathcal{B}}$ its spectral measure; H admits the spectral representation $H = \int_{\mathbf{R}} \omega dE(\omega)$. Let K be another densely defined operator on $\mathcal{H}$. The purpose here is to investigate spectral properties of $T := H + K$. For the Kuramoto model, $H = \mathcal{M}$, $K = \mathcal{P}$, and for a Schrödinger operator, H is the Laplacian, K is a potential function [6].

Let $\Omega \subset \mathbf{C}$ be a simply connected region included in the upper half plane, and let $\bar{I}$ be the intersection of the closure of Ω and the real axis (we assume that it is not empty and connected). Let I be an open interval obtained by removing the end points from $\bar{I}$ (Fig. 5). Later we will see that our setting makes sense when $\bar{I}$ is a (subset of) continuous spectrum of H. For a given operator $T = H + K$ on $\mathcal{H}$, we assume that there exists a locally convex Hausdorff topological vector space $X(\Omega)$ satisfying the following conditions.

(X1) $X(\Omega)$ is a dense subspace of $\mathcal{H}$.
(X2) The topology of $X(\Omega)$ is stronger than that of $\mathcal{H}$.
(X3) $X(\Omega)$ is a quasi-complete barreled space.

By (X1), (X2), the Gelfand triple

$$X(\Omega) \subset \mathcal{H} \subset X(\Omega)'$$

is well-defined. The definition of a barreled space is rather complicated [27]. It includes any Fréchet space, Banach space, Hilbert space, nuclear space and

Montel space [5]. If X is a barreled space, the Banach-Steinhaus theorem [6] holds and usual complex function theory (such as Cauchy theorem) is applicable for X'-valued functions [5].

Next, we need the assumptions for the spectral measure $E(B)$ of H.

(X4) For any $\phi \in X(\Omega)$, the spectral measure $(E(B)\phi, \phi)$ is absolutely continuous on the interval I [7]. Its density function denoted by $E[\phi, \phi](\omega)$ has an analytic continuation to the region $\Omega \cup I$.
(X5) For each point $\lambda \in I \cup \Omega$, the bilinear form $E[\,\cdot\,, \cdot\,](\lambda) : X(\Omega) \times X(\Omega) \to \mathbf{C}$ is separately continuous.

By (X4) with the aid of the polarization identity, we can verify that $(E(B)\phi, \psi)$ is absolutely continuous on I for any $\phi, \psi \in X(\Omega)$. We denote its density function as $E[\phi, \psi](\omega)$:

$$d(E(\omega)\phi, \psi) = E[\phi, \psi](\omega)d\omega, \quad \omega \in I.$$

Then, the function $E[\phi, \psi](\omega)$ is holomorphic in $\omega \in I \cup \Omega$. For simplicity, we use the notation $E[\phi, \psi](\omega)$ for any $\omega \in \mathbf{R}$.

Define a linear operator $A(\lambda) : X(\Omega) \to X(\Omega)'$ to be

$$\langle A(\lambda)\psi \,|\, \phi\rangle = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega)d\omega + 2\pi i E[\psi, \phi](\lambda) & (\lambda \in \Omega), \\ \displaystyle\lim_{y \to -0} \int_{\mathbf{R}} \frac{1}{x + iy - \omega} E[\psi, \phi](\omega)d\omega & (\lambda = x \in I), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - \omega} E[\psi, \phi](\omega)d\omega & (\mathrm{Im}(\lambda) < 0). \end{cases} \tag{5.1}$$

We can verify that the function $\langle A(\lambda)\psi \,|\, \phi\rangle$ is holomorphic in $\{\mathrm{Im}(\lambda) < 0\} \cup \Omega \cup I$. In particular, if $\mathrm{Im}(\lambda) < 0$ then $\langle A(\lambda)\psi \,|\, \phi\rangle = ((\lambda - H)^{-1}\psi, \phi)$ and $A(\lambda)$ coincides

[5] If a locally convex topological vector space is barreled and has the property that "any closed and bounded set is compact", then it is called a Montel space (this property is often called the Heine-Borel property). A Montel space has a nice property that any weakly convergent series is also strongly convergent. For a sufficient condition for a given space to be a Montel, refer to [16], [19]. For example, they are Montel spaces: the space of C^∞ functions, the space of C^∞ functions with compact support, the space of rapidly decreasing C^∞ functions, the space of holomorphic functions on an open region, and their dual spaces.

[6] **Banach-Steinhaus theorem.**
Let X be a barreled space and X' its dual space. For a subset $A \subset X'$, the following conditions are equivalent.
(i) A is bounded with respect to the weak dual topology.
(ii) A is bounded with respect to the strong dual topology.
(iii) A is equicontinuous as a family of mappings.
(iv) A is relatively compact with respect to the weak dual topology.
(i) $\Rightarrow$ (ii) is well-known as the uniform boundedness principle when X is a Banach space.

[7] Imagine the situation that $\bar{I}$ is a spectrum of H. Otherwise $(E(B)\phi, \phi) = 0$ on I.

with the resolvent of H. This means that $A(\lambda)\psi$ is an analytic continuation of the resolvent from the lower half plane to Ω through the interval I as an $X(\Omega)'$-valued function. We can show that $A(\lambda) : X(\Omega) \to X(\Omega)'$ is a continuous operator [8], if $X(\Omega)'$ is equipped with the weak dual topology.

We need some notation for the next assumptions. Let Q be a densely defined linear operator on $X(\Omega)$. Its dual operator $Q' : \mathsf{D}(Q') \to X(\Omega)'$ is defined as follows : The domain $\mathsf{D}(Q')$ of Q' is all elements $\mu \in X(\Omega)'$ so that the mapping $\phi \mapsto \langle \mu \,|\, Q\phi \rangle$ from $X(\Omega)$ to $\mathbf{C}$ is continuous, and Q' is defined through the equality $\langle Q'\mu \,|\, \phi \rangle = \langle \mu \,|\, Q\phi \rangle$. Next, for a densely defined operator Q on $\mathcal{H}$, its Hilbert-adjoint Q^* is defined through $(Q\phi, \psi) = (\phi, Q^*\psi)$. Moreover, if Q^* is densely defined on $X(\Omega)$, its dual operator $(Q^*)'$ can be considered and we denote is as $Q^\times$. Then, $Q^\times = (Q^*)'$ satisfies $Q = Q^\times|_{\mathsf{D}(Q)}$, which means that $Q^\times$ is a natural extension (lift) of Q from $\mathcal{H}$ to $X(\Omega)'$. For simplicity, we call $Q^\times$ the dual operator of Q.

For the operators H and K on $\mathcal{H}$, we assume the following.

(X6) $H\,(= H^*)$ is densely defined operator on $X(\Omega)$ (there is a dense subspace Y of $X(\Omega)$ such that $HY \subset X(\Omega)$).
(X7) K is H-bounded and K^* is densely defined on $X(\Omega)$.
(X8) For any $\lambda \in \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$, we have $K^\times A(\lambda) X(\Omega) \subset X(\Omega)$.

By (X6) and (X7), $H^\times, K^\times$ and $T^\times$ are densely defined on $X(\Omega)'$ (recall $T := H + K$). If H and K are continuous on $X(\Omega)$, so are $H^\times, K^\times$ and $T^\times$ on $X(\Omega)'$, but we do not assume it in general. An operator K is said to be H-bounded when $K(\lambda - H)^{-1}$ is a bounded operator on $\mathcal{H}$ for $\lambda \notin \sigma(H)$. Recalling that $A(\lambda)$ is an analytic continuation of $(\lambda - H)^{-1}$, (X8) is in some sense the analytic continuation version of (X7).

With these assumptions, we define a generalized eigenvalue. An eigenvalue and eigenvector in the usual sense are defined by $(\lambda - T)v = 0$. Since $T = H + K$ now, it is rewritten as $(id - (\lambda - H)^{-1}K)v = 0$. Recalling that the analytic continuation of $(\lambda - H)^{-1}$ in $X(\Omega)'$ is $A(\lambda)$, we make the following definition.

Definition 5.2 *If the equation*

$$(id - A(\lambda)K^\times)\mu = 0 \tag{5.2}$$

have a solution $0 \neq \mu \in X(\Omega)'$ *for some* $\lambda \in \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$, λ *and* μ *are called the* ***generalized eigenvalue*** *and* ***generalized eigenvector*** *of* T*, respectively.*

Applying $K^\times$ to (5.2), we obtain

$$(id - K^\times A(\lambda))K^\times \mu = 0. \tag{5.3}$$

[8] Note that since $X(\Omega)$ may not be a Banach space, there is a gap between a continuous operator and a bounded operator. The condition for two concepts to coincide other than a Banach space is complicated [1].

If $K^\times\mu = 0$, then (5.2) gives $\mu = 0$. Hence, λ is a generalized eigenvalue if and only if $id - K^\times A(\lambda)$ is not injective on $X(\Omega)$. Note that the operator $K^\times A(\lambda)$ on $X(\Omega)$ is well-defined because of (X8).

Theorem 5.3 *For a generalized eigenvalue λ of T and its generalized eigenvector μ, the equality*

$$T^\times\mu = \lambda\mu$$

holds.

Sketch of a Proof. By the operational calculus, we can show $\mathsf{D}(\lambda - H^\times) \supset \mathsf{R}(A(\lambda))$ and $(\lambda - H^\times)A(\lambda) = id|_{X(\Omega)}$. This yields

$$(\lambda - H^\times)(id - A(\lambda)K^\times)\mu = (\lambda - H^\times - K^\times)\mu = (\lambda - T^\times)\mu = 0.$$

Thus, a generalized eigenvalue is a true eigenvalue of the dual operator $T^\times$, although the converse statement is not true. An eigenvalue of $T^\times$ is not always a generalized eigenvalue. Since the dual space $X(\Omega)'$ is too large, typically any points in $\mathbf{C}$ become eigenvalues of $T^\times$.

5.2 Properties of $A(\lambda)$

For further discussion, let us investigate the properties of $A(\lambda)$ in detail. For $n = 1, 2, \cdots$, we define an linear operator $A^{(n)}(\lambda) : X(\Omega) \to X(\Omega)'$ to be

$$\langle A^{(n)}(\lambda)\psi \,|\, \phi\rangle = \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{(\lambda-\omega)^n} E[\psi,\phi](\omega)d\omega + 2\pi i \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\Big|_{z=\lambda} E[\psi,\phi](z), & (\lambda \in \Omega), \\ \displaystyle\lim_{y\to -0} \int_{\mathbf{R}} \frac{1}{(x+iy-\omega)^n} E[\psi,\phi](\omega)d\omega, & (\lambda = x \in I), \\ \displaystyle\int_{\mathbf{R}} \frac{1}{(\lambda-\omega)^n} E[\psi,\phi](\omega)d\omega. & (\mathrm{Im}(\lambda) < 0) \end{cases}$$

By integration by parts, it is easy to show that $\langle A^{(n)}(\lambda)\psi \,|\, \phi\rangle$ is an analytic continuation of $((\lambda - H)^{-n}\psi, \phi)$ from the lower half plane to the upper half plane. $A^{(1)}(\lambda)$ is also denoted by $A(\lambda)$ as before.

Proposition 5.4 *For any integer $j \geq n \geq 0$, the operator $A^{(j)}(\lambda)$ satisfies*

(i) $(\lambda - H^\times)^n A^{(j)}(\lambda) = A^{(j-n)}(\lambda)$, *where* $A^{(0)}(\lambda) := id$.

(ii) $A^{(j)}(\lambda)(\lambda - H^\times)^n = A^{(j-n)}(\lambda)$.
In particular, if $(\lambda - H^\times)\mu \in X(\Omega)$ *then* $A(\lambda)(\lambda - H^\times)\mu = \mu$.

(iii) $\dfrac{d^j}{d\lambda^j}\langle A(\lambda)\psi \,|\, \phi\rangle = (-1)^j j!\langle A^{(j+1)}(\lambda)\psi \,|\, \phi\rangle, \;\; j = 0, 1, \cdots.$

(iv) For any $\psi \in X(\Omega)$, $A(\lambda)\psi$ is expanded as

$$A(\lambda)\psi = \sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j A^{(j+1)}(\lambda_0)\psi, \tag{5.4}$$

and the right hand side converges with respect to the strong dual topology on $X(\Omega)'$.

Sketch of a Proof. (i) and (ii) are easily proved by the operational calculus. (iii) follows from the definition of $A(\lambda)$. Since $\langle A(\lambda)\psi \,|\, \phi\rangle$ is holomorphic, (iii) yields

$$\langle A(\lambda)\psi \,|\, \phi\rangle \quad = \quad \sum_{j=0}^{\infty}(\lambda_0 - \lambda)^j \langle A^{(j+1)}(\lambda_0)\psi \,|\, \phi\rangle, \tag{5.5}$$

which means that $A(\lambda)\psi$ is weakly holomorphic in $X(\Omega)'$. Since $X(\Omega)$ is barreled, a weakly holomorphic function is strongly holomorphic by the Banach-Steinhaus theorem.

Next, we define an eigenspace and the multiplicity of a generalized eigenvalue. In the usual spectral theory, the eigenspace of λ is defined as the space spanned by solutions of $(\lambda - T)^n v = 0$. For example $n = 2$, it is rearranged as

$$\begin{aligned} & (\lambda - H - K)(\lambda - H - K)v \\ = \; & (\lambda - H)^2(id - (\lambda - H)^{-2}K(\lambda - H)) \circ (id - (\lambda - H)^{-1}K)v = 0. \end{aligned}$$

Divided by $(\lambda - H)^2$, it gives

$$(id - (\lambda - H)^{-2}K(\lambda - H)) \circ (id - (\lambda - H)^{-1}K)v = 0.$$

Since the analytic continuation of $(\lambda - H)^{-n}$ is $A^{(n)}(\lambda)$, we may consider the equation

$$(id - A^{(2)}(\lambda)K^{\times}(\lambda - H^{\times})) \circ (id - A(\lambda)K^{\times})\,\mu = 0.$$

Thus, let us define an operator $B^{(n)}(\lambda) : \mathsf{D}(B^{(n)}(\lambda)) \subset X(\Omega)' \to X(\Omega)'$ by

$$B^{(n)}(\lambda) = id - A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1}. \tag{5.6}$$

Then, the above equation is simply written as $B^{(2)}(\lambda)B^{(1)}(\lambda)\mu = 0$. The domain of $B^{(n)}(\lambda)$ is the domain of $A^{(n)}(\lambda)K^{\times}(\lambda - H^{\times})^{n-1}$. The following equality

$$(\lambda - H^{\times})^k B^{(j)}(\lambda) = B^{(j-k)}(\lambda)(\lambda - H^{\times})^k, \quad j > k \tag{5.7}$$

is easily proved.

Definition 5.5 *The **generalized eigenspace** associated with a generalized eigenvalue λ is defined by*

$$V_\lambda = \bigcup_{m\geq 1} \mathrm{Ker}\, B^{(m)}(\lambda) \circ B^{(m-1)}(\lambda) \circ \cdots \circ B^{(1)}(\lambda),$$

and $\dim V_\lambda$ *is called the **multiplicity** of λ.*

In particular, an element of $\mathrm{Ker}\, B^{(1)}(\lambda)$ is a generalized eigenvector defined in Def. 5.2. In the same way as Thm. 5.3, we can prove the next theorem.

Theorem 5.6 *For any $\mu \in V_\lambda$, there exists an integer M such that $(\lambda - T^\times)^M \mu = 0$.*

The theorem means that V_λ is a subspace of an eigenspace $\bigcup_{m\geq 1} \mathrm{Ker}\,(\lambda - T^\times)^m$ of $T^\times$. Since the dual space $X(\Omega)'$ is too large, typically $\bigcup_{m\geq 1} \mathrm{Ker}\,(\lambda - T^\times)^m$ becomes an infinite dimensional, however, V_λ is finite dimensional for most applications (Thm. 5.16).

5.3 Generalized resolvent

Let $R_\lambda = (\lambda - T)^{-1}$ be the resolvent operator of T. Since

$$R_\lambda \psi = (\lambda - H)^{-1} \left(id - K(\lambda - H)^{-1}\right)^{-1} \psi \tag{5.8}$$

and the analytic continuation of $(\lambda - H)^{-1}$ in $X(\Omega)'$ is given by $A(\lambda)$, we make the following definition. In what follows, we put $\hat{\Omega} = \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$.

Definition 5.7 *When the inverse $(id - K^\times A(\lambda))^{-1}$ exists on $X(\Omega)$, the **generalized resolvent** $\mathcal{R}_\lambda : X(\Omega) \to X(\Omega)'$ of T is defined by*

$$\mathcal{R}_\lambda = A(\lambda) \circ (id - K^\times A(\lambda))^{-1} = (id - A(\lambda)K^\times)^{-1} \circ A(\lambda), \quad \lambda \in \hat{\Omega}. \tag{5.9}$$

The second equality follows from $(id - A(\lambda)K^\times)A(\lambda) = A(\lambda)(id - K^\times A(\lambda))$. Note that $id - K^\times A(\lambda)$ is an operator on $X(\Omega)$ because of (X8), and $id - A(\lambda)K^\times$ is an operator on $\mathsf{R}(A(\lambda))$. The former is injective if and only if so is the latter. Since $A(\lambda)$ is continuous as mentioned in Sec. 5.1, we require that $\mathcal{R}_\lambda : X(\Omega) \to X(\Omega)'$ is also continuous.

Definition 5.8 *The set of $\lambda \in \hat{\Omega}$ satisfying the following two conditions is called the **generalized resolvent set** $\hat{\varrho}(T)$; There exists a neighborhood $V_\lambda \subset \hat{\Omega}$ of λ such that*
(i) For any $\lambda' \in V_\lambda$, $\mathcal{R}_{\lambda'}$ is a densely defined continuous operator from $X(\Omega)$ into $X(\Omega)'$, where $X(\Omega)'$ is equipped with the weak dual topology.
(ii) For any $\psi \in X(\Omega)$, the set $\{\mathcal{R}_{\lambda'}(\psi)\}_{\lambda' \in V_\lambda}$ is a bounded set in $X(\Omega)'$ [9].

[9] Because of the Banach-Steinhaus theorem, a weakly bounded set is strongly bounded. Thus, we need not specify a topology here.

The complement $\hat{\sigma}(T) = \hat{\Omega}\backslash\hat{\varrho}(T)$ is called the **generalized spectrum set** of T. The **generalized point spectrum** $\hat{\sigma}_p(T)$ is the set of points $\lambda \in \hat{\sigma}(T)$ at which $id - K^{\times}A(\lambda)$ is not injective (this is the set of generalized eigenvalues). The **generalized residual spectrum** $\hat{\sigma}_r(T)$ is the set of points $\lambda \in \hat{\sigma}(T)$ such that the domain of $\mathcal{R}_\lambda$ is not dense in $X(\Omega)$. The **generalized continuous spectrum** is defined to be $\hat{\sigma}_c(T) = \hat{\sigma}(T)\backslash(\hat{\sigma}_p(T) \cup \hat{\sigma}_r(T))$.

By the definition, $\hat{\varrho}(T)$ is an open set. This definition looks rather complicated because $X(\Omega)$ is not a Banach space. To require the existence of the neighborhood $V_\lambda \subset \hat{\Omega}$ in the above definition was introduced by Waelbroeck [31] (see also Maeda [18]) for the spectral theory on locally convex spaces. If $\hat{\varrho}(T)$ were simply defined to be the set of points such that $\mathcal{R}_{\lambda'}$ is a densely defined continuous operator as in the Banach space theory, $\hat{\varrho}(T)$ is not an open set in general. If $X(\Omega)$ is a Banach space, the definition coincides with the usual definition of the resolvent set in a Banach space.

Theorem 5.9
(i) For each $\psi \in X(\Omega)$, $\mathcal{R}_\lambda(\psi)$ is an $X(\Omega)'$-valued holomorphic function on $\hat{\varrho}(T)$.
(ii) When $\mathrm{Im}(\lambda) < 0$, $\mathcal{R}_\lambda = i \circ (\lambda - T)^{-1}$ *(i is the embedding into $X(\Omega)'$).*

The second part (ii) implies that when $\mathrm{Im}(\lambda) < 0$, the equality $\langle \mathcal{R}_\lambda\psi \,|\, \phi\rangle = ((\lambda - T)^{-1}\psi, \phi)$ holds for $\psi, \phi \in X(\Omega)$. Thus, $\langle \mathcal{R}_\lambda\psi \,|\, \phi\rangle$ is an analytic continuation of $((\lambda - T)^{-1}\psi, \phi)$.

Sketch of a Proof of (i).
Put $\psi_\lambda = (id - K^{\times}A(\lambda))^{-1}(\psi)$. It is easy to confirm that

$$\begin{aligned} & \mathcal{R}_{\lambda+h}(\psi) - \mathcal{R}_\lambda(\psi) \\ = \; & (A(\lambda + h) - A(\lambda))(\psi_\lambda) + \mathcal{R}_{\lambda+h}K^{\times}(A(\lambda + h) - A(\lambda))(\psi_\lambda). \end{aligned}$$

We show that it tends to zero as $h \to 0$ with respect to the weak dual topology on $X(\Omega)'$. Since $A(\lambda)$ is holomorphic in λ, the first term is easy to treat. To estimate the second term, we need to estimate $\mathcal{R}_{\lambda+h}$ and $K^{\times}A(\lambda)$. For the latter one, we can verify that $K^{\times}A(\lambda)$ is also holomorphic in λ as an $X(\Omega)$-valued function, so that

$$\phi_h := K^{\times}(A(\lambda + h) - A(\lambda))(\psi_\lambda) \in X(\Omega)$$

tends to zero as $h \to 0$. For the former one, the set $\{\mathcal{R}_{\lambda+h}(\phi); |h| : \textit{small}\}$ is bounded for any $\phi \in X(\Omega)$ due to the condition (ii) in Def. 5.8. This shows that $\mathcal{R}_{\lambda+h}(\phi_h) \to 0$ weakly as $h \to 0$ by the condition (i). Hence, $\mathcal{R}_{\lambda+h}(\psi) \to \mathcal{R}_\lambda(\psi)$ as $h \to 0$.

Repeating the same procedure after dividing by h, it turns out that $\mathcal{R}_\lambda(\psi)$ is weakly holomorphic. Since $X(\Omega)$ is barreled, it is automatically strongly holomorphic.

Proposition 5.10 *$\mathcal{R}_\lambda$ satisfies that*
(i) $(\lambda - T^\times) \circ \mathcal{R}_\lambda = id|_{X(\Omega)}$,
(ii) when $\mu \in X(\Omega)'$ *satisfies* $(\lambda - T^\times)\mu \in X(\Omega)$, *then* $\mathcal{R}_\lambda \circ (\lambda - T^\times)\mu = \mu$,
(iii) $T^\times \circ \mathcal{R}_\lambda = \mathcal{R}_\lambda \circ T^\times$.

This proposition immediately follows from Prop. 5.4.

5.4 Generalized projection

Let $\Sigma \subset \hat{\sigma}(T)$ be a bounded subset of the generalized spectrum set which is separated from the rest of the spectrum by a simple closed curve $\gamma \subset \Omega \cup I \cup \{\lambda \,|\, \mathrm{Im}(\lambda) < 0\}$. Define an operator $\Pi_\Sigma : X(\Omega) \to X(\Omega)'$ by

$$\Pi_\Sigma \phi = \frac{1}{2\pi i} \int_\gamma \mathcal{R}_\lambda \phi \, d\lambda, \quad \phi \in X(\Omega), \tag{5.10}$$

where the integral is defined as the Pettis integral [10]. Since the composition $\Pi_\Sigma \circ \Pi_\Sigma$ cannot be defined, it is not a projection operator in the usual sense. Nevertheless, it is reasonable to call Π_Σ the **generalized projection** because of the following results.

Proposition 5.11 *The following hold*

$$\Pi_\Sigma(X(\Omega)) \cap (id - \Pi_\Sigma)(X(\Omega)) = \{0\}$$
$$X(\Omega) \subset \Pi_\Sigma(X(\Omega)) \oplus (id - \Pi_\Sigma)(X(\Omega)) \subset X(\Omega)'$$

In particular, for any $\phi \in X(\Omega)$, *there are* $\mu_1, \mu_2 \in X(\Omega)'$ *such that* ϕ *is uniquely decomposed as*

$$i(\phi) = \langle \phi | = \mu_1 + \mu_2, \quad \mu_1 \in \Pi_\Sigma(X(\Omega)), \ \mu_2 \in (id - \Pi_\Sigma)(X(\Omega)). \tag{5.11}$$

Proposition 5.12 Π_Σ *is* $T^\times$*-invariant :* $\Pi_\Sigma \circ T^\times = T^\times \circ \Pi_\Sigma$.

Theorem 5.13 *Let* λ_0 *be an isolated generalized eigenvalue,* Π_0 *be the generalized projection for* λ_0 *and* V_0 *be the generalized eigenspace of* λ_0 *(Def. 5.5). If* $\Pi_0 X(\Omega)$ *is finite dimensional,* $\Pi_0 X(\Omega) = V_0$.

[10] In general, let X be a topological vector space, X' its dual space with the strong dual topology, S compact Hausdorff space and μ be a finite Borel measure on S. For a mapping $f : S \to X'$, if there exists $I(f) \in X'$ satisfying

$$\langle I(f) \,|\, \phi \rangle = \int_S \langle f \,|\, \phi \rangle d\mu$$

for any $\phi \in X$, then f is said to be Pettis integrable and $I(f) = \int_S f d\mu$ is called the Pettis integral of f. If X is barreled and f is holomorphic, it is Pettis integrable [5].

In the usual spectral theory, these properties are proved by using $\Pi \circ \Pi = \Pi$ and the resolvent identity. In our case, since these formulae do not hold because Π and $\mathcal{R}_\lambda$ are mappings from $X(\Omega)$ into $X(\Omega)'$ (i.e. the composition of them is not defined), the proof is rather technical [5].

5.5 Properties of of the generalized spectrum

Obviously the definition of the generalized spectrum depends on the choice of the space $X(\Omega)$. When we want to emphasize the choice, we denote $\hat{\sigma}(T)$ as $\hat{\sigma}(T; X(\Omega))$. If we have two spaces $X_1(\Omega)$ and $X_2(\Omega)$ satisfying (X1) to (X8), there are two generalized spectra $\hat{\sigma}(T; X_1(\Omega))$ and $\hat{\sigma}(T; X_2(\Omega))$.

Proposition 5.14 ([6]) *Suppose that $X_2(\Omega)$ is a dense subspace of $X_1(\Omega)$ and the topology of $X_2(\Omega)$ is stronger than that of $X_1(\Omega)$. Then, the following statements hold.*
(i) $\hat{\sigma}(T; X_2(\Omega)) \subset \hat{\sigma}(T; X_1(\Omega))$,
(ii) Let $\Sigma \neq \emptyset$ be a bounded subset of $\hat{\sigma}(T; X_1(\Omega))$ which is separated from the rest of the spectrum $\hat{\sigma}(T; X_1(\Omega))$ by a simple closed curve γ. Then, there exists a point of $\hat{\sigma}(T; X_2(\Omega))$ inside γ. In particular, if λ is an isolated point of $\hat{\sigma}(T; X_1(\Omega))$, then $\lambda \in \hat{\sigma}(T; X_2(\Omega))$.

Sketch of a Proof. Because of the assumption of the topology, the generalized resolvent $\mathcal{R}_\lambda : X_2(\Omega) \to X_2(\Omega)'$ behaves "better" than $\mathcal{R}_\lambda : X_1(\Omega) \to X_1(\Omega)'$, which proves (i). For (ii), let Π_Σ be the generalized projection. By the assumption, $\Pi_\Sigma X_1(\Omega) \neq \{0\}$. Since $X_2(\Omega)$ is dense in $X_1(\Omega)$, we have $\Pi_\Sigma X_2(\Omega) \neq \{0\}$. □

Due to this theorem, the existence of isolated generalized eigenvalues is independent of the choice of $X(\Omega)$ [11].

For the next theorem, we define a uniformly compact operator. A linear operator L from a topological vector space X_1 to another topological vector space X_2 is said to be bounded if there exists a neighborhood $U \subset X_1$ of the origin such that $LU \subset X_2$ is a bounded set. When $L = L(\lambda)$ is parameterized by λ, it is said to be bounded uniformly in λ if such a neighborhood U is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$ is bounded uniformly in λ if and only if $L(\lambda)$ is continuous for each λ (U is taken to be the unit sphere).

Similarly, L is called compact if there exists a neighborhood $U \subset X_1$ of the origin such that $LU \subset X_2$ is relatively compact. When $L = L(\lambda)$ is parameterized by λ, it is said to be compact uniformly in λ if such a neighborhood U is independent of λ. When the domain X_1 is a Banach space, $L(\lambda)$ is compact uniformly in λ

[11] Historically, several definitions of the generalized eigenvalues had been proposed. It seems that their results are the same because of this theorem. For example, for the study of Schrödinger operators, the generalized eigenvalue is called the resonance pole. This is defined by the analytic continuation of a scattering matrix [24] or the method of complex deformation [17] and so on.

if and only if $L(\lambda)$ is compact for each λ. When the range X_2 is a Montel space, a (uniformly) bounded operator is (uniformly) compact because every bounded set in a Montel space is relatively compact.

Put $\hat{\Omega} := \{\mathrm{Im}(\lambda) < 0\} \cup I \cup \Omega$ as before. In many applications, $K^{\times}A(\lambda)$ is a bounded operator on $X(\Omega)$. In such a case, the following proposition is useful to estimate the generalized spectrum.

Proposition 5.15 *Suppose that for fixed $\lambda \in \hat{\Omega}$, there exists a neighborhood $U_\lambda \subset \hat{\Omega}$ of λ such that $K^{\times}A(\lambda') : X(\Omega) \to X(\Omega)$ is a bounded operator uniformly in $\lambda' \in U_\lambda$. If $id - K^{\times}A(\lambda)$ has a continuous inverse on $X(\Omega)$, then $\lambda \notin \hat{\sigma}(T)$.*

Sketch of a Proof. Check the condition of Def. 5.8. On the generalized resolvent $\mathcal{R}_\lambda = A(\lambda) \circ (id - K^{\times}A(\lambda))^{-1}$, since $A(\lambda) : X(\Omega) \to X(\Omega)'$ is continuous, it is sufficient to show that there exists a neighborhood V_λ of λ such that the set $\{(id - K^{\times}A(\lambda'))^{-1}\psi\}_{\lambda' \in V_\lambda}$ exists and is bounded in $X(\Omega)$ for any ψ. For this purpose, it is sufficient to show that the mapping $\lambda' \mapsto (id - K^{\times}A(\lambda'))^{-1}\psi$ is continuous in $\lambda' \in V_\lambda$. Since $A(\lambda)$ is holomorphic, there is an operator $D(\lambda, h)$ such that

$$\begin{aligned} id - K^{\times}A(\lambda + h) &= id - K^{\times}A(\lambda) - hD(\lambda, h) \\ &= \Big(id - hD(\lambda, h)(id - K^{\times}A(\lambda))^{-1}\Big) \circ (id - K^{\times}A(\lambda)) \end{aligned}$$

for small $h \in \mathbf{C}$. Since $K^{\times}A(\lambda)$ is uniformly bounded by the assumption, $D(\lambda, h)$ is a uniformly bounded operator in h. Further, $(id - K^{\times}A(\lambda))^{-1}$ is continuous by the assumption. Thus, $D(\lambda, h)(id - K^{\times}A(\lambda))^{-1}$ is a bounded operator. Then, Bruyn's theorem [2] is applicable to show that $id - hD(\lambda, h)(id - K^{\times}A(\lambda))^{-1}$ has a continuous inverse that is continuous in h (when $X(\Omega)$ is a Banach space, Bruyn's theorem is reduced to the existence of the Neumann series). This proves that $id - K^{\times}A(\lambda')$ has a continuous inverse which is continuous in λ'.

Theorem 5.16 *Suppose that $K^{\times}A(\lambda) : X(\Omega) \to X(\Omega)$ is a compact operator uniformly in $\lambda \in \hat{\Omega}$. Then, the following statements hold.*
(i) For any compact set $D \subset \hat{\Omega}$, the number of generalized eigenvalues in D is finite (thus $\hat{\sigma}_p(T)$ consists of a countable number of generalized eigenvalues and they may accumulate only on the boundary of $\hat{\Omega}$ or infinity).
(ii) For each $\lambda_0 \in \hat{\sigma}_p(T)$, the generalized eigenspace V_0 is of finite dimensional (in particular Thm. 5.13 holds).
(iii) $\hat{\sigma}_c(T) = \hat{\sigma}_r(T) = \emptyset$.

This kind of result is well-known as the Riesz-Schauder theory for a Banach space. Even if $X(\Omega)$ is not Banach but a general locally convex vector space, it is known that the Riesz-Schauder theory is valid [25], which is used to prove the above theorem.

5.6 Semigroup

Suppose that the operator $iT = i(H+K)$ generates a C^0-semigroup e^{iTt} on $\mathcal{H}$ (here, i is not the embedding but $\sqrt{-1}$). It is expressed by the Laplace inversion formula as

$$(e^{iTt}\psi, \phi) = \frac{1}{2\pi i} \lim_{x\to\infty} \int_{-x-iy}^{x-iy} e^{i\lambda t}((\lambda - T)^{-1}\psi, \phi)d\lambda, \quad x, y \in \mathbf{R}, \tag{5.12}$$

for $\phi, \psi \in \mathcal{H}$, where the integral path is the horizontal straight line below the spectrum of T. If T has a continuous spectrum on the real axis, we cannot deform the integral path from the lower to the upper half plane and it is difficult to estimate the asymptotic behavior of the semigroup as $t \to \infty$. However, for $\phi, \psi \in X(\Omega)$, we can rewrite (5.12) as

$$(e^{iTt}\psi, \phi) = \frac{1}{2\pi i} \lim_{x\to\infty} \int_{-x-iy}^{x-iy} e^{i\lambda t}\langle \mathcal{R}_\lambda \psi \,|\, \phi\rangle d\lambda.$$

As a result, the path can be deformed toward the Riemann surface of $\mathcal{R}_\lambda$. In many applications, the set of singularities of $\langle \mathcal{R}_\lambda \psi \,|\, \phi\rangle$ consists of isolated generalized eigenvalues, and we can estimate the Laplace inversion formula by the residue theorem. A residue is calculated by using the generalized projection. Let Π_0 be the projection associated with an isolated generalized eigenvalue λ_0 with the multiplicity M. The residue of it is given by

$$\frac{1}{2\pi i}\int_{\gamma_0} e^{i\lambda t}\langle \mathcal{R}_\lambda \psi \,|\, \phi\rangle d\lambda = \sum_{k=0}^{M-1} e^{i\lambda_0 t}\frac{(-it)^k}{k!}\langle (\lambda_0 - T^\times)^k \Pi_0 \psi \,|\, \phi\rangle,$$

where γ_0 is a small simple closed curve enclosing λ_0, see Fig. 6. In particular, if λ_0 lies on the upper half plane, it induces an exponentially decaying term with respect to the weak topology on $X(\Omega)'$, not the topology on $\mathcal{H}$. This kind of decay in the weak dual topology induced by the generalized eigenvalues is known in plasma physics as the Landau damping [14] and in Schrödinger equations as a tunnel effect [17, 24, 6].

In general, the decay of a semigroup occurs only during a transient state. To see it, let λ_0 be a generalized eigenvalue on the upper half plane and $\mu_0 \in X(\Omega)'$ its generalized eigenvector. Let $(e^{iTt})^\times$ be the dual operator. Since the equality $(e^{iTt})^\times \mu_0 = e^{i\lambda_0 t}\mu_0$ holds, if we consider μ_0 as the initial condition, actually the semigroup decays to zero exponentially. However, μ_0 is an element of the dual space, which may be not a suitable choice for an application. Since $X(\Omega)$ is a dense subspace of $X(\Omega)'$, for any $\varepsilon > 0$, there are $\tau > 0$ and a function $\phi_0 \in X(\Omega)$ such that for $0 \le t \le \tau$, the inequality

$$|\langle (e^{iTt})^\times \phi_0 \,|\, \psi\rangle - \langle (e^{iTt})^\times \mu_0 \,|\, \psi\rangle| < \varepsilon$$

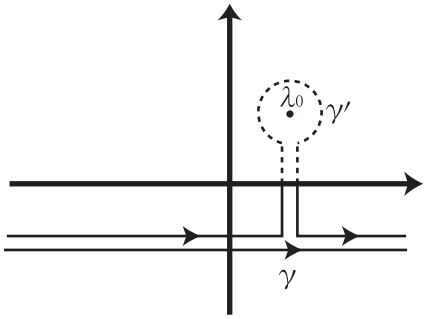

Figure 6: Deformation of the integral path γ to γ'. The solid curve denotes the path on the original complex plane, and the dotted one denotes the path on the second Riemann sheet.

holds. This means that for a finite time interval $0 \leq t \leq \tau$, we have

$$(e^{iTt}\phi_0, \psi) \sim e^{i\lambda_0 t}\langle \mu_0 \,|\, \psi \rangle,$$

and the generalized eigenvalue gives a transient behavior.

6 Application to the Kuramoto model

6.1 The stability of the de-synchronization state

Let us apply the previous results to the linear operator $T_1 = i\mathcal{M} + K\mathcal{P}/2$ obtained by the linearization of the Kuramoto model (Sec. 2). Since the self-adjoint operator $\mathcal{M}$ is multiplied by $i = \sqrt{-1}$, the right (resp. left) half plane play the same role as the lower (resp. upper) half plane in the previous sections.

Since the continuous spectrum of T_1 is the whole imaginary axis, we cannot determine the stability of the de-synchronization state within Hilbert space theory. Thus, we suitably introduce the Gelfand triplet $X \subset L^2(\mathbf{R}, g(\omega)d\omega) \subset X'$ so that if the resolvent $(\lambda - T_1)^{-1}$ is regarded as an operator from X into X', it has an analytic continuation from the right half plane to the left half plane across the continuous spectrum. We can show that the resolvent is calculated as

$$((\lambda - T_1)^{-1}\phi, \psi^*) = D[\phi, \psi](\lambda) + \frac{K}{2 - KD[P_0, P_0](\lambda)} D[\phi, P_0](\lambda) \cdot D[P_0, \psi](\lambda),$$

$$D[\phi, \psi](\lambda) := \int_{\mathbf{R}} \frac{1}{\lambda - i\omega} \phi(\omega)\psi(\omega)g(\omega)d\omega,$$

where $P_0(\omega) \equiv 1$ is a constant function and $\psi^*(x) := \overline{\psi(\overline{x})}$. If $D[\phi, \psi](\lambda)$ has an

analytic continuation from the right to the left half plane, it is given by

$$\int_{\mathbf{R}} \frac{1}{\lambda - i\omega} \phi(\omega)\psi(\omega)g(\omega)d\omega + 2\pi\phi(-i\lambda)\psi(-i\lambda)g(-i\lambda).$$

For the existence of the second term, ϕ and ψ should be holomorphic on the upper half plane. Thus, we define X to be some class of functions that are holomorphic on the upper half plane and are in $L^2(\mathbf{R}, g(\omega)d\omega)$ on the real axis (see [3] for the precise definition). With a suitable topology on X, T_1 and $X \subset L^2(\mathbf{R}, g(\omega)d\omega) \subset X'$ satisfy the assumptions (X1) to (X8). Further, they satisfy the assumption for Thm. 5.16 ; the generalized spectrum consists of discrete generalized eigenvalues with finite multiplicities (actually the multiplicities are 1).

The generalized eigenvalues are defined by (5.2). More convenient way to obtain them is as follows: An eigenvalue in the usual sense is given by the root of the equation (3.2). The analytic continuation of this equation from the right to the left half plane is given by

$$\int_{\mathbf{R}} \frac{1}{\lambda - i\omega} g(\omega)d\omega + 2\pi g(-i\lambda) = \frac{2}{K}, \quad (\mathrm{Re}(\lambda) < 0), \tag{6.1}$$

whose root gives a generalized eigenvalue. When g is the Gaussian distribution, there are infinitely many generalized eigenvalues on the left half plane. Recall that when $K > K_c$, there exists a unique eigenvalue $\lambda_0 = \lambda_0(K)$, which is a root of (3.2), on the positive real axis. For $K \leq K_c$, it becomes a root of (6.1). This implies that at $K = K_c$, $\lambda_0(K)$ crosses the imaginary axis (now it is a branch cut of the Riemann surface), goes to the second Riemann sheet and becomes a generalized eigenvalue (Fig. 4).

A generalized eigenvector is calculated in a similar manner. When $K > K_c$, the eigenvector $v \in L^2(\mathbf{R}, g(\omega)d\omega)$ of the usual eigenvalue λ is given by (3.3). By the embedding $i : L^2(\mathbf{R}, g(\omega)d\omega) \to X'$ we regard v as an element in X', which is denoted by μ_λ. Its action on X is defined by

$$\langle \mu_\lambda \,|\, \phi^* \rangle = (v, \phi^*) = \int_{\mathbf{R}} \frac{1}{\lambda - i\omega} \phi(\omega)g(\omega)d\omega, \quad (\mathrm{Re}(\lambda) > 0).$$

The generalized eigenvector μ_λ associated with a generalized eigenvalue on the left half plane λ is given by the analytic continuation, that is

$$\langle \mu_\lambda \,|\, \phi^* \rangle = \int_{\mathbf{R}} \frac{1}{\lambda - i\omega} \phi(\omega)g(\omega)d\omega + 2\pi\phi(-i\lambda)g(-i\lambda), \quad (\mathrm{Re}(\lambda) < 0).$$

It is not an element of $L^2(\mathbf{R}, g(\omega)d\omega)$ but of X'.

Let $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ be the set of generalized eigenvalues and their generalized eigenvectors, respectively. By deforming the integral path of the Laplace inversion formula as in Fig. 3 and using the residue theorem, we can prove the next theorem.

Theorem 6.1 (Spectral decomposition) *For any $\phi \in X$, the dual operator of the semigroup $e^{T_1 t}$ generated by T_1 is expanded as*

$$(e^{T_1 t})^{\times}\phi = \sum_{n=0}^{\infty} e^{\lambda_n t}\langle \mu_n \,|\, \phi\rangle \mu_n, \tag{6.2}$$

where the right hand side converses in X' with respect to the strong dual topology. When, $0 < K < K_c$, $\mathrm{Re}(\lambda_n) < 0$ for all $n = 0, 1, \cdots$, which proves that $(e^{T_1 t})^{\times}\phi$ converges to 0 as $t \to \infty$ in X' (asymptotic stability of the de-synchronization).

Note that T_1 is not a self-adjoint nor compact operator. Thus, a spectral decomposition does not hold within Hilbert space theory. Nevertheless, it is possible by using elements in X'.

6.2 Bifurcation to the synchronized state

The remaining task is to show a bifurcation from the de-synchro state to the synchro state. To investigate a bifurcation in dynamical systems, one of the most effective ways is to apply the center manifold reduction. In our problem, there is a continuous spectrum on the imaginary axis, so that the center manifold is not well-defined within Hilbert space. To treat this difficulty, by using generalized eigenvalues λ_n and their generalized eigenvectors μ_n, we define the generalized center subspace $\mathbf{E^c}$ by

$$\mathbf{E}^c := \mathrm{span}\{\mu_n \,|\, \lambda_n \in i\mathbf{R}\} \subset X'.$$

This is the range of the generalized projection associated with generalized eigenvalues on the imaginary axis. For the Kuramoto model, when $K = K_c$ there is a generalized eigenvalue $\lambda_0 = 0$ with the multiplicity 1. Thus, there exists a corresponding 1-dimensional center subspace $\mathbf{E^c}$. Furthermore, we can prove the existence of a center manifold in X' which is tangent to $\mathbf{E^c}$.

In Sec.2, we derive the system of equations for the Fourier coefficients Z_j. For example, the equation of Z_1 is

$$\frac{dZ_1}{dt} = i\omega Z_1 + \frac{K}{2}\eta(t) - \frac{K}{2}\overline{\eta(t)}Z_2 = T_1 Z_1 - \frac{K}{2}\overline{(Z_1, P_0)}Z_2.$$

This defines an equation on $L^2(\mathbf{R}, g(\omega)d\omega)$, however, it is difficult to investigate it in Hilbert space because of the continuous spectrum of T_1. Thus, by using the embedding $i : L^2(\mathbf{R}, g(\omega)d\omega) \to X'$, we regard this equation as an equation given on X'. To this end, the operator T_1 is replaced by its dual, the inner product (Z_1, P_0) is replaced by the paring $\langle Z_1 \,|\, P_0\rangle$:

$$\frac{dZ_1}{dt} = T_1^{\times} Z_1 - \frac{K}{2}\overline{\langle Z_1 \,|\, P_0\rangle}Z_2, \quad Z_j \in X'.$$

On the dual space, the center subspace $\mathbf{E^c}$ is well-defined. Let fix an element $\mu_0 \in \mathbf{E^c}$ (generalized eigenvector of $\lambda_0 = 0$). Let α be a coordinate of $\mathbf{E^c}$ and we decompose Z_1 into the direction of the center subspace and its complement as $Z_1 = \alpha(t)\mu_0 + Y_1$, where Y_1 is given by $Y_1 = (id - \Pi_0)Z_1$ by using the projection Π_0. Since Y_1 and $Z_2, Z_3, \cdots$ are vector that are outside the center subspace, we can assume that they are of order $O(\alpha^2)$. By substituting $Z_1 = \alpha(t)\mu_0 + Y_1$ into the equation and after a long calculation, the dynamical system on the center manifold is obtained as

$$\frac{d}{dt}\alpha = (K - K_c)p_1\alpha + p_3\alpha|\alpha|^2 + O(\alpha^5), \tag{6.3}$$

where p_1 and p_3 are constants given by

$$p_1 = \frac{D_0}{K_c}, \quad p_3 = \frac{\pi D_0 K_c^3 g''(0)}{16},$$

and D_0 is a constant related to the residue around $\lambda_0 = 0$. This equation is a normal form of the pitchfork bifurcation. Hence, it is easy to see that when $-p_1/p_3 > 0$ and $K > K_c$, there exists a steady state (fixed point) approximately given by

$$|\alpha| = \sqrt{\frac{-p_1}{p_3}}\sqrt{K - K_c} + O(K - K_c).$$

Since $r = |\alpha| + O(\alpha^2)$, this result gives Kuramoto's bifurcation diagram (Fig. 2).

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