

Extension and Unification of Singular Perturbation Methods for ODEs Based on the Renormalization Group Method

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Abstract

The renormalization group (RG) method is one of the singular perturbation methods which is used in search for asymptotic behavior of solutions of differential equations. In this article, time-independent vector fields and time (almost) periodic vector fields are considered. Theorems on error estimates for approximate solutions, existence of approximate invariant manifolds and their stability, inheritance of symmetries from those for the original equation to those for the RG equation, are proved. Further it is proved that the RG method unifies traditional singular perturbation methods, such as the averaging method, the multiple time scale method, the (hyper-) normal forms theory, the center manifold reduction, the geometric singular perturbation method and the phase reduction. A necessary and sufficient condition for the convergence of the infinite order RG equation is also investigated.

1 Introduction

Differential equations form a fundamental topic in mathematics and its application to natural sciences. In particular, perturbation methods occupy an important place in the theory of differential equations. Although most of the differential equations can not be solved exactly, some of them are close to solvable problems in some sense, so that perturbation methods, which provide techniques to handle such class of problems, have been long studied.

This article deals with a system of ordinary differential equations (ODEs) on a manifold M of the form

$$\frac{dx}{dt} = \varepsilon g(t, x, \varepsilon), \quad x \in M, \quad (1.1)$$

which is almost periodic in t with appropriate assumptions (see the assumption (A) in

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Sec.2.1), where $\varepsilon \in \mathbf{R}$ or \mathbf{C} is a small parameter.

Since ε is small, it is natural to try to construct a solution of this system as a power series in ε of the form

$$x = \hat{x}(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots . \quad (1.2)$$

Substituting Eq.(1.2) into Eq.(1.1) yields a system of ODEs on x_0, x_1, x_2, \cdots . The method to construct $\hat{x}(t)$ in this manner is called the *regular perturbation method*.

It is known that if the function $g(t, x, \varepsilon)$ is analytic in ε , the series (1.2) converges to an exact solution of (1.1), while if it is not analytic, (1.2) diverges and no longer provides an exact solution. However, the problem arising immediately is that one can not calculate infinite series like (1.2) in general whether it converges or not, because it involves infinitely many ODEs on x_0, x_1, x_2, \cdots . If the series is truncated at a finite-order term in ε , another problem arises. For example, suppose that Eq.(1.1) admits an exact solution $x(t) = \sin(\varepsilon t)$, and that we do not know the exact solution. In this case, the regular perturbation method provides a series of the form

$$\hat{x}(t) = \varepsilon t - \frac{1}{3!}(\varepsilon t)^3 + \frac{1}{5!}(\varepsilon t)^5 + \cdots . \quad (1.3)$$

If truncated, the series becomes a polynomial in t , which diverges as $t \rightarrow \infty$ although the exact solution is periodic in t . Thus, the perturbation method fails to predict qualitative properties of the exact solution. Methods which handle such a difficulty and provide acceptable approximate solutions are called *singular perturbation methods*. Many singular perturbation methods have been proposed so far [1,3,6,38,39,43,44,47,50] and many authors reported that some of them produced the same results though procedures to construct approximate solutions were different from one another [9,38,43,44,47].

The renormalization group (RG) method is the relatively new method proposed by Chen, Goldenfeld and Oono [8,9], which reduces a problem to a more simple equation called the *RG equation*, based on an idea of the renormalization group in quantum field theory. In their papers, it is shown (without a proof) that the RG method unifies conventional singular perturbation methods such as the multiple time scale method, the boundary layer technique, the WKB analysis and so on.

After their works, many studies of the RG method have been done [10-13,16,21,22,26-32,35,36,42,45,46,48,49,53,55-59]. Kunihiro [28,29] interpreted an approximate solution obtained by the RG method as the envelope of a family of regular perturbation solutions. Nozaki *et al.* [45,46] proposed the proto-RG method to derive the RG equation effectively. Ziane [53], DeVille *et al.* [32] and Chiba [10] gave error estimates for approximate solutions.

Chiba [10] defined the higher order RG equation and the RG transformation to improve error estimates. He also proved that the RG method could provide approximate vector fields and approximate invariant manifolds as well as approximate solutions. Ei, Fujii and Kunihiro [16] applied the RG method to obtain approximate center manifolds and their method was rigorously formulated by Chiba [12]. DeVille *et al.* [32] showed that lower order RG equations are equivalent to normal forms of vector fields, and this fact was extended to higher order RG equations by Chiba [11]. Applications to partial differential equations are appeared in [16,31,42,45,46,55-59].

One of the purposes of this paper is to give basic theorems on the RG method extending author's previous works [10-12], in which the RG method is discussed for more restricted problems than Eq.(1.1). At first, definitions of the higher order RG equations for Eq.(1.1) are given and properties of them are investigated. It is proved that the RG method provides approximate vector fields (Thm.2.5) and approximate solutions (Thm.2.7) along with error estimates. Further, it is shown that if the RG equation has a normally hyperbolic invariant manifold N , the original equation (1.1) also has an invariant manifold N_ε which is diffeomorphic to N (Thms.2.9, 2.14). The RG equation proves to have the same symmetries (action of Lie groups) as those for the original equation (Thm.2.12). In addition, if the original equation is an autonomous system, the RG equation is shown to have an additional symmetry (Thm.2.15). These facts imply that the RG equation is easier to analyze than the original equation. An illustrative example to verify these theorems is also given (Sec.2.5).

The other purpose of this paper is to show that the RG method extends and unifies other traditional singular perturbation methods, such as the averaging method (Sec.4.1), the multiple time scale method (Sec.4.2), the (hyper-) normal forms theory (Sec.4.3), the center manifold reduction (Sec.3.2), the geometric singular perturbation method (Sec.3.3), the phase reduction (Sec.3.4), and Kunihiro's method [28,29] based on envelopes (Sec.4.4). A few of these results were partially obtained by many authors [9,38,43,44,47]. The present arguments will greatly reveal the relations among these methods.

Some properties of the infinite order RG equation are also investigated. It is proved that the infinite order RG equation converges if and only if the original equation is invariant under an appropriate torus action (Thm.5.1). This result extends Zung's theorem [54] which gives a necessary and sufficient condition for the convergence of normal forms of infinite order. The infinite RG equation for a time-dependent linear system proves to be convergent (Thm.5.6) and be related to monodromy matrices in Floquet theory.

Throughout this paper, solutions of differential equations are supposed to be defined for all $t \in \mathbf{R}$.

2 Renormalization group method

In this section, we give the definition of the RG (renormalization group) equation and main theorems on the RG method, such as the existence of invariant manifolds and inheritance of symmetries. An illustrative example and comments on symbolic computation of the RG equation are also provided.

2.1 Setting, definitions and basic lemmas

Let M be an n dimensional manifold and U an open set in M whose closure \bar{U} is compact. Let $g(t, \cdot, \varepsilon)$ be a vector field on U parameterized by $t \in \mathbf{R}$ and $\varepsilon \in \mathbf{C}$. We consider the system of differential equations

$$\frac{dx}{dt} = \dot{x} = \varepsilon g(t, x, \varepsilon). \quad (2.1)$$

For this system, we make the following assumption.

(A) The vector field $g(t, x, \varepsilon)$ is C^1 with respect to time $t \in \mathbf{R}$ and C^∞ with respect to $x \in U$ and $\varepsilon \in \mathcal{E}$, where $\mathcal{E} \subset \mathbf{C}$ is a small neighborhood of the origin. Further, g is an almost periodic function with respect to t uniformly in $x \in \bar{U}$ and $\varepsilon \in \bar{\mathcal{E}}$, the set of whose Fourier exponents has no accumulation points on \mathbf{R} .

In general, a function $h(t, x)$ is called *almost periodic with respect to t uniformly in $x \in \bar{U}$* if the set

$$T(h, \delta) := \{\tau \mid \|h(t + \tau, x) - h(t, x)\| < \delta, \quad \forall t \in \mathbf{R}, \forall x \in \bar{U}\} \subset \mathbf{R}$$

is relatively dense for any $\delta > 0$; that is, there exists a positive number L such that $[a, a + L] \cap T(h, \delta) \neq \emptyset$ for all $a \in \mathbf{R}$. It is known that an almost periodic function is expanded in a Fourier series as $h(t, x) \sim \sum a_n(x) e^{i\lambda_n t}$, ($i = \sqrt{-1}$), where $\lambda_n \in \mathbf{R}$ is called a *Fourier exponent*. See Fink [20] for basic facts on almost periodic functions. The condition for Fourier exponents in the above assumption (A) is essentially used to prove Lemma 2.1 below. We denote $\text{Mod}(h)$ the smallest additive group of real numbers that contains the Fourier exponents λ_n of an almost periodic function $h(t)$ and call it the *module* of h .

Let $\sum_{k=1}^{\infty} \varepsilon^k g_k(t, x)$ be the formal Taylor expansion of $\varepsilon g(t, x, \varepsilon)$ in ε :

$$\dot{x} = \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots . \quad (2.2)$$

By the assumption (A), we can show that $g_i(t, x)$ ($i = 1, 2, \cdots$) are almost periodic functions with respect to $t \in \mathbf{R}$ uniformly in $x \in \bar{U}$ such that $\text{Mod}(g_i) \subset \text{Mod}(g)$.

Though Eq.(2.1) is mainly considered in this paper, we note here that Eqs.(2.3) and (2.5) below are reduced to Eq.(2.1): Consider the system of the form

$$\dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon), \quad (2.3)$$

where $f(t, \cdot)$ is a C^∞ vector field on U and g satisfies the assumption (A). Let φ_t be the flow of f ; that is, $\varphi_t(x_0)$ is a solution of the equation $\dot{x} = f(t, x)$ whose initial value is x_0 at the initial time $t = 0$. For this system, changing the coordinates by $x = \varphi_t(X)$ provides

$$\dot{X} = \varepsilon \left(\frac{\partial \varphi_t}{\partial X}(X) \right)^{-1} g(t, \varphi_t(X), \varepsilon) := \varepsilon \tilde{g}(t, X, \varepsilon). \quad (2.4)$$

We suppose that

(B) the vector field g satisfies the assumption (A) and there exists an open set $W \subset U$ such that $\varphi_t(W) \subset U$ and $\varphi_t(x)$ is almost periodic with respect to t uniformly in $x \in \bar{W}$, the set of whose Fourier exponents has no accumulation points.

Under the assumption (B), we can show that the vector field $\tilde{g}(t, X, \varepsilon)$ in the right hand side of Eq.(2.4) satisfies the assumption (A), in which g is replaced by \tilde{g} . Thus Eq.(2.3) is reduced to Eq.(2.1) by the transformation $x \mapsto X$.

In many applications, Eq.(2.3) is of the form

$$\begin{aligned} \dot{x} &= Fx + \varepsilon g(x, \varepsilon) \\ &= Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots , \quad x \in \mathbf{C}^n, \end{aligned} \quad (2.5)$$

where

(C1) the matrix F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues are on the imaginary axis,

(C2) each $g_i(x)$ is a polynomial vector field on \mathbf{C}^n .

Then, the assumptions (C1) and (C2) imply the assumption (B) because $\varphi_t(x) = e^{Ft}x$ is almost periodic. Therefore the coordinate transformation $x = e^{Ft}X$ brings Eq.(2.5) into the form of Eq.(2.1) : $\dot{X} = \varepsilon e^{-Ft}g(e^{Ft}X, \varepsilon) := \varepsilon \tilde{g}(t, X, \varepsilon)$. In this case, $\text{Mod}(\tilde{g})$ is generated by the absolute

values of the eigenvalues of F . Note that any equations $\dot{x} = f(x)$ with C^∞ vector fields f such that $f(0) = 0$ take the form (2.5) if we put $x \mapsto \varepsilon x$ and expand the equations in ε .

In what follows, we consider Eq.(2.1) with the assumption (A). We suppose that the system (2.1) is defined on an open set U on Euclidean space $M = \mathbf{C}^n$. However, all results to be obtained below can be easily extended to those for a system on an arbitrary manifold by taking local coordinates. Let us substitute $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ into the right hand side of Eq.(2.2) and expand it with respect to ε . We write the resultant as

$$\sum_{k=1}^{\infty} \varepsilon^k g_k(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = \sum_{k=1}^{\infty} \varepsilon^k G_k(t, x_0, x_1, \cdots, x_{k-1}). \quad (2.6)$$

For instance, G_1, G_2, G_3 and G_4 are given by

$$G_1(t, x_0) = g_1(t, x_0), \quad (2.7)$$

$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(t, x_0)x_1 + g_2(t, x_0), \quad (2.8)$$

$$G_3(t, x_0, x_1, x_2) = \frac{1}{2} \frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_1}{\partial x}(t, x_0)x_2 + \frac{\partial g_2}{\partial x}(t, x_0)x_1 + g_3(t, x_0), \quad (2.9)$$

$$G_4(t, x_0, x_1, x_2, x_3) = \frac{1}{6} \frac{\partial^3 g_1}{\partial x^3}(t, x_0)x_1^3 + \frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1x_2 + \frac{\partial g_1}{\partial x}(t, x_0)x_3 \\ + \frac{1}{2} \frac{\partial^2 g_2}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_2}{\partial x}(t, x_0)x_2 + \frac{\partial g_3}{\partial x}(t, x_0)x_1 + g_4(t, x_0), \quad (2.10)$$

respectively. Note that G_i ($i = 1, 2, \cdots$) are almost periodic functions with respect to t uniformly in $x \in \bar{U}$ such that $\text{Mod}(G_i) \subset \text{Mod}(g)$. With these G_i 's, we define the C^∞ maps $R_i, u_t^{(i)} : U \rightarrow M$ to be

$$R_1(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t G_1(s, y) ds, \quad (2.11)$$

$$u_t^{(1)}(y) = \int^t (G_1(s, y) - R_1(y)) ds, \quad (2.12)$$

and

$$R_i(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t (G_i(s, y, u_s^{(1)}(y), \cdots, u_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} \frac{\partial u_s^{(k)}}{\partial y}(y) R_{i-k}(y)) ds, \quad (2.13)$$

$$u_t^{(i)}(y) = \int^t (G_i(s, y, u_s^{(1)}(y), \cdots, u_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} \frac{\partial u_s^{(k)}}{\partial y}(y) R_{i-k}(y) - R_i(y)) ds, \quad (2.14)$$

for $i = 2, 3, \cdots$, respectively, where \int^t denotes the indefinite integral, whose integral constants are fixed arbitrarily (see also Remark 2.4 and Section 2.4).

Lemma 2.1. (i) The maps R_i ($i = 1, 2, \dots$) are well-defined (i.e. the limits exist).

(ii) The maps $u_t^{(i)}(y)$ ($i = 1, 2, \dots$) are almost periodic functions with respect to t uniformly in $y \in \bar{U}$ such that $\text{Mod}(u^{(i)}) \subset \text{Mod}(g)$. In particular, $u_t^{(i)}$ are bounded in $t \in \mathbf{R}$.

Proof. We prove the lemma by induction. Since $G_1(t, y) = g_1(t, y)$ is almost periodic, it is expanded in a Fourier series of the form

$$g_1(t, y) = \sum_{\lambda_n \in \text{Mod}(g_1)} a_n(y) e^{i\lambda_n t}, \quad \lambda_n \in \mathbf{R}, \quad (2.15)$$

where $\lambda_0 = 0$. Clearly $R_1(y)$ coincides with $a_0(y)$. Thus $u_t^{(1)}(y)$ is written as

$$u_t^{(1)}(y) = \int \sum_{\lambda_n \neq 0} a_n(y) e^{i\lambda_n s} ds. \quad (2.16)$$

In general, it is known that the primitive function $\int h(t, y) dt$ of a uniformly almost periodic function $h(t, y)$ is also uniformly almost periodic if the set of Fourier exponents of $h(t, y)$ is bounded away from zero (see Fink [20]). Since the set of Fourier exponents of $g_1(t, y) - R_1(y)$ is bounded away from zero by the assumption (A), $u_t^{(1)}(y)$ is almost periodic and calculated as

$$u_t^{(1)}(y) = \sum_{\lambda_n \neq 0} \frac{1}{i\lambda_n} a_n(y) e^{i\lambda_n t} + (\text{integral constant}). \quad (2.17)$$

This proves Lemma 2.1 for $i = 1$.

Suppose that Lemma 2.1 holds for $i = 1, 2, \dots, k-1$. Since $G_k(t, x_0, \dots, x_{k-1})$ and $u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)$ are uniformly almost periodic functions, the composition $G_k(t, y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y))$ is also a uniformly almost periodic function whose module is included in $\text{Mod}(g)$ (see Fink [20]). Since the sum, the product and the derivative with respect to a parameter y of uniformly almost periodic functions are also uniformly almost periodic (see Fink [20]), the integrand in Eq.(2.13) is a uniformly almost periodic function, whose module is included in $\text{Mod}(g)$. The $R_k(y)$ coincides with its Fourier coefficient associated with the zero Fourier exponent. By the assumption (A), the set of Fourier exponents of the integrand in Eq.(2.13) has no accumulation points. Thus it turns out that the set of Fourier exponents of the integrand in Eq.(2.14) is bounded away from zero. This proves that $u_t^{(k)}(y)$ is uniformly almost periodic and the proof of Lemma 2.1 is completed. ■

Before introducing the RG equation, we want to explain how it is derived according to Chen, Goldenfeld and Oono [8,9]. The reader who is not interested in formal arguments can skip the next paragraph and go to Definition 2.2.

At first, let us try to construct a formal solution of Eq.(2.1) by the regular perturbation method; that is, substitute Eq.(1.2) into Eq.(2.1). Then we obtain a system of ODEs

$$\begin{cases} \dot{x}_0 = 0, \\ \dot{x}_1 = G_1(t, x_0), \\ \vdots \\ \dot{x}_n = G_n(t, x_0, \dots, x_{n-1}), \\ \vdots \end{cases}$$

Let $x_0(t) = y \in \mathbf{C}^n$ be a solution of the zero-th order equation. Then, the first order equation is solved as

$$x_1(t) = \int^t G_1(s, y) ds = R_1(y)t + \int^t (G_1(s, y) - R_1(y)) ds = R_1(y)t + u_t^{(1)}(y),$$

where we decompose $x_1(t)$ into the bounded term $u_t^{(1)}(y)$ and the divergence term $R_1(y)t$ called the *secular term*. In a similar manner, we solve the equations on x_2, x_3, \dots step by step. We can show that solutions are expressed as

$$x_n(t) = u_t^{(n)}(y) + \left(R_n(y) + \sum_{k=1}^{n-1} \frac{\partial u^{(k)}}{\partial y}(y) R_{n-k}(y) \right) t + O(t^2),$$

(see Chiba [10] for the proof). In this way, we obtain a formal solution of the form

$$\hat{x}(t) := \hat{x}(t, y) = y + \sum_{n=1}^{\infty} \varepsilon^n u_t^{(n)}(y) + \sum_{n=1}^{\infty} \varepsilon^n \left(R_n(y) + \sum_{k=1}^{n-1} \frac{\partial u^{(k)}}{\partial y}(y) R_{n-k}(y) \right) t + O(t^2).$$

Now we introduce a dummy parameter $\tau \in \mathbf{R}$ and replace polynomials t^j in the above by $(t - \tau)^j$. Next, we regard $y = y(\tau)$ as a function of τ to be determined so that we recover the formal solution $\hat{x}(t, y)$:

$$\hat{x}(t, y) = y(\tau) + \sum_{n=1}^{\infty} \varepsilon^n u_t^{(n)}(y(\tau)) + \sum_{n=1}^{\infty} \varepsilon^n \left(R_n(y(\tau)) + \sum_{k=1}^{n-1} \frac{\partial u^{(k)}}{\partial y}(y(\tau)) R_{n-k}(y(\tau)) \right) (t - \tau) + O((t - \tau)^2).$$

Since $\hat{x}(t, y)$ has to be independent of the dummy parameter τ , we impose the condition

$$\left. \frac{d}{d\tau} \right|_{\tau=t} \hat{x}(t, y) = 0,$$

which is called the *RG condition*. This condition provides

$$\begin{aligned} 0 &= \frac{dy}{dt} + \sum_{n=1}^{\infty} \varepsilon^n \frac{\partial u_t^{(n)}}{\partial y}(y) \frac{dy}{dt} - \sum_{n=1}^{\infty} \varepsilon^n \left(R_n(y) + \sum_{k=1}^{n-1} \frac{\partial u_t^{(k)}}{\partial y}(y) R_{n-k}(y) \right) \\ &= \left(id + \sum_{n=1}^{\infty} \varepsilon^n \frac{\partial u_t^{(n)}}{\partial y}(y) \right) \frac{dy}{dt} - \left(id + \sum_{n=1}^{\infty} \varepsilon^n \frac{\partial u_t^{(n)}}{\partial y}(y) \right) \sum_{k=1}^{\infty} \varepsilon^k R_k(y). \end{aligned}$$

Thus we see that $y(t)$ has to satisfy the equation $dy/dt = \sum_{k=1}^{\infty} \varepsilon^k R_k(y)$, which gives the RG equation. Motivated this formal argument, we define the RG equation as follows:

Definition 2.2. Along with R_i and $u_i^{(i)}$, we define the m -th order RG equation for Eq.(2.1) to be

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots + \varepsilon^m R_m(y), \quad (2.18)$$

and the m -th order RG transformation to be

$$\alpha_t^{(m)}(y) = y + \varepsilon u_t^{(1)}(y) + \cdots + \varepsilon^m u_t^{(m)}(y). \quad (2.19)$$

Domains of Eq.(2.18) and the map $\alpha_t^{(m)}$ are shown in the next lemma.

Lemma 2.3. If $|\varepsilon|$ is sufficiently small, there exists an open set $V = V(\varepsilon) \subset U$ such that $\alpha_t^{(m)}(y)$ is a diffeomorphism from V into U , and the inverse $(\alpha_t^{(m)})^{-1}(x)$ is also an almost periodic function with respect to t uniformly in x .

Proof. Since the vector field $g(t, x, \varepsilon)$ is C^∞ with respect to x and ε , so is the map $\alpha_t^{(m)}$. Since $\alpha_t^{(m)}$ is close to the identity map if $|\varepsilon|$ is small, there is an open set $V_t \subset U$ such that $\alpha_t^{(m)}$ is a diffeomorphism on V_t . Since V_t 's are ε -close to each other and since $\alpha_t^{(m)}$ is almost periodic, the set $\tilde{V} := \bigcap_{t \in \mathbf{R}} V_t$ is not empty. We can take the subset $V \subset \tilde{V}$ if necessary so that $\alpha_t^{(m)}(V) \subset U$.

Next thing to do is to prove that $(\alpha_t^{(m)})^{-1}$ is an uniformly almost periodic function. Since $\alpha_t^{(m)}$ is uniformly almost periodic, the set

$$T(\alpha_t^{(m)}, \delta) = \{\tau \mid \|\alpha_{t+\tau}^{(m)}(y) - \alpha_t^{(m)}(y)\| < \delta, \quad \forall t \in \mathbf{R}, \forall y \in V\} \quad (2.20)$$

is relatively dense for any small $\delta > 0$. For $y \in V$, put $x = \alpha_t^{(m)}(y)$. Then

$$\begin{aligned} \|(\alpha_{t+\tau}^{(m)})^{-1}(x) - (\alpha_t^{(m)})^{-1}(x)\| &= \|(\alpha_{t+\tau}^{(m)})^{-1}(\alpha_t^{(m)}(y)) - (\alpha_{t+\tau}^{(m)})^{-1}(\alpha_{t+\tau}^{(m)}(y))\| \\ &\leq L_{t+\tau} \|\alpha_t^{(m)}(y) - \alpha_{t+\tau}^{(m)}(y)\| < L_{t+\tau} \delta, \end{aligned} \quad (2.21)$$

if $\tau \in T(\alpha_t^{(m)}, \delta)$, where L_t is the Lipschitz constant of the map $(\alpha_t^{(m)})^{-1}|_U$. Since $\alpha_t^{(m)}$ is almost periodic, we can prove that there exists the number $L := \max_{t \in \mathbf{R}} L_t$. Now the inequality

$$\|(\alpha_{t+\tau}^{(m)})^{-1}(x) - (\alpha_t^{(m)})^{-1}(x)\| < L\delta \quad (2.22)$$

holds for any small $\delta > 0$, $\tau \in T(\alpha_t^{(m)}, \delta)$ and $x \in \alpha_t^{(m)}(V)$. This proves that $(\alpha_t^{(m)})^{-1}$ is an almost periodic function with respect to t uniformly in $x \in \alpha_t^{(m)}(V)$. ■

In what follows, we suppose that the m -th order RG equation and the m -th order RG transformation are defined on the set V above. Note that the smaller $|\varepsilon|$ is, the larger set V we may take.

Remark 2.4. Since the integral constants in Eqs.(2.11) to (2.14) are left undetermined, the m -th order RG equations and the m -th order RG transformations are not unique although $R_1(y)$ is uniquely determined. However, the theorems described below hold for any choice of integral constants unless otherwise noted. Good choices of integral constants simplify the RG equations and it will be studied in Section 2.4.

2.2 Main theorems

Now we are in a position to state our main theorems.

Theorem 2.5. Let $\alpha_t^{(m)}$ be the m -th order RG transformation for Eq.(2.1) defined on V as Lemma 2.3. If $|\varepsilon|$ is sufficiently small, there exists a vector field $S(t, y, \varepsilon)$ on V parameterized by t and ε such that

(i) by changing the coordinates as $x = \alpha_t^{(m)}(y)$, Eq.(2.1) is transformed into the system

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots + \varepsilon^m R_m(y) + \varepsilon^{m+1} S(t, y, \varepsilon), \quad (2.23)$$

(ii) S is an almost periodic function with respect to t uniformly in $y \in V$ with $\text{Mod}(S) \subset \text{Mod}(g)$,

(iii) $S(t, y, \varepsilon)$ is C^1 with respect to t and C^∞ with respect to y and ε . In particular, S and its derivatives are bounded as $\varepsilon \rightarrow 0$ and $t \rightarrow \infty$.

Proof. The proof is done by simple calculation. By putting $x = \alpha_t^{(m)}(y)$, the left hand side of Eq.(2.1) is calculated as

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \alpha_t^{(m)}(y) \\ &= \dot{y} + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \dot{y} + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial t}(y) \\ &= \left(id + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right) \dot{y} + \sum_{k=1}^m \varepsilon^k \left(G_k(t, y, u_t^{(1)}, \dots, u_t^{(k-1)}) - \sum_{j=1}^{k-1} \frac{\partial u_t^{(j)}}{\partial y}(y) R_{k-j}(y) - R_k(y) \right). \end{aligned} \quad (2.24)$$

On the other hand, the right hand side is calculated as

$$\begin{aligned}\varepsilon g(t, \alpha_t^{(m)}(y), \varepsilon) &= \sum_{k=1}^{\infty} \varepsilon^k g_k(t, y + \varepsilon u_t^{(1)}(y) + \varepsilon^2 u_t^{(2)}(y) + \cdots) \\ &= \sum_{k=1}^{\infty} \varepsilon^k G_k(t, y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)).\end{aligned}\quad (2.25)$$

Thus Eq.(2.1) is transformed into

$$\begin{aligned}\dot{y} &= \left(id + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^{-1} \sum_{k=1}^m \varepsilon^k \left(R_k(y) + \sum_{j=1}^{k-1} \frac{\partial u_t^{(j)}}{\partial y}(y) R_{k-j}(y) \right) \\ &\quad + \left(id + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^{-1} \sum_{k=m+1}^{\infty} \varepsilon^k G_k(t, y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)) \\ &= \left(id + \sum_{j=1}^{\infty} (-1)^j \left(\sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^j \right) \left(\sum_{k=1}^m \varepsilon^k R_k(y) + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \sum_{j=1}^{m-k} \varepsilon^j R_j(y) \right) \\ &\quad + \left(id + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^{-1} \sum_{k=m+1}^{\infty} \varepsilon^k G_k(t, y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)) \\ &= \sum_{k=1}^m \varepsilon^k R_k(y) + \sum_{j=1}^{\infty} (-1)^j \left(\sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^j \sum_{i=m-k+1}^m \varepsilon^i R_i(y) \\ &\quad + \left(id + \sum_{k=1}^m \varepsilon^k \frac{\partial u_t^{(k)}}{\partial y}(y) \right)^{-1} \sum_{k=m+1}^{\infty} \varepsilon^k G_k(t, y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)).\end{aligned}\quad (2.26)$$

The last two terms above are of order $O(\varepsilon^{m+1})$ and almost periodic functions because they consist of almost periodic functions $u_t^{(i)}$ and G_i . This proves Theorem 2.5. \blacksquare

Remark 2.6. To prove Theorem 2.5 (i),(iii), we do not need the assumption of almost periodicity for $g(t, x, \varepsilon)$ as long as $R_i(y)$ are well-defined and $g, u_t^{(i)}$ and their derivatives are bounded in t so that the last two terms in Eq.(2.26) are bounded. In Chiba [10], Theorem 2.5 (i) and (iii) for $m = 1$ are proved without the assumption (A) but assumptions on boundedness of $g, u_t^{(i)}$ and their derivatives.

Thm.2.5 (iii) implies that we can use the m -th order RG equation to construct approximate solutions of Eq.(2.1). Indeed, a curve $\alpha_t^{(m)}(y(t))$, a solution of the RG equation transformed by the RG transformation, gives an approximate solution of Eq.(2.1).

Theorem 2.7 (Error estimate). Let $y(t)$ be a solution of the m -th order RG equation and $\alpha_t^{(m)}$ the m -th order RG transformation. There exist positive constants ε_0, C and T such that a

solution $x(t)$ of Eq.(2.1) with $x(0) = \alpha_0^{(m)}(y(0))$ satisfies the inequality

$$\|x(t) - \alpha_t^{(m)}(y(t))\| < C|\varepsilon|^m, \quad (2.27)$$

as long as $|\varepsilon| < \varepsilon_0$, $y(t) \in V$ and $0 \leq t \leq T/|\varepsilon|$.

Remark 2.8. Since the velocity of $y(t)$ is of order $O(\varepsilon)$, $y(0) \in V$ implies $y(t) \in V$ for $0 \leq t \leq T/|\varepsilon|$ unless $y(0)$ is ε -close to the boundary of V . If we define $u_t^{(i)}$ so that the indefinite integrals in Eqs.(2.12, 14) are replaced by the definite integrals \int_0^t , $\alpha_0^{(m)}$ is the identity and $\alpha_0^{(m)}(y(0)) = y(0)$.

Proof of Thm.2.7. Since $\alpha_t^{(m)}$ is a diffeomorphism on V and bounded in $t \in \mathbf{R}$, it is sufficient to prove that a solution $y(t)$ of Eq.(2.18) and a solution $\tilde{y}(t)$ of Eq.(2.23) with $y(0) = \tilde{y}(0)$ satisfy the inequality

$$\|\tilde{y}(t) - y(t)\| < \tilde{C}|\varepsilon|^m, \quad 0 \leq t \leq T/|\varepsilon|, \quad (2.28)$$

for some positive constant \tilde{C} .

Let $L_1 > 0$ be the Lipschitz constant of the function $R_1(y) + \varepsilon R_2(y) + \dots + \varepsilon^{m-1} R_m(y)$ on \bar{V} and $L_2 > 0$ a constant such that $\sup_{t \in \mathbf{R}, y \in \bar{V}} \|S(t, y, \varepsilon)\| \leq L_2$. Then, by Eq.(2.18) and Eq.(2.23), $y(t)$ and $\tilde{y}(t)$ prove to satisfy

$$\|\tilde{y}(t) - y(t)\| \leq \varepsilon L_1 \int_0^t \|\tilde{y}(s) - y(s)\| ds + L_2 \varepsilon^{m+1} t. \quad (2.29)$$

Now the Gronwall inequality proves that

$$\|\tilde{y}(t) - y(t)\| \leq \frac{L_2}{L_1} \varepsilon^m (e^{\varepsilon L_1 t} - 1). \quad (2.30)$$

The right hand side is of order $O(\varepsilon^m)$ if $0 \leq t \leq T/\varepsilon$. ■

In the same way as this proof, we can show that if $R_1(y) = \dots = R_k(y) = 0$ holds with $k \leq m$, the inequality (2.27) holds for the longer time interval $0 \leq t \leq T/|\varepsilon|^{k+1}$. This fact is proved by Murdock and Wang [41] for the case $k = 1$ in terms of the multiple time scale method.

We can also detect existence of invariant manifolds. Note that introducing the new variable s , we can rewrite Eq.(2.1) as the autonomous system

$$\begin{cases} \frac{dx}{dt} = \varepsilon g(s, x, \varepsilon), \\ \frac{ds}{dt} = 1. \end{cases} \quad (2.31)$$

Then we say that Eq.(2.31) is defined on the (s, x) space.

Theorem 2.9 (Existence of invariant manifolds). Suppose that $R_1(y) = \cdots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the RG equation for Eq.(2.1). If the vector field $R_k(y)$ has a boundaryless compact normally hyperbolic invariant manifold N , then for sufficiently small $\varepsilon > 0$, Eq.(2.31) has an invariant manifold N_ε on the (s, x) space which is diffeomorphic to $\mathbf{R} \times N$. In particular, the stability of N_ε coincides with that of N .

To prove this theorem, we need Fenichel's theorem :

Theorem (Fenichel [18]). Let M be a C^1 manifold and $\mathcal{X}(M)$ the set of C^1 vector fields on M with the C^1 topology. Suppose that $f \in \mathcal{X}(M)$ has a boundaryless compact normally hyperbolic f -invariant manifold $N \subset M$. Then, the following holds:

- (i) There is a neighborhood $\mathcal{U} \subset \mathcal{X}(M)$ of f such that there exists a normally hyperbolic g -invariant manifold $N_g \subset M$ for any $g \in \mathcal{U}$. The N_g is diffeomorphic to N .
- (ii) If $\|f - g\| \sim O(\varepsilon)$, N_g lies within an $O(\varepsilon)$ neighborhood of N uniquely.
- (iii) The stability of N_ε coincides with that of N .

Note that for the case of a compact normally hyperbolic invariant manifold with boundary, Fenichel's theorem is modified as follows : If a vector field f has a compact normally hyperbolic invariant manifold N with boundary, then a vector field g , which is C^1 close to f , has a *locally* invariant manifold N_g which is diffeomorphic to N . In this case, an orbit of the flow of g on N_g may go out from N_g through its boundary. According to this theorem, Thm.2.9 has to be modified so that N_ε is *locally* invariant if N has boundary.

See [18,24,51] for the proof of Fenichel's theorem and the definition of normal hyperbolicity.

Proof of Thm.2.9. Changing the time scale as $t \mapsto t/\varepsilon^k$ and introducing the new variable s , we rewrite the k -th order RG equation as

$$\begin{cases} \frac{dy}{dt} = R_k(y), \\ \frac{ds}{dt} = 1, \end{cases} \quad (2.32)$$

and Eq.(2.23) as

$$\begin{cases} \frac{dy}{dt} = R_k(y) + \varepsilon R_{k+1}(y) + \cdots + \varepsilon^{m-k} R_m(y) + \varepsilon^{m+1-k} S(s/\varepsilon^k, y, \varepsilon), \\ \frac{ds}{dt} = 1, \end{cases} \quad (2.33)$$

respectively. Suppose that $m \geq 2k$. Since S is bounded in s and since

$$\frac{\partial}{\partial y} \varepsilon^{m+1-k} S(s/\varepsilon^k, y, \varepsilon) \sim O(\varepsilon^{k+1}), \quad \frac{\partial}{\partial s} \varepsilon^{m+1-k} S(s/\varepsilon^k, y, \varepsilon) \sim O(\varepsilon), \quad (2.34)$$

Eq.(2.33) is ε -close to Eq.(2.32) on the (s, y) space in the C^1 topology.

By the assumption, Eq.(2.32) has a normally hyperbolic invariant manifold $\mathbf{R} \times N$ on the (s, y) space. At this time, Fenichel's theorem is not applicable because $\mathbf{R} \times N$ is not compact. To handle this difficulty, we do as follows:

Since S is almost periodic, the set

$$T(S, \delta) := \{\tau \mid \|S((s - \tau)/\varepsilon^k, y, \varepsilon) - S(s/\varepsilon^k, y, \varepsilon)\| < \delta, \quad \forall s \in \mathbf{R}\} \quad (2.35)$$

is relatively dense for any small $\delta > 0$. Let us fix δ so that it is sufficiently smaller than ε and fix $\tau \in T(S, \delta)$ arbitrarily. Then $W := [0, \tau] \times N$ is a compact locally invariant manifold of Eq.(2.32) with boundaries $\{0\} \times N$ and $\{\tau\} \times N$ (see Fig.1).

Now Fenichel's theorem proves that Eq.(2.33) has a locally invariant manifold W_ε which is diffeomorphic to W and lies within an $O(\varepsilon)$ neighborhood of W uniquely.

To extend W_ε along the s axis, consider the system

$$\begin{cases} \dot{y} = R_k(y) + \varepsilon R_{k+1}(y) + \cdots + \varepsilon^{m-k} R_m(y) + \varepsilon^{m+1-k} S((s - \tau)/\varepsilon^k, y, \varepsilon), \\ \dot{s} = 1. \end{cases} \quad (2.36)$$

Since the above system is δ -close to Eq.(2.33), it has a locally invariant manifold $W_{\varepsilon, \delta}$, which is diffeomorphic to W_ε . By putting $\tilde{s} = s - \tau$, Eq.(2.36) is rewritten as

$$\begin{cases} \dot{y} = R_k(y) + \varepsilon R_{k+1}(y) + \cdots + \varepsilon^{m-k} R_m(y) + \varepsilon^{m+1-k} S(\tilde{s}/\varepsilon^k, y, \varepsilon), \\ \dot{\tilde{s}} = 1, \end{cases} \quad (2.37)$$

and it takes the same form as Eq.(2.33). This means that the set

$$K := \{(s, y) \mid (s - \tau, y) \in W_{\varepsilon, \delta}\}$$

is a locally invariant manifold of Eq.(2.33). Since $W_{\varepsilon, \delta}$ is δ -close to W_ε and since $\delta \ll \varepsilon$, both of $W_\varepsilon \cap \{s = \tau\}$ and $K \cap \{s = \tau\}$ are ε -close to W . Since an invariant manifold of Eq.(2.33) which lies within an $O(\varepsilon)$ neighborhood of W is unique by Fenichel's theorem, $K \cap \{s = \tau\}$ has to coincide with $W_\varepsilon \cap \{s = \tau\}$. This proves that K is connected to W_ε and $K \cup W_\varepsilon$ gives a locally invariant manifold of Eq.(2.33).

This procedure is done for any $\tau \in T(S, \delta)$. Thus it turns out that W_ε is extended along the s axis and it gives an invariant manifold $\tilde{N}_\varepsilon \simeq \mathbf{R} \times N$ of Eq.(2.33). An invariant manifold N_ε of Eq.(2.1) is obtained by transforming \tilde{N}_ε by $\alpha_t^{(m)}$.

Note that by the construction, projections of the sets $\tilde{N}_\varepsilon \cap \{s = \tau\}$, $\tau \in T(S, \delta)$ on to the y space are δ -close to each other. This fact is used to prove the next corollary. ■

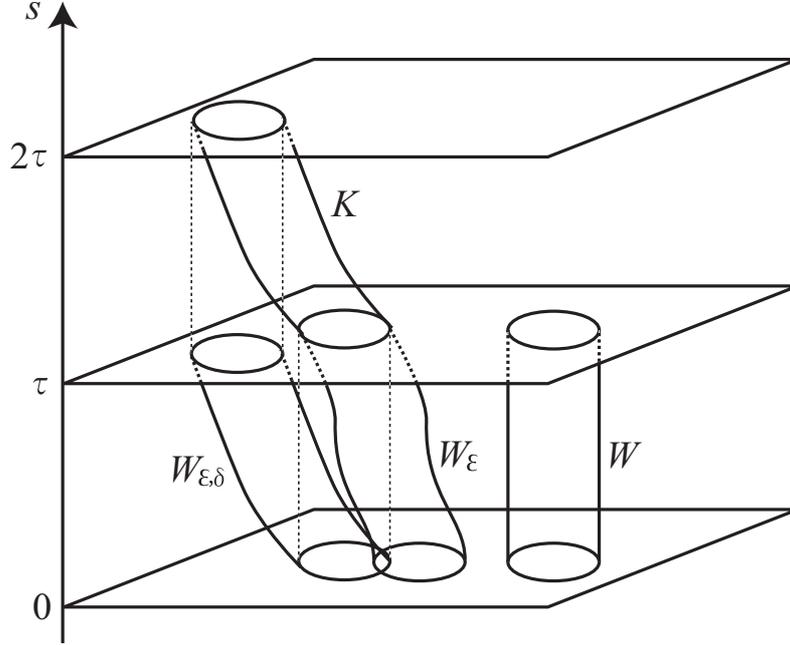


Fig. 1 A schematic view of the proof for the case that N is a circle. The W_ε is ε -close to W and $W_{\varepsilon, \delta}$ is δ -close to W_ε . The K is the “copy” of $W_{\varepsilon, \delta}$.

The next corollary (ii) and (iii) for $k = 1$ are proved in Bogoliubov, Mitropolsky [6] and Fink [20] and immediately follow from Thm.2.9.

Corollary 2.10. Suppose that $R_1(y) = \dots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the RG equation for Eq.(2.1). For sufficiently small $\varepsilon > 0$,

- (i) if the vector field $R_k(y)$ has a hyperbolic periodic orbit $\gamma_0(t)$, then Eq.(2.1) has an almost periodic solution with the same stability as $\gamma_0(t)$,
- (ii) if the vector field $R_k(y)$ has a hyperbolic fixed point γ_0 , then Eq.(2.1) has an almost periodic solution $\gamma_\varepsilon(t)$ with the same stability as γ_0 such that $\text{Mod}(\gamma_\varepsilon) \subset \text{Mod}(g)$,
- (iii) if the vector field $R_k(y)$ has a hyperbolic fixed point γ_0 and if g is periodic in t with a period T , then Eq.(2.1) has a periodic solution $\gamma_\varepsilon(t)$ with the same stability as γ_0 and the period T (it need not be the least period).

Proof. If $R_k(y)$ has a periodic orbit, Eq.(2.33) has an invariant cylinder \tilde{N}_ε on the (s, y) space as is represented in Fig.1. To prove Corollary 2.10 (i), at first we suppose that $g(t, x, \varepsilon)$ is

periodic with a period T . In this case, since $S(t, y, \varepsilon)$ is a periodic function with the period T , \tilde{N}_ε is periodic along the s axis in the sense that the projections $S^1 := \tilde{N}_\varepsilon \cap \{s = mT\}$ give the same circle for all integers m . Let $y = \gamma(t)$, $s = t$ be a solution of Eq.(2.33) on the cylinder. Then $\gamma(mT)$, $m = 0, 1, \dots$ gives a discrete dynamics on S^1 . If $\gamma(mT)$ converges to a fixed point or a periodic orbit as $m \rightarrow \infty$, then $\gamma(t)$ converges to a periodic function as $t \rightarrow \infty$. Otherwise, the orbit of $\gamma(mT)$ is dense on S^1 and in this case $\gamma(t)$ is an almost periodic function. A solution of Eq.(2.1) is obtained by transforming $\gamma(t)$ by the almost periodic map $\alpha_t^{(m)}$. This proves (i) of Corollary 2.10 for the case that g is periodic.

If g is almost periodic, the sets $\tilde{N}_\varepsilon \cap \{s = \tau\}$ give circles for any $\tau \in T(S, \delta)$ and they are δ -close to each other as is mentioned in the end of the proof of Thm.2.9. In this case, there exists a coordinate transformation $Y = \varphi(y, t)$ such that the cylinder \tilde{N}_ε is straightened along the s axis. The function φ is almost periodic in t because $\|\varphi(y, t + \tau) - \varphi(y, t)\|$ is of order $O(\delta)$ for any $\tau \in T(S, \delta)$. Now the proof is reduced to the case that g is periodic.

The proofs of (ii) and (iii) of Corollary 2.10 are done in the same way as (i), details of which are left to the reader. ■

Remark 2.11. Suppose that the first order RG equation $\varepsilon R_1(y) \neq 0$ does not have normally hyperbolic invariant manifolds but the second order RG equation $\varepsilon R_1(y) + \varepsilon^2 R_2(y)$ does. Then can we conclude that the original system (2.1) has an invariant manifold with the same stability as that of the second order RG equation? Unfortunately, it is not true in general. For example, suppose that the RG equation for some system is a linear equation of the form

$$\dot{y}/\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y - \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} y + \dots, \quad y \in \mathbf{R}^2. \quad (2.38)$$

The origin is a fixed point of this system, however, the first term has zero eigenvalues and we can not determine the stability up to the first order RG equation. If we calculate up to the second order, the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.39)$$

are $-\varepsilon$ (double root), so that $y = 0$ is a stable fixed point of the second order RG equation if $\varepsilon > 0$. Unlike Corollary 2.10 (ii), this *does not* prove that the original system has a stable almost periodic solution. Indeed, if we calculate the third order RG equation, the eigenvalues of the matrix in the right hand side of Eq.(2.38) are 3ε and $-\varepsilon$. Therefore the origin is an unstable fixed point of the third order RG equation. This example shows that if we truncate higher order terms of the RG equation, stability of an invariant manifold may change and

we can not use $\varepsilon R_1(y) + \varepsilon^2 R_2(y)$ to investigate stability of an invariant manifold as long as $R_1(y) \neq 0$. This is because Fenichel's theorem does not hold if the vector field f in his theorem depends on the parameter ε .

Theorems 2.7 and 2.9 mean that the RG equation is useful to understand the properties of the flow of the system (2.1). Since the RG equation is an autonomous system while Eq.(2.1) is not, it seems that the RG equation is easier to analyze than the original system (2.1). Actually, we can show that the RG equation does not lose symmetries the system (2.1) has.

Recall that integral constants in Eqs.(2.12, 14) are left undetermined and they can depend on y (see Remark 2.4). To express the integral constants $B_i(y)$ in Eqs.(2.12, 14) explicitly, we rewrite them as

$$u_t^{(1)}(y) = B_1(y) + \int^t (G_1(s, y) - R_1(y)) ds,$$

and

$$u_t^{(i)}(y) = B_i(y) + \int^t (G_i(s, y, u_s^{(1)}(y), \dots, u_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} \frac{\partial u_s^{(k)}}{\partial y}(y) R_{i-k}(y) - R_i(y)) ds,$$

for $i = 2, 3, \dots$, where integral constants of the indefinite integrals in the above formulas are chosen to be zero.

Theorem 2.12 (Inheritance of symmetries). Suppose that an ε -independent Lie group H acts on $U \subset M$. If the vector field g and integral constants $B_i(y)$, $i = 1, \dots, m-1$ in Eqs.(2.12, 14) are invariant under the action of H ; that is, they satisfy

$$g(t, hy, \varepsilon) = \frac{\partial h}{\partial y}(y) g(t, y, \varepsilon), \quad B_i(hy) = \frac{\partial h}{\partial y}(y) B_i(y), \quad (2.40)$$

for any $h \in H, y \in U, t \in \mathbf{R}$ and ε , then the m -th order RG equation for Eq.(2.1) is also invariant under the action of H .

Proof. Since $h \in H$ is independent of ε , Eq.(2.40) implies

$$g_i(t, hy) = \frac{\partial h}{\partial y}(y) g_i(t, y), \quad (2.41)$$

for $i = 1, 2, \dots$. We prove by induction that $R_i(y)$ and $u_t^{(i)}(y)$, $i = 1, 2, \dots$, are invariant under the action of H . At first, $R_1(hy)$, $h \in H$ is calculated as

$$\begin{aligned} R_1(hy) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t G_1(t, hy) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \frac{\partial h}{\partial y}(y) G_1(t, y) ds = \frac{\partial h}{\partial y}(y) R_1(y). \end{aligned} \quad (2.42)$$

Next, $u_t^{(1)}$ is calculated in a similar way:

$$\begin{aligned} u_t^{(1)}(hy) &= B_1(hy) + \int^t (G_1(s, hy) - R_1(hy)) ds \\ &= \frac{\partial h}{\partial y}(y)B_1(y) + \frac{\partial h}{\partial y}(y) \int^t (G_1(s, y) - R_1(y)) ds \\ &= \frac{\partial h}{\partial y}(y)u_t^{(1)}(y). \end{aligned}$$

Suppose that R_k and $u_t^{(k)}$ are invariant under the action of H for $k = 1, 2, \dots, i-1$. Then, it is easy to verify that

$$\frac{\partial u_t^{(k)}}{\partial y}(hy) = \frac{\partial h}{\partial y}(y) \frac{\partial u_t^{(k)}}{\partial y}(y) \left(\frac{\partial h}{\partial y}(y) \right)^{-1}, \quad (2.43)$$

$$G_k(hy, u_t^{(1)}(hy), \dots, u_t^{(k-1)}(hy)) = \frac{\partial h}{\partial y}(y) G_k(y, u_t^{(1)}(y), \dots, u_t^{(k-1)}(y)), \quad (2.44)$$

for $k = 1, 2, \dots, i-1$. These equalities and Eqs.(2.13), (2.14) prove Theorem 2.12 by a similar calculation to Eq.(2.42). \blacksquare

2.3 Main theorems for autonomous systems

In this subsection, we consider an autonomous system of the form

$$\begin{aligned} \dot{x} &= f(x) + \varepsilon g(x, \varepsilon) \\ &= f(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \dots, \quad x \in U \subset M, \end{aligned} \quad (2.45)$$

where the flow φ_t of f is assumed to be almost periodic due to the assumption (B) so that Eq.(2.45) is transformed into the system of the form of (2.1). For this system, we restate definitions and theorems obtained so far in the present notation for convenience. We also show a few additional theorems.

Definition 2.13. Let φ_t be the flow of the vector field f . For Eq.(2.45), define the C^∞ maps $R_i, h_t^{(i)} : U \rightarrow M$ to be

$$R_1(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t (D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) ds, \quad (2.46)$$

$$h_t^{(1)}(y) = (D\varphi_t)_y \int^t \left((D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) - R_1(y) \right) ds, \quad (2.47)$$

and

$$R_i(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left((D\varphi_s)_y^{-1} G_i(s, \varphi_s(y), h_s^{(1)}(y), \dots, h_s^{(i-1)}(y)) \right. \\ \left. - (D\varphi_s)_y^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) \right) ds, \quad (2.48)$$

$$h_t^{(i)}(y) = (D\varphi_t)_y \int^t \left((D\varphi_s)_y^{-1} G_i(s, \varphi_s(y), h_s^{(1)}(y), \dots, h_s^{(i-1)}(y)) \right. \\ \left. - (D\varphi_s)_y^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) - R_i(y) \right) ds, \quad (2.49)$$

for $i = 2, 3, \dots$, respectively, where $(Dh_t^{(k)})_y$ is the derivative of $h_t^{(k)}(y)$ with respect to y , $(D\varphi_t)_y$ is the derivative of $\varphi_t(y)$ with respect to y , and where G_i are defined through Eq.(2.6). With these R_i and $h_t^{(i)}$, define the m -th order RG equation for Eq.(2.45) to be

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \dots + \varepsilon^m R_m(y), \quad (2.50)$$

and define the m -th order RG transformation to be

$$\alpha_t^{(m)}(y) = \varphi_t(y) + \varepsilon h_t^{(1)}(y) + \dots + \varepsilon^m h_t^{(m)}(y), \quad (2.51)$$

respectively.

In the present notation, Theorems 2.5 and 2.7 are true though the relation $\text{Mod}(S) \subset \text{Mod}(g)$ in Thm.2.5 (ii) is replaced by $\text{Mod}(S) \subset \text{Mod}(\varphi_t)$. Note that even if Eq.(2.45) is autonomous, the function S depends on t as long as the flow φ_t depends on t .

Theorem 2.9 is refined as follows:

Theorem 2.14 (Existence of invariant manifolds). Suppose that $R_1(y) = \dots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the RG equation for Eq.(2.45). If the vector field $R_k(y)$ has a boundaryless compact normally hyperbolic invariant manifold N , then for sufficiently small $\varepsilon > 0$, Eq.(2.45) has an invariant manifold N_ε , which is diffeomorphic to N . In particular, the stability of N_ε coincides with that of N .

Note that unlike Thm.2.9, we need not prepare the (s, x) space, and the invariant manifold N_ε lies on M not $\mathbf{R} \times M$. This theorem immediately follows from the proof of Thm.2.9. Indeed, Eq.(2.33) has an invariant manifold $\tilde{N}_\varepsilon \simeq \mathbf{R} \times N$ on the (s, y) space as is shown in the proof of Thm.2.9. An invariant manifold of Eq.(2.45) on the (s, x) is obtained by transforming \tilde{N}_ε by the RG transformation. However, it has to be straight along the s axis

because Eq.(2.45) is autonomous. Thus its projection onto the x space gives the invariant manifold N_ε of Eq.(2.45) (see Fig.2).

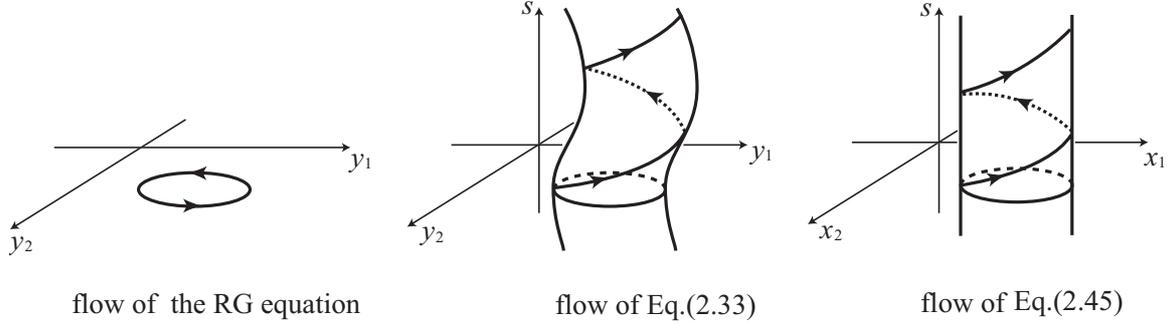


Fig. 2 A schematic view of the proof for the case that N is a circle. The projection of the straight cylinder on to the x space gives an invariant manifold of Eq.(2.45).

For the case of Eq.(2.1), the RG equation is simpler than the original system (2.1) in the sense that it has the same symmetries as (2.1) and further it is an autonomous system while (2.1) is not. In the present situation of (2.45), Theorem 2.12 of inheritance of symmetries still holds as long as the assumption for g is replaced as “the vector field f and g are invariant under the action of a Lie group H ”. However, since Eq.(2.45) is originally an autonomous system, it is not clear that the RG equation for Eq.(2.45) is easier to analyze than the original system (2.45). The next theorem shows that the RG equation for Eq.(2.45) has larger symmetries than Eq.(2.45).

To express the integral constants $B_i(y)$ in Eqs.(2.47, 49) explicitly, we rewrite them as

$$h_t^{(1)}(y) = (D\varphi_t)_y B_1(y) + (D\varphi_t)_y \int^t \left((D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) - R_1(y) \right) ds,$$

and

$$h_t^{(i)}(y) = (D\varphi_t)_y B_i(y) + (D\varphi_t)_y \int^t \left((D\varphi_s)_y^{-1} G_i(s, \varphi_s(y), h_s^{(1)}(y), \dots, h_s^{(i-1)}(y)) \right. \\ \left. - (D\varphi_s)_y^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) - R_i(y) \right) ds,$$

for $i = 2, 3, \dots$, where integral constants of the indefinite integrals in the above formulas are chosen to be zero.

Theorem 2.15 (Additional symmetry). Let φ_t be the flow of the vector field f defined on $U \subset M$. If the integral constants B_i in Eqs.(2.47, 49) are chosen so that they are invariant

under the action of the one-parameter group $\{\varphi_t : U \rightarrow M \mid t \in \mathbf{R}\}$, then the RG equation for Eq.(2.45) is also invariant under the action of the group. In other words, R_i satisfies the equality

$$R_i(\varphi_t(y)) = (D\varphi_t)_y R_i(y), \quad (2.52)$$

for $i = 1, 2, \dots$.

Proof. Since Eq.(2.45) is autonomous, the function $G_k(t, x_0, \dots, x_{k-1})$ defined through Eq.(2.6) is independent of t and we write it as $G_k(x_0, \dots, x_{k-1})$. We prove by induction that equalities $R_i(\varphi_t(y)) = (D\varphi_t)_y R_i(y)$ and $h_t^{(i)}(\varphi_{t'}(y)) = h_{t+t'}^{(i)}(y)$ hold for $i = 1, 2, \dots$. For all $s' \in \mathbf{R}$, $R_1(\varphi_{s'}(y))$ takes the form

$$\begin{aligned} R_1(\varphi_{s'}(y)) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int (D\varphi_s)_{\varphi_{s'}(y)}^{-1} G_1(\varphi_s \circ \varphi_{s'}(y)) ds \\ &= (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int (D\varphi_{s+s'})_y^{-1} G_1(\varphi_{s+s'}(y)) ds. \end{aligned}$$

Putting $s + s' = s''$, we verify that

$$\begin{aligned} R_1(\varphi_{s'}(y)) &= (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int^{t+s'} (D\varphi_{s''})_y^{-1} G_1(\varphi_{s''}(y)) ds'' \\ &= (D\varphi_{s'})_y R_1(y) + (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int_t^{t+s'} (D\varphi_{s''})_y^{-1} G_1(\varphi_{s''}(y)) ds'' \\ &= (D\varphi_{s'})_y R_1(y). \end{aligned}$$

The $h_t^{(1)}(\varphi_{s'}(y))$ is calculated in a similar way as

$$\begin{aligned} h_t^{(1)}(\varphi_{s'}(y)) &= (D\varphi_t)_y B_1(\varphi_{s'}(y)) + (D\varphi_t)_{\varphi_{s'}(y)} \int^t \left((D\varphi_s)_{\varphi_{s'}(y)}^{-1} G_1(\varphi_s \circ \varphi_{s'}(y)) - R_1(\varphi_{s'}(y)) \right) ds \\ &= (D\varphi_{t+s'})_y B_1(y) + (D\varphi_{t+s'})_y \int^t \left((D\varphi_{s+s'})_y^{-1} G_1(\varphi_{s+s'}(y)) - R_1(y) \right) ds. \end{aligned}$$

Putting $s + s' = s''$ provides

$$\begin{aligned} h_t^{(1)}(\varphi_{s'}(y)) &= (D\varphi_{t+s'})_y B_1(y) + (D\varphi_{t+s'})_y \int^{t+s'} \left((D\varphi_{s''})_y^{-1} G_1(\varphi_{s''}(y)) - R_1(y) \right) ds'' \\ &= h_{t+s'}^{(1)}(y). \end{aligned} \quad (2.53)$$

Suppose that $R_k(\varphi_t(y)) = (D\varphi_t)_y R_k(y)$ and $h_t^{(k)}(\varphi_{t'}(y)) = h_{t+t'}^{(k)}(y)$ hold for $k = 1, 2, \dots, i-1$.

Then, $R_i(\varphi_{s'}(y))$ is calculated as

$$\begin{aligned}
R_i(\varphi_{s'}(y)) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left((D\varphi_s)_{\varphi_{s'}(y)}^{-1} G_i(\varphi_s \circ \varphi_{s'}(y), h_s^{(1)}(\varphi_{s'}(y)), \dots, h_s^{(i-1)}(\varphi_{s'}(y))) \right. \\
&\quad \left. - (D\varphi_s)_{\varphi_{s'}(y)}^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_{\varphi_{s'}(y)} R_{i-k}(\varphi_{s'}(y)) \right) ds \\
&= (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left((D\varphi_{s+s'})_y^{-1} G_i(\varphi_{s+s'}(y), h_{s+s'}^{(1)}(y), \dots, h_{s+s'}^{(i-1)}(y)) \right. \\
&\quad \left. - (D\varphi_{s+s'})_y^{-1} \sum_{k=1}^{i-1} (Dh_{s+s'}^{(k)})_y R_{i-k}(y) \right) ds.
\end{aligned}$$

Putting $s + s' = s''$ provides

$$\begin{aligned}
R_i(\varphi_{s'}(y)) &= (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int^{t+s'} \left((D\varphi_{s''})_y^{-1} G_i(\varphi_{s''}(y), h_{s''}^{(1)}(y), \dots, h_{s''}^{(i-1)}(y)) \right. \\
&\quad \left. - (D\varphi_{s''})_y^{-1} \sum_{k=1}^{i-1} (Dh_{s''}^{(k)})_y R_{i-k}(y) \right) ds'' \\
&= (D\varphi_{s'})_y R_i(y) + (D\varphi_{s'})_y \lim_{t \rightarrow \infty} \frac{1}{t} \int_t^{t+s'} \left((D\varphi_{s''})_y^{-1} G_i(\varphi_{s''}(y), h_{s''}^{(1)}(y), \dots, h_{s''}^{(i-1)}(y)) \right. \\
&\quad \left. - (D\varphi_{s''})_y^{-1} \sum_{k=1}^{i-1} (Dh_{s''}^{(k)})_y R_{i-k}(y) \right) ds'' \\
&= (D\varphi_{s'})_y R_i(y).
\end{aligned}$$

We can show the equality $h_t^{(i)}(\varphi_{s'}(y)) = h_{t+s'}^{(i)}(y)$ in a similar way. ■

2.4 Simplified RG equation

Recall that the definitions of R_i and $u_t^{(i)}$ given in Eqs.(2.11) to (2.14) include the indefinite integrals and we have left the integral constants undetermined (see Rem.2.4). In this subsection, we investigate how different choices of the integral constants change the forms of RG equations and RG transformations.

Let us fix definitions of R_i and $u_t^{(i)}$, $i = 1, 2, \dots$ by fixing integral constants in Eqs.(2.12, 14) arbitrarily (note that R_i and $u_t^{(i)}$ are independent of the integral constants in Eqs.(2.11, 13)). For these R_i and $u_t^{(i)}$, we define \tilde{R}_i and $\tilde{u}_t^{(i)}$ to be

$$\tilde{R}_1(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t G_1(s, y) ds, \quad (2.54)$$

$$\tilde{u}_t^{(1)}(y) = B_1(y) + \int^t (G_1(s, y) - \tilde{R}_1(y)) ds, \quad (2.55)$$

and

$$\widetilde{R}_i(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(G_i(s, y, \widetilde{u}_s^{(1)}(y), \dots, \widetilde{u}_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} \frac{\partial \widetilde{u}_s^{(k)}}{\partial y}(y) \widetilde{R}_{i-k}(y) \right) ds, \quad (2.56)$$

$$\widetilde{u}_t^{(i)}(y) = B_i(y) + \int^t \left(G_i(s, y, \widetilde{u}_s^{(1)}(y), \dots, \widetilde{u}_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} \frac{\partial \widetilde{u}_s^{(k)}}{\partial y}(y) \widetilde{R}_{i-k}(y) - \widetilde{R}_i(y) \right) ds, \quad (2.57)$$

for $i = 2, 3, \dots$, respectively, where integral constants in the definitions of $\widetilde{u}_t^{(i)}$ are the same as those of $u_t^{(i)}$ and where B_i , $i = 1, 2, \dots$ are arbitrary vector fields which imply other choice of integral constants. Along with these functions, We define the m -th order RG equation and the m -th order RG transformation to be

$$\dot{y} = \varepsilon \widetilde{R}_1(y) + \dots + \varepsilon^m \widetilde{R}_m(y), \quad (2.58)$$

and

$$\widetilde{\alpha}_t^{(m)}(y) = y + \varepsilon \widetilde{u}_t^{(1)}(y) + \dots + \varepsilon^m \widetilde{u}_t^{(m)}(y), \quad (2.59)$$

respectively. Now we have two pairs of RG equations-transformations, Eqs.(2.18),(2.19) and Eqs.(2.58),(2.59). Main theorems described so far hold for both of them for any choices of $B_i(y)$'s except to Thms. 2.11 and 2.15, in which we need additional assumptions for $B_i(y)$'s as was stated.

Let us examine relations between R_i , $u_t^{(i)}$ and \widetilde{R}_i , $\widetilde{u}_t^{(i)}$. Clearly \widetilde{R}_1 coincides with R_1 . Thus $\widetilde{u}_t^{(1)}$ is given as $\widetilde{u}_t^{(1)}(y) = u_t^{(1)}(y) + B_1(y)$. According to the definition of G_2 (Eq.(2.8)), \widetilde{R}_2 is calculated as

$$\begin{aligned} \widetilde{R}_2(y) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(\frac{\partial g_1}{\partial y}(s, y) \widetilde{u}_s^{(1)}(y) + g_2(s, y) - \frac{\partial \widetilde{u}_s^{(1)}}{\partial y} \widetilde{R}_1(y) \right) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(\frac{\partial g_1}{\partial y}(s, y) u_s^{(1)}(y) + g_2(s, y) - \frac{\partial u_s^{(1)}}{\partial y} R_1(y) \right) ds \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \int^t \left(\frac{\partial g_1}{\partial y}(s, y) B_1(y) - \frac{\partial B_1}{\partial y}(y) R_1(y) \right) ds \\ &= R_2(y) + \frac{\partial R_1}{\partial y}(y) B_1(y) - \frac{\partial B_1}{\partial y}(y) R_1(y). \end{aligned} \quad (2.60)$$

If we define the commutator $[\cdot, \cdot]$ of vector fields to be

$$[B_1, R_1](y) = \frac{\partial B_1}{\partial y}(y) R_1(y) - \frac{\partial R_1}{\partial y}(y) B_1(y), \quad (2.61)$$

Eq.(2.60) is rewritten as

$$\widetilde{R}_2(y) = R_2(y) - [B_1, R_1](y). \quad (2.62)$$

Similar calculation proves that \widetilde{R}_i , $i = 2, 3, \dots$ are expressed as

$$\widetilde{R}_i(y) = R_i(y) + P_i(R_1, \dots, R_{i-1}, B_1, \dots, B_{i-2})(y) - [B_{i-1}, R_1](y), \quad (2.63)$$

where P_i is a function of R_1, \dots, R_{i-1} and B_1, \dots, B_{i-2} . See Chiba[11] for the proof. Thus appropriate choices of vector fields $B_i(y)$, $i = 1, 2, \dots$ may simplify \widetilde{R}_i , $i = 2, 3, \dots$ through Eq.(2.63).

Suppose that R_i , $i = 1, 2, \dots$ are elements of some finite dimensional vector space V and the commutator $[\cdot, R_1]$ defines the linear map on V . For example if the function $g(t, x, \varepsilon)$ in Eq.(2.1) is polynomial in x , V is the space of polynomial vector fields. If Eq.(2.1) is an n -dimensional linear equation, then V is the space of all $n \times n$ constant matrices. Let us take a complementary subspace C to $\text{Im}[\cdot, R_1]$ into V arbitrarily: $V = \text{Im}[\cdot, R_1] \oplus C$. Then, there exist $B_i \in V$, $i = 1, 2, \dots, m-1$ such that $\widetilde{R}_i \in C$ for $i = 2, 3, \dots, m$ because of Eq.(2.63). If $\widetilde{R}_i \in C$ for $i = 2, 3, \dots, m$, we call Eq.(2.58) the m -th order simplified RG equation. See Chiba[11] for explicit forms of the simplified RG equation for the cases that Eq.(2.1) is polynomial in x or a linear system. In particular, they are quite related to the simplified normal forms theory (hyper-normal forms theory) [2,39,40]. See also Section 4.3.

Since integral constants B_i 's in Eqs.(2.55) and (2.57) are independent of t , we can show the next claim, which will be used to prove Thm.5.1.

Claim 2.16. RG equations and RG transformations are not unique in general because of undetermined integral constants in Eqs.(2.12) and (2.14). Let $\alpha_t^{(m)}$ and $\widetilde{\alpha}_t^{(m)}$ be two different RG transformations for a given system (2.1). Then, there exists a *time-independent* transformation $\phi(y, \varepsilon)$, which is C^∞ with respect to y and ε , such that $\widetilde{\alpha}_t^{(m)}(y) = \alpha_t^{(m)} \circ \phi(y, \varepsilon)$. Conversely, $\alpha_t^{(m)} \circ \phi(y, \varepsilon)$ gives one of the RG transformations for any C^∞ maps $\phi(y, \varepsilon)$.

Note that the map $\phi(y, \varepsilon)$ is independent of t because it brings one of the RG equations into the other RG equation, both of which are autonomous systems. According to Claim 2.16, one of the simplest way to achieve simplified RG equations is as follows: At first, we calculate R_i and $u_t^{(i)}$ by fixing integral constants arbitrarily and obtain the RG equation. It may be convenient in practice to choose zeros as integral constants (see Prop.2.18 below). Then, any other RG equations are given by transforming the present RG equation by time-independent C^∞ maps.

2.5 An example

In this subsection, we give an example to verify the main theorems. See Chiba[10,11] for more examples. Consider the system on \mathbf{R}^2

$$\begin{cases} \dot{X}_1 = X_2 + X_2^2 + \varepsilon^2 k \sin(\omega t), \\ \dot{X}_2 = -X_1 + \varepsilon^2 X_2 - X_1 X_2 + X_2^2, \end{cases} \quad (2.64)$$

where $\varepsilon > 0$, $k \geq 0$ and $\omega > 0$ are parameters. Changing the coordinates by $(X_1, X_2) = (\varepsilon x_1, \varepsilon x_2)$ yields

$$\begin{cases} \dot{x}_1 = x_2 + \varepsilon x_2^2 + \varepsilon k \sin(\omega t), \\ \dot{x}_2 = -x_1 + \varepsilon(x_2^2 - x_1 x_2) + \varepsilon^2 x_2. \end{cases} \quad (2.65)$$

Diagonalizing the unperturbed term by introducing the complex variable z as $x_1 = z + \bar{z}$, $x_2 = i(z - \bar{z})$ may simplify our calculation :

$$\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2} (i(z - \bar{z})^2 - 2z^2 + 2z\bar{z} + k \sin(\omega t)) + \frac{\varepsilon^2}{2}(z - \bar{z}), \\ \dot{\bar{z}} = -i\bar{z} + \frac{\varepsilon}{2} (-i(z - \bar{z})^2 - 2\bar{z}^2 + 2z\bar{z} + k \sin(\omega t)) - \frac{\varepsilon^2}{2}(z - \bar{z}), \end{cases} \quad (2.66)$$

where $i = \sqrt{-1}$. Let us calculate the RG equation for the system (2.65) or (2.66). In this example, all integral constants in Eqs.(2.47, 49) are chosen to be zero.

(i) When $\omega \neq 1, 2$, the second order RG equation for Eq.(2.66) is given as

$$\begin{cases} \dot{y}_1 = \frac{1}{2}\varepsilon^2(y_1 - 3y_1^2 y_2 - \frac{16i}{3}y_1^2 y_2), \\ \dot{y}_2 = \frac{1}{2}\varepsilon^2(y_2 - 3y_1 y_2^2 + \frac{16i}{3}y_1 y_2^2), \end{cases} \quad (2.67)$$

where the first order RG equation R_1 vanishes. Note that it is independent of the time periodic external force $k \sin(\omega t)$. Thus this RG equation coincides with that of the autonomous system obtained by putting $k = 0$ in Eq.(2.66) and Theorem 2.15 is applicable to Eq.(2.67). Indeed, since the above RG equation is invariant under the rotation group $(y_1, y_2) \mapsto (e^{i\tau} y_1, e^{-i\tau} y_2)$, putting $y_1 = r e^{i\theta}$, $y_2 = r e^{-i\theta}$ results in

$$\begin{cases} \dot{r} = \frac{1}{2}\varepsilon^2 r(1 - 3r^2), \\ \dot{\theta} = -\frac{8}{3}\varepsilon^2 r^2, \end{cases} \quad (2.68)$$

and it is easily solved. We can verify that this RG equation has a stable periodic orbit $r = \sqrt{1/3}$ if $\varepsilon > 0$. Now Corollary 2.10 (i) proves that the original system (2.65) has a stable

almost periodic solution if $\varepsilon > 0$ is small (see Fig.3). If $k = 0$ and Eq.(2.65) is autonomous, then Thm.2.14 is applied to conclude that Eq.(2.65) has a stable periodic orbit.

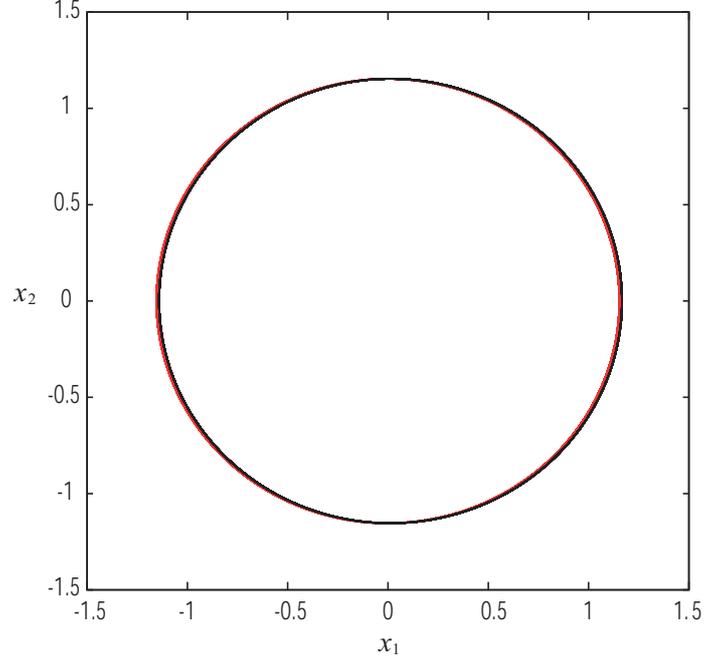


Fig. 3 Numerical results of the system (2.65) and its RG equation (2.68) for $\omega = 3$, $k = 1.8$ and $\varepsilon = 0.01$. The red curve denotes the stable periodic orbit of the RG equation and the black curve denotes the almost periodic solution of Eq.(2.65). They almost overlap with one another.

(ii) When $\omega = 2$, we prefer Eq.(2.65) to Eq.(2.66) to avoid complex numbers. The second order RG equation for Eq.(2.65) is given by

$$\begin{cases} \dot{y}_1 = \frac{\varepsilon^2}{24} (12y_1 - 9y_1^3 - 16y_1^2y_2 - 9y_1y_2^2 - 16y_2^3 - k(6y_1 + 4y_2)), \\ \dot{y}_2 = \frac{\varepsilon^2}{24} (12y_2 - 9y_2^3 + 16y_1y_2^2 - 9y_1^2y_2 + 16y_1^3 - k(4y_1 - 6y_2)). \end{cases} \quad (2.69)$$

Since it depends on the time periodic term $k \sin(\omega^2 t)$, Thm.2.15 is no longer applicable and to analyze this RG equation is rather difficult. However, numerical simulation shows that Eq.(2.69) undergoes a typical homoclinic bifurcations (see Chow, Li and Wang [14]) whose phase portraits are given as Fig.4.

Let us consider the case $k = 1.8$. In this case, the RG equation has one stable periodic orbit and two stable fixed points. Thus Corollary 2.10 proves that the original system (2.65) has one almost periodic solution $\gamma(t)$ and two periodic solutions with the period π (see Fig.5).

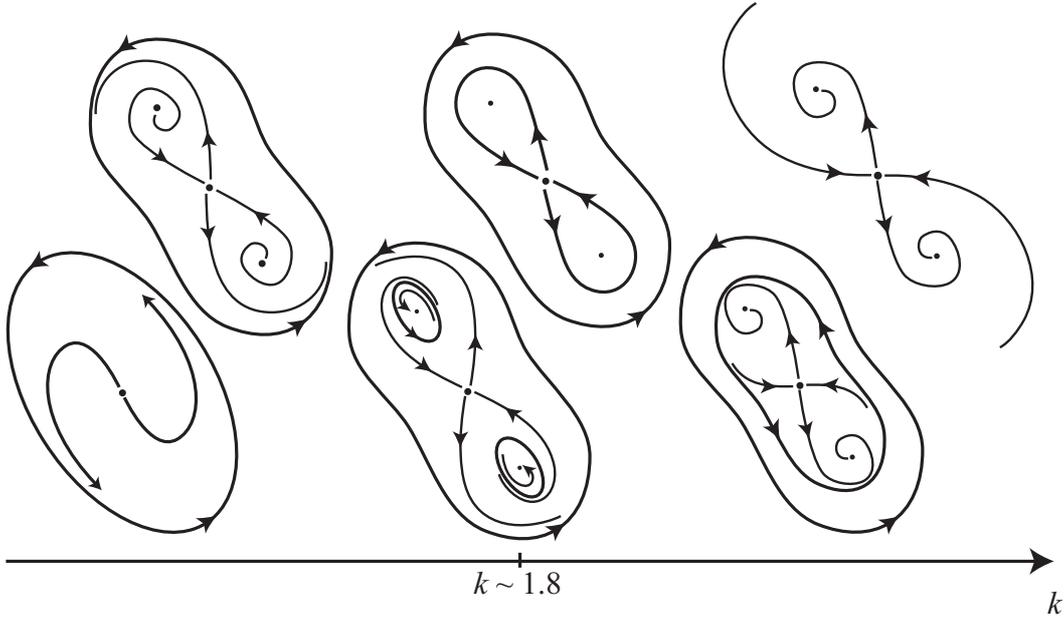


Fig. 4 Phase portraits of the RG equation (2.69).

(iii) When $\omega = 1$, the second RG equation for Eq.(2.65) is given by

$$\begin{cases} \dot{y}_1 = \frac{\varepsilon^2}{24} (12y_1 - 9y_1^3 - 16y_1^2y_2 - 9y_1y_2^2 - 16y_2^3), \\ \dot{y}_2 = \frac{\varepsilon k}{2} + \frac{\varepsilon^2}{24} (12y_2 - 9y_2^3 + 16y_1y_2^2 - 9y_1^2y_2 + 16y_1^3). \end{cases} \quad (2.70)$$

In this case, the first order RG equation does not vanish. For small ε , Eq.(2.70) has a stable fixed point $y_0 = y_0(k, \varepsilon)$, which tends to infinity as $\varepsilon \rightarrow 0$. For example if $k = 1.8$ and $\varepsilon = 0.01$, y_0 is given by $y_0 \sim (-4.35, 2.31)$ (see Fig.6). Therefore, the original system has a stable periodic orbit whose radius tends to infinity as $\varepsilon \rightarrow 0$.

2.6 A few remarks on symbolic computation

One can calculate the RG equation and the RG transformation by using symbolic computation softwares, such as Mathematica and Maple, with formulas (2.11) to (2.14). In this subsection, we provide a few remarks which may be convenient for symbolic computation.

It is not comfortable to compute the limits in formulas (2.11) and (2.13) directly by symbolic computation softwares because it takes too much time. The next proposition is useful to compute them.

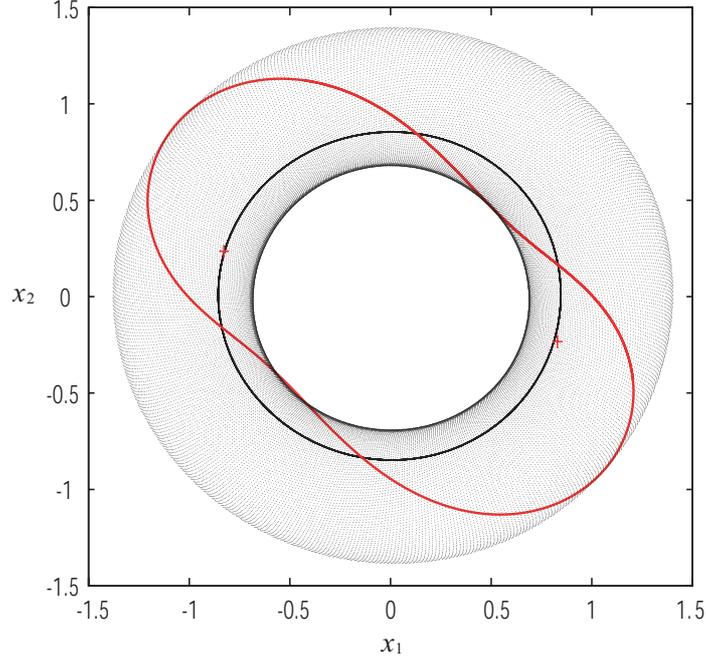


Fig. 5 Numerical results of the system (2.65) and its RG equation (2.69) for $\omega = 2$, $k = 1.8$ and $\varepsilon = 0.01$. The red curve and the red cross points denote the stable periodic orbit and the stable fixed points of the RG equation, respectively. The black dots represent points $\gamma(n/2)$, $n = 1, 2, \dots$ on the almost periodic solution $\gamma(t)$ of Eq.(2.65). The black curves denote the two periodic solutions of Eq.(2.65), which almost overlap with one another.

Proposition 2.17. Suppose that $F(t, y)$ and its primitive function are almost periodic functions in t . Then, $\lim_{t \rightarrow \infty} \frac{1}{t} \int^t F(t, y) dt$ gives a coefficient of a linear term of $\int^t F(t, y) dt$ with respect to t .

This proposition is easily proved because F is expanded in a Fourier series as $F(t, y) \sim \sum a_n(y) e^{i\lambda_n t}$. Thus, to obtain $R_i(y)$'s, we compute integrals in Eqs.(2.11,13) and extract linear terms with respect to t . To do so, for example in Mathematica, the command `Coefficient[Integrate[F[t,y],t],t]` is available, where `F[t,y]` is the integrand in Eqs.(2.11, 13).

When computing the integrals in Eqs.(2.11) to (2.14), we recommend that the integrands are expressed by exponential functions with respect to t such as $e^{i\lambda t}$ if they include trigonometric functions such as $\sin \lambda t$, $\cos \lambda t$ (in Mathematica, it is done by using the command `TrigToExp`). It is because if they include trigonometric functions, softwares may choose unexpected integral constants while if they consist of exponential functions, then zeros are

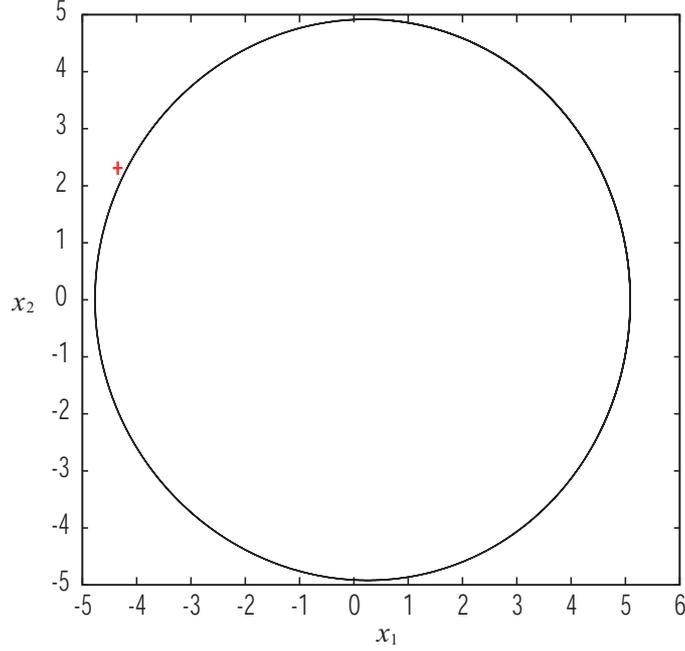


Fig. 6 Numerical results of the system (2.65) and its RG equation (2.70) for $\omega = 1$, $k = 1.8$ and $\varepsilon = 0.01$. The red cross point denotes the stable fixed point of the RG equation. The black curve denotes the periodic solution of Eq.(2.65).

chosen as integral constants. If all integral constants in Eqs.(2.12), (2.14) are zeros, the second term in the integrand in Eq.(2.13) does not include a constant term with respect to t . Thus we obtain the next proposition, which reduces the amount of calculation.

Proposition 2.18. If we choose zeros as integral constants in the formulas (2.11) to (2.14) for $i = 1, \dots, k - 1$, then Eq.(2.13) for $i = k$ is written as

$$R_k(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t G_k(s, y, u_s^{(1)}(y), \dots, u_s^{(k-1)}(y)) ds. \quad (2.71)$$

If a system for which we calculate the RG equation includes parameters such as k and ω in Eq.(2.65), then softwares automatically assume that it is in a generic case. For example if we compute the RG equation for Eq.(2.65) by using Mathematica, it is assumed that $\omega \neq 0, 1, 2$. We can verify that the RG equation (2.67) obtained by softwares is invalid when $\omega = 0, 1, 2$ by computing the second order RG transformation. Indeed, it includes the factors $\omega, \omega - 1, \omega - 2$ in denominators. The explicit form of the RG transformation for Eq.(2.65) is too complicated to show here. To obtain the RG equation for $\omega = 0, 1, 2$, substitute them into the original system (2.65) and compute the RG equation again for each case.

3 Restricted RG method

Suppose that Eq.(2.3) defined on $U \subset M$ does not satisfy the assumption (B) on U but satisfies it on a submanifold $N_0 \subset U$. Then, the RG method discussed in the previous section is still valid if domains of the RG equation and the RG transformation are restricted to N_0 . This method to construct an approximate flow on some restricted region is called the *restricted RG method* and it gives extension of the center manifold theory, the geometric singular perturbation method and the phase reduction.

3.1 Main results of the restricted RG method

Consider the system of the form

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \quad (3.1)$$

defined on $U \subset M$. For this system, we suppose that

(D1) the vector field f is C^∞ and it has a compact attracting normally hyperbolic invariant manifold $N_0 \subset U$. The flow $\varphi_t(x)$ of f on N_0 is an almost periodic function with respect to $t \in \mathbf{R}$ uniformly in $x \in N_0$, the set of whose Fourier exponents has no accumulation points.

(D2) there exists an open set $V \supset N_0$ in U such that the vector field g is C^1 in $t \in \mathbf{R}$, C^∞ in $x \in V$ and small ε , and that g is an almost periodic function with respect to $t \in \mathbf{R}$ uniformly in $x \in V$ and small ε , the set of whose Fourier exponents has no accumulation points (i.e. the assumption (A) is satisfied on V).

If g is independent of t and Eq.(3.1) is autonomous, Fenichel's theorem proves that Eq.(3.1) has an attracting invariant manifold N_ε near N_0 . If g depends on t , we rewrite Eq.(3.1) as

$$\begin{cases} \dot{x} = f(x) + \varepsilon g(s, x, \varepsilon), \\ \dot{s} = 1, \end{cases} \quad (3.2)$$

so that the unperturbed term $(f(x), 1)$ has an attracting normally hyperbolic invariant manifold $\mathbf{R} \times N_0$ on the (s, x) space. Then, a similar argument to the proof of Thm.2.9 proves that Eq.(3.2) has an attracting invariant manifold N_ε on the (s, x) space, which is diffeomorphic to $\mathbf{R} \times N_0$.

In both cases, since N_ε is attracting, long time behavior of the flow of Eq.(3.1) is well described by the flow on N_ε . Thus a central issue is to construct N_ε and the flow on it

approximately. To do so, we establish the restricted RG method on N_0 .

Definition 3.1. For Eq.(3.1), we define C^∞ maps $R_i, h_t^{(i)} : N_0 \rightarrow M$ to be

$$R_1(y) = \lim_{t \rightarrow -\infty} \frac{1}{t} \int (D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) ds, \quad (3.3)$$

$$h_t^{(1)}(y) = (D\varphi_t)_y \int_{-\infty}^t \left((D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) - R_1(y) \right) ds, \quad (3.4)$$

and

$$R_i(y) = \lim_{t \rightarrow -\infty} \frac{1}{t} \int \left((D\varphi_s)_y^{-1} G_i(s, \varphi_s(y), h_s^{(1)}(y), \dots, h_s^{(i-1)}(y)) \right. \\ \left. - (D\varphi_s)_y^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) \right) ds, \quad (3.5)$$

$$h_t^{(i)}(y) = (D\varphi_t)_y \int_{-\infty}^t \left((D\varphi_s)_y^{-1} G_i(s, \varphi_s(y), h_s^{(1)}(y), \dots, h_s^{(i-1)}(y)) \right. \\ \left. - (D\varphi_s)_y^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) - R_i(y) \right) ds, \quad (3.6)$$

for $i = 2, 3, \dots$, respectively. Note that $\lim_{t \rightarrow \infty}$ in Eqs.(2.46, 48) are replaced by $\lim_{t \rightarrow -\infty}$ and the indefinite integrals in Eqs.(2.47, 49) are replaced by the definite integrals. With these R_i and $h_t^{(i)}$, define the *restricted m -th order RG equation* for Eq.(3.1) to be

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \dots + \varepsilon^m R_m(y), \quad y \in N_0, \quad (3.7)$$

and define the *restricted m -th order RG transformation* to be

$$\alpha_t(y) = \varphi_t(y) + \varepsilon h_t^{(1)}(y) + \dots + \varepsilon^m h_t^{(m)}(y), \quad y \in N_0, \quad (3.8)$$

respectively.

Note that the domains of them are restricted to N_0 . To see that they make sense, we prove the next lemma, which corresponds to Lemma 2.1.

Lemma 3.2. (i) The maps R_i ($i = 1, 2, \dots$) are well-defined and $R_i(y) \in T_y N_0$ for any $y \in N_0$. (ii) The maps $h_t^{(i)}$ ($i = 1, 2, \dots$) are almost periodic functions with respect to t uniformly in $y \in N_0$. In particular, $h_t^{(i)}$ are bounded in $t \in \mathbf{R}$ if $y \in N_0$.

Proof. Let π_s be the projection from $T_y M$ to the stable subspace E_s of N_0 and π_{N_0} the projection from $T_y M$ to the tangent space $T_y N_0$. Note that $\pi_s + \pi_{N_0} = id$. By the definition of an attracting hyperbolic invariant manifold, there exist positive constants C_1 and α such that

$$\|\pi_s(D\varphi_t)_y^{-1} v\| < C_1 e^{\alpha t} \|v\|, \quad (3.9)$$

for any $t < 0, y \in N_0$ and $v \in T_y M$. Since $\varphi_t(y) \in N_0$ for all $t \in \mathbf{R}$ and since G_1 is almost periodic in t , there exists a positive constant C_2 such that $\|G_1(t, \varphi_t(y))\| < C_2$. Then, $\pi_s R_1(y)$ proves to satisfy

$$\begin{aligned} \|\pi_s R_1(y)\| &\leq \lim_{t \rightarrow -\infty} \frac{1}{t} \int_{-\infty}^t \|\pi_s (D\varphi_s)_y^{-1} G_1(s, \varphi_s(y))\| ds \\ &< \lim_{t \rightarrow -\infty} \frac{1}{t} \int_{-\infty}^t C_1 C_2 e^{\alpha s} ds = 0. \end{aligned} \quad (3.10)$$

This means that $R_1(y) \in T_y N_0$.

To prove (ii) of Lemma 3.2, note that the set

$$T(\delta) = \{\tau \mid \|G_1(s + \tau, y) - G_1(s, y)\| < \delta, \|\varphi_{s+\tau}(y) - \varphi_s(y)\| < \delta, \quad \forall s \in \mathbf{R}, \forall y \in N_0\} \quad (3.11)$$

is relatively dense. For $\tau \in T(\delta)$, $h_t^{(1)}(\varphi_\tau(y))$ is calculated as

$$\begin{aligned} h_t^{(1)}(\varphi_\tau(y)) &= (D\varphi_t)_{\varphi_\tau(y)} \int_{-\infty}^t \left((D\varphi_s)_{\varphi_\tau(y)}^{-1} G_1(s, \varphi_s \circ \varphi_\tau(y)) - R_1(\varphi_\tau(y)) \right) ds \\ &= \int_{-\infty}^t \left((D\varphi_{s-t})_y^{-1} G_1(s, \varphi_{s+\tau}(y)) - (D\varphi_t)_{\varphi_\tau(y)} R_1(\varphi_\tau(y)) \right) ds. \end{aligned} \quad (3.12)$$

Putting $s' = s + \tau$ yields

$$h_t^{(1)}(\varphi_\tau(y)) = \int_{-\infty}^{t+\tau} \left((D\varphi_{s'-(t+\tau)})_y^{-1} G_1(s' - \tau, \varphi_{s'}(y)) - (D\varphi_t)_{\varphi_\tau(y)} R_1(\varphi_\tau(y)) \right) ds'. \quad (3.13)$$

Since the space $T_y N_0$ is $(D\varphi_t)_y$ -invariant, $\pi_s (D\varphi_t)_y R_1(y) = 0$. This and Eq.(3.13) provide that

$$\begin{aligned} \|\pi_s h_{t+\tau}^{(1)}(y) - \pi_s h_t^{(1)}(\varphi_\tau(y))\| &\leq \int_{-\infty}^{t+\tau} \|\pi_s (D\varphi_{s-(t+\tau)})_y^{-1}\| \cdot \|G_1(s, \varphi_s(y)) - G_1(s - \tau, \varphi_s(y))\| ds \\ &\leq \int_{-\infty}^{t+\tau} \delta C_1 e^{\alpha(s-(t+\tau))} ds = \delta C_1 / \alpha. \end{aligned} \quad (3.14)$$

Thus we obtain

$$\begin{aligned} \|\pi_s h_{t+\tau}^{(1)}(y) - \pi_s h_t^{(1)}(y)\| &\leq \|\pi_s h_{t+\tau}^{(1)}(y) - \pi_s h_t^{(1)}(\varphi_\tau(y))\| + \|\pi_s h_t^{(1)}(\varphi_\tau(y)) - \pi_s h_t^{(1)}(y)\| \\ &\leq (C_1 / \alpha + L_t) \delta, \end{aligned}$$

where L_t , which is bounded in t , is the Lipschitz constant of $\pi_s h_t^{(1)}|_{N_0}$. This proves that $\pi_s h_t^{(1)}$ is an almost periodic function with respect to t . On the other hand, $\pi_{N_0} h_t^{(1)}$ is written as

$$\pi_{N_0} h_t^{(1)}(y) = \pi_{N_0} (D\varphi_t)_y \int_{-\infty}^t \left(\pi_{N_0} (D\varphi_s)_y^{-1} G_1(s, \varphi_s(y)) - R_1(y) \right) ds. \quad (3.15)$$

Since $\pi_{N_0} (D\varphi_t)_y$ is almost periodic, we can show that $\pi_{N_0} h_t^{(1)}(y)$ is almost periodic in t in the same way as the proof of Lemma 2.1 (ii). This proves that $h_t^{(1)} = \pi_s h_t^{(1)} + \pi_{N_0} h_t^{(1)}$ is an almost

periodic function in t . The proof of Lemma 3.2 for $i = 2, 3, \dots$ is done in a similar way by induction and we omit it here. ■

Since $R_i(y) \in T_y N_0$, Eq.(3.7) defines a $\dim N_0$ -dimensional differential equation on N_0 .

Remark 3.3. Even if $h_t^{(i)}$ are defined by using indefinite integrals as Eqs.(2.47, 49), we can show that $h_t^{(i)}$ is bounded as $t \rightarrow \infty$, though it is not bounded as $t \rightarrow -\infty$. In this case, the theorems listed below are true for large t .

Now we are in a position to state main theorems of the restricted RG method, all proofs of which are the same as before and omitted.

Theorem 3.4. Let $\alpha_t^{(m)}$ be the restricted m -th order RG transformation for Eq.(3.1). If $|\varepsilon|$ is sufficiently small, there exists a function $S(t, y, \varepsilon)$, $y \in N_0$ such that

(i) by changing the coordinates as $x = \alpha_t^{(m)}(y)$, Eq.(3.1) is transformed into the system

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \dots + \varepsilon^m R_m(y) + \varepsilon^{m+1} S(t, y, \varepsilon), \quad (3.16)$$

(ii) S is an almost periodic function with respect to t uniformly in $y \in N_0$,

(iii) $S(t, y, \varepsilon)$ is C^1 with respect to t and C^∞ with respect to $y \in N_0$ and ε .

Theorem 3.5 (Error estimate). Let $y(t)$ be a solution of the restricted m -th order RG equation and $\alpha_t^{(m)}$ the restricted m -th order RG transformation. There exist positive constants ε_0, C and T such that a solution $x(t)$ of Eq.(3.1) with $x(0) = \alpha_0^{(m)}(y(0)) \in \alpha_0^{(m)}(N_0)$ satisfies the inequality

$$\|x(t) - \alpha_t^{(m)}(y(t))\| < C|\varepsilon|^m, \quad (3.17)$$

for $|\varepsilon| < \varepsilon_0$ and $0 \leq t \leq T/|\varepsilon|$.

Theorem 3.6 (Existence of invariant manifolds). Suppose that $R_1(y) = \dots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the restricted RG equation for Eq.(3.1). If the vector field $R_k(y)$ has a boundaryless compact normally hyperbolic invariant manifold $L \subset N_0$, then for sufficiently small $\varepsilon > 0$, the system (3.2) has an invariant manifold L_ε on the (s, x) space which is diffeomorphic to $\mathbf{R} \times L$. In particular, the stability of L_ε coincides with that of L .

Theorem 3.7 (Inheritance of symmetries). Suppose that an ε -independent Lie group H acts on N_0 . If the vector field f and g are invariant under the action of H , then the restricted m -th order RG equation for Eq.(3.1) is also invariant under the action of H .

Recall that our purpose is to construct the invariant manifold N_ε of Eq.(3.1) and the flow on N_ε approximately. The flow on N_ε is well understood by Theorems 3.4 to 3.6, and N_ε is

given by the next theorem.

Theorem 3.8. Let $\alpha_t^{(m)}$ be the restricted m -th order RG transformation for Eq.(3.1). Then, the set $\{(t, x) \mid x \in \alpha_t^{(m)}(N_0)\}$ lies within an $O(\varepsilon^{m+1})$ neighborhood of the attracting invariant manifold N_ε of Eq.(3.2).

Proof. Though the maps $R_i(y)$ and $\alpha_t^{(m)}$ are defined on N_0 , we can extend them to the maps defined on $V \supset N_0$ so that Eq.(3.7) is C^1 close to Eq.(3.16) on V and that N_0 is an attracting normally hyperbolic invariant manifold of Eq.(3.7). Then the same argument as the proof of Thm.2.9 proves Theorem 3.8. ■

If the vector field g is independent of t and Eq.(3.1) is autonomous, we can prove the next theorems.

Theorem 3.9 (Existence of invariant manifolds). Suppose that $R_1(y) = \cdots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the restricted RG equation for Eq.(3.1) with t -independent g . If the vector field $R_k(y)$ has a boundaryless compact normally hyperbolic invariant manifold L , then for sufficiently small $\varepsilon > 0$, Eq.(3.1) has an invariant manifold L_ε , which is diffeomorphic to L . In particular, the stability of L_ε coincides with that of L .

Theorem 3.10 (Additional symmetry). The restricted RG equation for Eq.(3.1) with t -independent g is invariant under the action of the one-parameter group $\{\varphi_t : N_0 \rightarrow N_0 \mid t \in \mathbf{R}\}$. In other words, R_i satisfies the equality

$$R_i(\varphi_t(y)) = (D\varphi_t)_y R_i(y), \quad y \in N_0, \quad (3.18)$$

for $i = 1, 2, \dots$.

For autonomous systems, Thm.3.8 is restated as follows: Recall that if the function g depends on t , the attracting invariant manifold N_ε of Eq.(3.1) and the approximate invariant manifold described in Thm.3.8 depend on t in the sense that they lie on the (s, x) space. If Eq.(3.1) is autonomous, its attracting invariant manifold N_ε lies on M and is independent of t . Thus we want to construct an approximate invariant manifold of N_ε so that it is also independent of t .

Theorem 3.11. Let $\alpha_t^{(m)}$ be the restricted m -th order RG transformation for Eq.(3.1). If g is independent of t , the set $\alpha_t^{(m)}(N_0) = \{\alpha_t^{(m)}(y) \mid y \in N_0\}$ is independent of t and lies within an $O(\varepsilon^{m+1})$ neighborhood of the attracting invariant manifold N_ε of Eq.(3.1).

Proof. We have to show that the set $\alpha_t^{(m)}(N_0)$ is independent of t . Indeed, we know the equality $h_t^{(i)}(\varphi_{t'}(y)) = h_{t+t'}^{(i)}(y)$ as is shown in the proof of the Thm.2.15. This proves that

$$\alpha_{t+t'}^{(m)}(N_0) = \alpha_t^{(m)}(\varphi_{t'}(N_0)) = \alpha_t^{(m)}(N_0). \quad (3.19)$$

The rest of the proof is the same as the proofs of Thm.2.14 and Thm.3.8. ■

3.2 Center manifold reduction

The restricted RG method recovers the approximation theory of center manifolds (Carr [7]). Consider a system of the form

$$\begin{aligned} \dot{x} &= Fx + \varepsilon g(x, \varepsilon) \\ &= Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \quad x \in \mathbf{R}^n, \end{aligned} \quad (3.20)$$

where unperturbed term Fx is linear. For this system, we suppose that

(E1) all eigenvalues of the $n \times n$ constant matrix F are on the imaginary axis or the left half plane. The Jordan block corresponding to eigenvalues on the imaginary axis is diagonalizable.

(E2) g is C^∞ with respect to x and ε such that $g(0, \varepsilon) = 0$.

If all eigenvalues of F are on the left half plane, the origin is a stable fixed point and the flow near the origin is trivial. In what follows, we suppose that at least one eigenvalue is on the imaginary axis. In this case, Eq.(3.20) has a center manifold which is tangent to the center subspace N_0 at the origin. The center subspace N_0 , which is spanned by eigenvectors associated with eigenvalues on the imaginary axis, is an attracting normally hyperbolic invariant manifold of the unperturbed term Fx , and the flow of Fx on N_0 is almost periodic. However, since N_0 is not compact, we take an n -dimensional closed ball K including the origin and consider $N_0 \cap K$. Then, we obtain the next theorem as a corollary of Thm.3.9 and Thm.3.11.

Theorem 3.12 (Approximation of Center Manifolds, [12]). Let $\alpha_t^{(m)}$ be the restricted m -th order RG transformation for Eq.(3.20) and K a small compact neighborhood of the origin. Then, the set $\alpha_t^{(m)}(K \cap N_0)$ lies within an $O(\varepsilon^{m+1})$ neighborhood of the center manifold of Eq.(3.20). The flow of Eq.(3.20) on the center manifold is well approximated by those of the restricted RG equation. In particular, suppose that $R_1(y) = \cdots = R_{k-1}(y) = 0$ and $\varepsilon^k R_k(y)$ is the first non-zero term in the restricted RG equation. If the vector field $R_k(y)$ has a boundaryless compact normally hyperbolic invariant manifold L , then for sufficiently small $\varepsilon > 0$,

Eq.(3.20) has an invariant manifold L_ε on the center manifold, which is diffeomorphic to L . The stability of L_ε coincides with that of L .

See Chiba [12] for the detail of the proof and examples.

3.3 Geometric singular perturbation method

The restricted RG method can also recover the geometric singular perturbation method proposed by Fenichel [19].

Consider the autonomous system

$$\dot{x} = f(x) + \varepsilon g(x, \varepsilon), \quad x \in \mathbf{R}^n \quad (3.21)$$

on \mathbf{R}^n with the assumption that

(F) suppose that f and g are C^∞ with respect to x and ε , and that f has an m -dimensional attracting normally hyperbolic invariant manifold N_0 which consists of fixed points of f , where $m < n$.

Note that this system satisfies the assumptions (D1) and (D2), so that Thm.3.9 to Thm.3.11 hold. The invariant manifold N_0 consisting of fixed points of the unperturbed system is called the *critical manifold* (see [1]). For this system, Fenichel[19] proved that there exist local coordinates (u, v) such that the system (3.21) is expressed as

$$\begin{cases} \dot{u} = \varepsilon \hat{g}_1(u, v, \varepsilon), & u \in \mathbf{R}^m, \\ \dot{v} = \hat{f}(u, v) + \varepsilon \hat{g}_2(u, v, \varepsilon), & v \in \mathbf{R}^{n-m}, \end{cases} \quad (3.22)$$

where $\hat{f}(u, 0) = 0$ for any $u \in \mathbf{R}^m$. In this coordinate, the critical manifold N_0 is locally given as the u -plane. Further he proved the next theorem.

Theorem 3.13 (Fenichel [19]). Suppose that the system $\dot{u} = \varepsilon \hat{g}_1(u, 0, 0)$ has a compact normally hyperbolic invariant manifold L . If $\varepsilon > 0$ is sufficiently small, the system (3.22) has an invariant manifold L_ε which is diffeomorphic to L .

This method to obtain an invariant manifold of (3.22) is called the *geometric singular perturbation method*. By using the fact that $\varphi_t(u) = u$ for $u \in N_0$, it is easy to verify that the system $\dot{u} = \varepsilon \hat{g}_1(u, 0, 0)$ described above is just the restricted first order RG equation for Eq.(3.22). Thus Thm.3.13 immediately follows from Thm.3.9. Note that in our method, we

need not change the coordinates so that Eq.(3.21) is transformed into the form of Eq.(3.22).

Example 3.14. Consider the system on \mathbf{R}^2

$$\begin{cases} \dot{x}_1 = -x_1 + (x_1 + c)x_2, \\ \varepsilon \dot{x}_2 = x_1 - (x_1 + 1)x_2, \end{cases} \quad (3.23)$$

where $0 < c < 1$ is a constant. This system arises from a model of the kinetics of enzyme reactions (see Carr [7]). Set $t = \varepsilon s$ and denote differentiation with respect to s by $'$. Then, the above system is rewritten as

$$\begin{cases} x_1' = \varepsilon(-x_1 + cx_2 + x_1x_2), \\ x_2' = x_1 - x_2 - x_1x_2. \end{cases} \quad (3.24)$$

The attracting critical manifold N_0 of this system is expressed as the graph of the function

$$x_2 = h(x_1) := \frac{x_1}{1 + x_1}. \quad (3.25)$$

Since the restricted first order RG transformation for Eq.(3.24) is given by

$$\alpha_t^{(1)}(y_1) = \begin{pmatrix} y_1 \\ y_1/(1 + y_1) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ -(c - 1)y_1/(1 + y_1)^4 \end{pmatrix}, \quad (3.26)$$

Theorem 3.11 proves that the attracting invariant manifold of Eq.(3.24) is given as the graph of

$$x_2 = \frac{x_1}{1 + x_1} - \varepsilon \frac{(c - 1)x_1}{(1 + x_1)^4} + O(\varepsilon^2). \quad (3.27)$$

If $|x_1|$ is sufficiently small, it is expanded as

$$\begin{aligned} x_2 &= x_1(1 - x_1) - \varepsilon(c - 1)x_1(1 - 4x_1) + O(x_1^3, \varepsilon^2) \\ &= (1 - \varepsilon(c - 1))x_1 - (1 - 4\varepsilon(c - 1))x_1^2 + O(x_1^3, \varepsilon^2). \end{aligned} \quad (3.28)$$

This result coincides with the result obtained by the local center manifold theory (see Carr [7]). The restricted first order RG equation on N_0 is given by

$$y_1' = \varepsilon \frac{(c - 1)y_1}{1 + y_1}. \quad (3.29)$$

This RG equation describes a motion on the invariant manifold (3.27) approximately. Since it has the stable fixed point $y_1 = 0$ if $c < 1$, the system (3.24) also has a stable fixed point $(x_1, x_2) = (0, 0)$ by virtue of Theorem 3.9.

3.4 Phase reduction

Consider a system of the form

$$\dot{x} = f(t, x) + \varepsilon g_1(t, x), \quad x \in \mathbf{R}^n. \quad (3.30)$$

For this system, we suppose that

(G1) the vector fields f and g are C^∞ in x , C^1 in t , and T -periodic in t . It need not be the least period. In particular, f and g are allowed to be independent of t .

(G2) the unperturbed system $\dot{x} = f(t, x)$ has a k -parameter family of T -periodic solutions which constructs an attracting invariant torus $\mathbb{T}^k \subset \mathbf{R}^n$.

Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be coordinates on \mathbb{T}^k , so-called *phase variables*. It is called the *phase reduction* to derive equations on α which govern the dynamics of Eq.(3.30) on the invariant torus. The phase reduction was first introduced by Malkin [33,34] and rediscovered by many authors. In this subsection, we show that the RG method can recover the phase reduction method.

The next theorem is due to Malkin [33,34]. See also Hahn [23], Blekhman [4], and Hoppensteadt and Izhikevich [25].

Theorem 3.15 (Malkin [33,34]). Consider the system (3.30) with the assumptions (G1) and (G2). Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be phase variables and $U(t; \alpha)$ the periodic solutions of the unperturbed system parameterized by α . Suppose that the adjoint equation

$$\frac{dQ_i}{dt} = - \left(\frac{\partial f}{\partial x}(t, U(t; \alpha)) \right)^\top Q_i \quad (3.31)$$

has exactly k independent T -periodic solutions $Q_1(t; \alpha), \dots, Q_k(t; \alpha)$, where A^\top denotes the transpose matrix of a matrix A . Let $Q = Q(t; \alpha)$ be the $k \times n$ matrix whose columns are these solutions such that

$$Q^\top \frac{\partial U}{\partial \alpha}(t; \alpha) = id. \quad (3.32)$$

Then, Eq.(3.30) has a solution of the form

$$x(t) = U(t, \alpha(t)) + O(\varepsilon), \quad (3.33)$$

where $\alpha(t)$ is a solution of the system

$$\frac{d\alpha}{dt} = \frac{\varepsilon}{T} \int_0^T Q(s; \alpha)^\top g_1(s, U(s; \alpha)) ds. \quad (3.34)$$

Now we show that the system (3.34) of the phase variables is just the first order RG equation. Note that the system (3.30) satisfies the assumptions (D1) and (D2) with $N_0 = \mathbb{T}^k$ and the RG method is applicable. The restricted first order RG equation for Eq.(3.30) is given by

$$\dot{y} = \varepsilon \lim_{t \rightarrow -\infty} \int_0^t \left(\frac{\partial \varphi_s}{\partial y}(y) \right)^{-1} g_1(s, \varphi_s(y)) ds, \quad y \in \mathbb{T}^k. \quad (3.35)$$

Let us change the coordinates by using the k -parameter family of periodic solutions as $y = U(0; \alpha)$. Then, Eq.(3.35) is rewritten as

$$\frac{\partial U}{\partial \alpha}(0; \alpha) \dot{\alpha} = \varepsilon \lim_{t \rightarrow -\infty} \int_0^t \left(\frac{\partial \varphi_s}{\partial y}(U(0; \alpha)) \right)^{-1} g_1(s, U(s; \alpha)) ds. \quad (3.36)$$

Since $U(t; \alpha) = \varphi_t(U(0; \alpha))$, the equality

$$\frac{\partial U}{\partial \alpha}(t; \alpha) = \frac{\partial \varphi_t}{\partial y}(U(0; \alpha)) \frac{\partial U}{\partial \alpha}(0; \alpha) \quad (3.37)$$

holds. Then, Eqs.(3.32), (3.36) and (3.37) are put together to obtain

$$\begin{aligned} \dot{\alpha} &= \varepsilon \lim_{t \rightarrow -\infty} \int_0^t \left(\frac{\partial U}{\partial \alpha}(0; \alpha) \right)^{-1} \left(\frac{\partial \varphi_s}{\partial y}(U(0; \alpha)) \right)^{-1} g_1(s, U(s; \alpha)) ds \\ &= \varepsilon \lim_{t \rightarrow -\infty} \int_0^t \left(\frac{\partial U}{\partial y}(s; \alpha) \right)^{-1} g_1(s, U(s; \alpha)) ds \\ &= \varepsilon \lim_{t \rightarrow -\infty} \int_0^t Q(s; \alpha)^\top g_1(s, U(s; \alpha)) ds. \end{aligned} \quad (3.38)$$

Since the integrand in the above equation is T -periodic, Eq.(3.38) is reduced to Eq.(3.34).

4 Relation to other singular perturbation methods

In the previous section, we have seen that the restricted RG method unifies the center manifold reduction, the geometric singular perturbation method and the phase reduction. In this section, we show that the RG method described in Section 2 unifies the traditional singular perturbation methods, such as the averaging method, the multiple time scale method and the normal forms theory. We will also give explicit formulas for regular perturbation solutions and show how the RG equation is derived from the regular perturbation method through the envelope theory.

4.1 Averaging method

The averaging method is one of the most traditional and famous singular perturbation methods based on an idea of averaging a given equation, which is periodic in time t , with respect to t to obtain an autonomous equation (see Sanders, Verhulst and Murdock [47]). In most literature, only first and second order averaging equations are given and they coincide with first and second order RG equations, respectively. Our m -th order RG equation gives a generalization of such lower order averaging equations. In Chiba and Paz [13], the third and fifth order RG equations are used to unfold a phase diagram of the Kuramoto model of coupled oscillators.

Many authors formulated the averaging method for time-periodic differential equations. However, the RG equation (the averaging equation) can be defined as long as limits in Eqs.(2.11, 13) exist and $R_i(y)$ are well-defined even if a system (2.1) does not satisfy the assumption (A). If R_1, \dots, R_m are well-defined for a system (2.1), which need not satisfy the assumption (A), we say (2.1) satisfies the *KBM condition up to order m* (KBM stands for Krylov, Bogoliubov and Mitropolsky, see Bogoliubov and Mitropolsky [6]). We showed that a system (2.1) with the assumption (A) satisfies the KBM condition up to all order (Lemma 2.1). If the assumption (A) is violated, the error estimate (2.27) for approximate solutions may get worse, such as $\|x(t) - \alpha_t^{(m)}(y(t))\| < C\sqrt{\varepsilon}$, even if the KBM condition is satisfied and thus the RG equation is defined. See Sanders *et al.* [47] and DeVille *et al.* [32] for such examples.

Even if (2.1) satisfies the KBM condition, we need additional assumptions to prove Thm.2.9. It is because to apply Fenichel's theorem, which was used in the proof of Thm.2.9, we have to show that the error function $S(t, y, \varepsilon)$ in Eq.(2.23) is C^1 with respect to t, x and bounded in $t \in \mathbf{R}$. See Chiba [10] for more detail.

4.2 Multiple time scale method

The multiple time scale method [3,38,44] was perhaps introduced in the early 20th century and now it is one of the most used perturbation techniques along with the averaging method. In this subsection, we give a brief review of the method and show that it yields the same results as the RG method.

Consider the system (2.1) satisfying the assumption (A). Let us introduce the new time

scales t_m as

$$t_m = \varepsilon^m t, \quad m = 0, 1, 2, \dots, \quad (4.1)$$

and consider t and t_0, t_1, t_2, \dots to be independent of each other. Then, the “total” derivative d/dt is rewritten as

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \dots. \quad (4.2)$$

Let us expand the dependent variable x as

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots, \quad (4.3)$$

where $x_i = x_i(t_0, t_1, t_2, \dots)$. Substituting Eqs.(4.2) and (4.3) into (2.1), we obtain

$$\left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \dots \right) (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) = \sum_{k=1}^{\infty} \varepsilon^k g_k(t_0, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots). \quad (4.4)$$

Expanding the both sides of the above in ε and equating the coefficients of each ε^k , we obtain ODEs on x_0, x_1, x_2, \dots ;

$$\left\{ \begin{array}{l} \frac{\partial x_0}{\partial t_0} = 0, \\ \frac{\partial x_1}{\partial t_0} = G_1(t_0, x_0) - \frac{\partial x_0}{\partial t_1}, \\ \frac{\partial x_2}{\partial t_0} = G_2(t_0, x_0, x_1) - \frac{\partial x_1}{\partial t_1} - \frac{\partial x_0}{\partial t_2}, \\ \vdots \\ \frac{\partial x_m}{\partial t_0} = G_m(t_0, x_0, \dots, x_{m-1}) - \sum_{j=1}^m \frac{\partial x_{m-j}}{\partial t_j}, \end{array} \right. \quad (4.5)$$

where the functions G_k , $k = 1, 2, \dots$ are defined through Eq.(2.6). Let $x_0 = y = y(t_1, t_2, \dots)$ be a solution of the zeroth order equation. Then, a general solution of the first order equation of (4.5) is given as

$$x_1 = B_1 + \int^{t_0} \left(G_1(s, y) - \frac{\partial y}{\partial t_1} \right) ds = B_1 + u_{t_0}^{(1)}(y) + R_1(y)t_0 - \frac{\partial y}{\partial t_1} t_0, \quad (4.6)$$

where $B_1 = B_1(y; t_1, t_2, \dots)$ is independent of t_0 and where R_1 and $u_{t_0}^{(1)}$ are defined by Eqs.(2.11) and (2.12), respectively. Now we define $\partial y / \partial t_1$ so that x_1 above is bounded in t_0 :

$$R_1(y)t_0 - \frac{\partial y}{\partial t_1} t_0 = 0. \quad (4.7)$$

This condition is called the *non-secularity condition* and it yields

$$\frac{\partial y}{\partial t_1} = R_1(y), \quad x_1 = x_1(y) = B_1 + u_{t_0}^{(1)}(y). \quad (4.8)$$

Next thing to do is calculating x_2 . The equation on x_2 is written as

$$\begin{aligned} \frac{\partial x_2}{\partial t_0} &= G_2(t_0, y, B_1 + u_{t_0}^{(1)}(y)) - \frac{\partial}{\partial t_1}(B_1 + u_{t_0}^{(1)}(y)) - \frac{\partial y}{\partial t_2} \\ &= \frac{\partial g_1}{\partial y}(t_0, y)(B_1 + u_{t_0}^{(1)}(y)) + g_2(t_0, y) - \frac{\partial u_{t_0}^{(1)}}{\partial y}(y)R_1(y) - \frac{\partial B_1}{\partial t_1} - \frac{\partial B_1}{\partial y}R_1(y) - \frac{\partial y}{\partial t_2}, \end{aligned} \quad (4.9)$$

a general solution of which is given by

$$\begin{aligned} x_2 &= B_2 + \int^{t_0} \left(\frac{\partial g_1}{\partial y}(s, y)(B_1 + u_s^{(1)}(y)) + g_2(s, y) - \frac{\partial u_s^{(1)}}{\partial y}(y)R_1(y) - \frac{\partial B_1}{\partial t_1} - \frac{\partial B_1}{\partial y}R_1(y) - \frac{\partial y}{\partial t_2} \right) ds \\ &= B_2 + u_{t_0}^{(2)}(y) + \frac{\partial u_{t_0}^{(1)}}{\partial y}(y)B_1 \\ &\quad + \frac{\partial R_1}{\partial y}(y)B_1 t_0 + R_2(y)t_0 - \frac{\partial B_1}{\partial t_1}t_0 - \frac{\partial B_1}{\partial y}R_1(y)t_0 - \frac{\partial y}{\partial t_2}t_0, \end{aligned} \quad (4.10)$$

where $B_2 = B_2(y; t_1, t_2, \dots)$ is independent of t_0 . If we impose the non-secularity condition so that x_2 is bounded in t_0 , we obtain

$$\frac{\partial y}{\partial t_2} = R_2(y) - \frac{\partial B_1}{\partial t_1} - [B_1, R_1](y), \quad (4.11)$$

$$x_2 = x_2(y) = B_2 + u_{t_0}^{(2)}(y) + \frac{\partial u_{t_0}^{(1)}}{\partial y}(y)B_1, \quad (4.12)$$

where the commutator $[\cdot, \cdot]$ is defined in Eq.(2.61).

By proceeding in a similar manner, we can show that the non-resonance condition at each stage yields equations on y of the form

$$\frac{\partial y}{\partial t_i} = R_i(y) + Q_i(R_1, \dots, R_{i-1}, B_1, \dots, B_{i-2})(y) - \frac{\partial B_{i-1}}{\partial t_1} - [B_{i-1}, R_1](y), \quad (4.13)$$

for $i = 2, 3, \dots$, where Q_i is a function of R_1, \dots, R_{i-1} and B_1, \dots, B_{i-2} . Combining these equations on y , we obtain

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial y}{\partial t_0} + \varepsilon \frac{\partial y}{\partial t_1} + \varepsilon^2 \frac{\partial y}{\partial t_2} + \dots \\ &= \varepsilon R_1(y) + \varepsilon^2 \left(R_2(y) - \frac{\partial B_1}{\partial t_1} - [B_1, R_1](y) \right) + \dots \end{aligned} \quad (4.14)$$

The functions B_1, B_2, \dots are still left undetermined. We propose two different ways to determine them.

(i) If we define B_i 's as solutions of the linear inhomogeneous differential equations

$$\frac{\partial B_i}{\partial t_1} = -[B_i, R_1](y) + R_{i+1}(y) + Q_{i+1}(R_1, \dots, R_i, B_1, \dots, B_{i-1})(y), \quad (4.15)$$

then Eq.(4.14) is reduced to the equation

$$\frac{dy}{dt} = \varepsilon R_1(y). \quad (4.16)$$

Let $y = y(t)$ be a solution of this equation. Then, we can prove that a curve defined by

$$\begin{aligned} \alpha_t^{(m)}(y(t)) &:= y(t) + \varepsilon x_1(y(t)) + \varepsilon^2 x_2(y(t)) + \dots + \varepsilon^m x_m(y(t)) \\ &= y + \varepsilon \left(B_1 + u_t^{(1)}(y) \right) + \varepsilon^2 \left(B_2 + u_t^{(2)}(y) + \frac{\partial u_t^{(1)}}{\partial y}(y) B_1 \right) + \dots \Big|_{y=y(t)} \end{aligned} \quad (4.17)$$

provides an approximate solution for Eq.(2.1) and satisfies the same inequality as (2.27) (see Murdock[38]).

(ii) Otherwise, we suppose that B_i 's are independent of t_1, t_2, \dots so that Eq.(4.14) is independent of t_1, t_2, \dots . Then Eq.(4.14) takes the form

$$\frac{dy}{dt} = \varepsilon R_1(y) + \varepsilon^2 (R_2(y) - [B_1, R_1](y)) + \dots, \quad (4.18)$$

which coincides with the (simplified) RG equation (see Sec.2.4).

4.3 Normal forms

Let us consider the system (2.5) with assumptions (C1) and (C2). The technique of normal forms is used to analyze local dynamics of such a system and had been well developed in the last century (Murdock [39]). Let us recall the definition of normal forms. For a given system (2.5), there exist a *time-independent* local coordinate transformation $x = h(z)$ defined near the origin which brings Eq.(2.5) into the form

$$\dot{z} = Fz + \varepsilon \tilde{g}_1(z) + \dots + \varepsilon^m \tilde{g}_m(z) + \varepsilon^{m+1} \tilde{S}(z, \varepsilon) \quad (4.19)$$

with the properties that $\tilde{g}_i(z)$, $i = 1, \dots, m$ satisfy $\tilde{g}_i(e^{Ft}z) = e^{Ft} \tilde{g}_i(z)$ and that \tilde{S} is C^∞ in z and ε . Then the truncated system

$$\dot{z} = Fz + \varepsilon \tilde{g}_1(z) + \dots + \varepsilon^m \tilde{g}_m(z) \quad (4.20)$$

is called the *normal form of Eq.(2.5) up to order m*.

Note that if the matrix F is *not* diagonalizable, different definitions are adopted for normal forms.

Now we consider the RG equation and the RG transformation for Eq.(2.5), respectively, defined by Eqs.(2.50) and (2.51) with Eqs.(2.46) to (2.49). The flow $\varphi_t(y)$ in Eqs.(2.46) to (2.49) is given as $\varphi_t(y) = e^{Ft}y$ by using the fundamental matrix e^{Ft} in our situation. Thm.2.5 shows that the RG transformation $x = \alpha_t^{(m)}(y)$ brings Eq.(2.5) into the form of Eq.(2.23). Because of Thm.2.15, further change of variables as $y = e^{-Ft}z$ transforms Eq.(2.23) into the system

$$\dot{z} = Fz + \varepsilon R_1(z) + \cdots + \varepsilon^m R_m(z) + \varepsilon^{m+1} e^{Ft} S(t, e^{-Ft}z, \varepsilon) \quad (4.21)$$

with the property that $R_i(e^{Ft}z) = e^{Ft}R_i(z)$ for $i = 1, \dots, m$.

Now we need the next lemma.

Lemma 4.1. For the autonomous system (2.5), the RG transformation satisfies the equality $\alpha_t^{(m)}(e^{Ft'}y) = \alpha_{t+t'}^{(m)}(y)$ for all t, t', y .

This lemma immediately follows from the equality $h_t^{(i)}(\varphi_{t'}(y)) = h_{t+t'}^{(i)}(y)$, $i = 1, 2, \dots$ proved in the proof of Thm.2.15. Thus it turns out that the composition of two transformations $x = \alpha_t^{(m)}(y)$ and $y = e^{-Ft}z$ is independent of t :

$$x = \alpha_t^{(m)}(e^{-Ft}z) = \alpha_0^{(m)}(z). \quad (4.22)$$

This proves that Eq.(4.21) is obtained from Eq.(2.5) by the time-independent transformation, and thus the truncated system

$$\dot{z} = Fz + \varepsilon R_1(z) + \cdots + \varepsilon^m R_m(z) \quad (4.23)$$

is a normal form of Eq.(2.5) up to order m .

Note that normal forms of a given system are not unique in general and the simplest form among them is called a *hyper-normal form* ([2,39,40]). The RG method can also provide hyper-normal forms by choosing undetermined integral constants in Eqs.(2.47) and (2.49) appropriately as was discussed in Sec.2.4. See also Chiba [11].

4.4 Regular perturbation and envelopes

In this subsection, we give some properties of the regular perturbation method. The RG equation is obtained from regular perturbation solutions through Kunihiro's idea based on envelopes [28,29].

Consider the system (2.1) on \mathbf{R}^n with the assumption (A). Let us construct a formal solution of it of the form

$$x = \hat{x} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots . \quad (4.24)$$

Substituting Eq.(4.24) into Eq.(2.1) and equating the coefficients of each ε^k , we obtain a system of ODEs :

$$\begin{cases} \dot{x}_0 = 0, \\ \dot{x}_1 = G_1(t, x_0), \\ \vdots \\ \dot{x}_k = G_k(t, x_0, \cdots, x_{k-1}), \\ \vdots \end{cases} \quad (4.25)$$

where the functions G_k are defined through Eq.(2.6). Solving these equations and substituting solutions $x_k = x_k(t)$ into Eq.(4.24), we obtain a formal solution $x = \hat{x}(t)$. If Eq.(2.1) is analytic in $\varepsilon \in \mathcal{E} \subset \mathbf{C}$, Eq.(4.24) converges for $\varepsilon \in \mathcal{E}$ and gives an exact solution of Eq.(2.1). However, in this section, we need not such an assumption and regard Eq.(4.24) as a formal power series in ε . To construct a formal solution $\hat{x}(t)$ as above is called the *regular perturbation method*.

Let us define $R_i(y)$ and $u_t^{(i)}(y)$, $i = 1, 2, \cdots$ by Eqs.(2.11) to (2.14). By using these functions, regular perturbation solutions (4.24) are given as follows:

Proposition 4.2. Solutions of the system of ODEs (4.25) are given as

$$x_k = x_k(t, y) = u_t^{(k)}(y) + p_1^{(k)}(t, y)t + p_2^{(k)}(t, y)t^2 + \cdots + p_k^{(k)}(t, y)t^k, \quad (4.26)$$

for $k = 1, 2, \cdots$, where $y \in \mathbf{R}^n$ is an arbitrary constant (a solution of $\dot{x}_0 = 0$) and where $p_i^{(j)}$ are defined by

$$\begin{cases} p_1^{(1)}(t, y) = p_1^{(1)}(y) = R_1(y) \\ p_1^{(i)}(t, y) = R_i(y) + \sum_{k=1}^{i-1} \frac{\partial u_t^{(k)}}{\partial y}(y) R_{i-k}(y), \quad i = 2, 3, \cdots, \\ p_j^{(i)}(t, y) = \frac{1}{j} \sum_{k=1}^{i-1} \frac{\partial p_{j-1}^{(k)}}{\partial y}(t, y) R_{i-k}(y), \quad j = 2, 3, \cdots, i-1, \\ p_i^{(i)}(t, y) = p_i^{(i)}(y) = \frac{1}{i} \frac{\partial p_{i-1}^{(i-1)}}{\partial y}(y) R_1(y), \quad i = 2, 3, \cdots, \\ p_j^{(i)}(t, y) = 0, \quad j > i. \end{cases} \quad (4.27)$$

In particular, $p_j^{(i)}(t, y)$ are almost periodic functions with respect to t for all $i, j \in \mathbf{N}$.

Proof. This proposition is proved in Chiba [10]. ■

Now we derive the RG equation (2.18) according to an idea of Kunihiro [28,29]. We can show that regular perturbation solutions are also expressed as

$$x = \hat{x}(t, \tau, y) = y + \varepsilon x_1(t, \tau, y) + \varepsilon^2 x_2(t, \tau, y) + \cdots \quad (4.28)$$

with

$$x_k(t, \tau, y) = u_t^{(k)}(y) + p_1^{(k)}(t, y)(t - \tau) + p_2^{(k)}(t, y)(t - \tau)^2 + \cdots + p_k^{(k)}(t, y)(t - \tau)^k, \quad (4.29)$$

by choosing initial times and initial values appropriately when solving the system (4.25) so that a formal solution $\hat{x}(t, \tau, y)$ passes through $y + \sum_{k=1}^{\infty} \varepsilon^k u_{\tau}^{(k)}(y)$ at $t = \tau$, where $\tau \in \mathbf{R}$ is an arbitrary constant. Further, we regard $y = y(\tau)$ as a function of τ to be determined, and consider the family $\{\hat{x}(t, \tau, y(\tau))\}_{\tau \in \mathbf{R}}$ of regular perturbation solutions parameterized by $\tau \in \mathbf{R}$. It is known that regular perturbation solutions are close to exact solutions only for a short time interval $|t| \sim O(1)$. However, if we move the initial value $y + \sum_{k=1}^{\infty} \varepsilon^k u_{\tau}^{(k)}(y)$ along an exact solution by varying τ as Fig. 7, it seems that the envelope of the family $\{\hat{x}(t, \tau, y(\tau))\}_{\tau \in \mathbf{R}}$ gives the exact solution of Eq.(2.1).

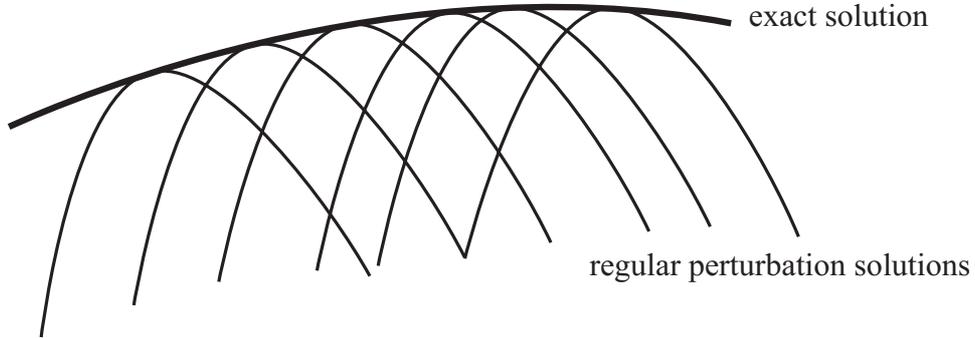


Fig. 7 Exact solution of Eq.(2.1) and a family of regular perturbation solutions $\hat{x}(t, \tau, y(\tau))$.

The envelope is calculated as follows: At first, we differentiate $\hat{x}(t, \tau, y(\tau))$ with respect to $\tau = t$ and equate it to zero :

$$\left. \frac{d}{d\tau} \right|_{\tau=t} \hat{x}(t, \tau, y(\tau)) = 0. \quad (4.30)$$

Substituting Eqs.(4.28) and (4.29) into the above yields

$$\begin{aligned}
0 &= \frac{dy}{dt} + \sum_{k=1}^{\infty} \varepsilon^k \frac{d}{d\tau} \Big|_{\tau=t} x_k(t, \tau, y(\tau)) \\
&= \frac{dy}{dt} + \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial u_t^{(k)}}{\partial y}(y) \frac{dy}{dt} - p_1^{(k)}(t, y) \right) \\
&= \frac{dy}{dt} + \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial u_t^{(k)}}{\partial y}(y) \frac{dy}{dt} - R_k(y) - \sum_{j=1}^{k-1} \frac{\partial u_t^{(j)}}{\partial y}(y) R_{k-j}(y) \right). \tag{4.31}
\end{aligned}$$

This equality is formally satisfied if we define $y = y(t)$ as a solution of the ODE

$$\frac{dy}{dt} = \sum_{k=1}^{\infty} \varepsilon^k R_k(y). \tag{4.32}$$

This recovers the m -th order RG equation if truncated at order ε^m . With this $y(t)$, the envelope for the family $\{\hat{x}(t, \tau, y(\tau))\}_{\tau \in \mathbf{R}}$ is given by $\hat{x}(t, t, y(t))$. Again Eqs.(4.28) and (4.29) prove that

$$\begin{aligned}
\hat{x}(t, t, y(t)) &= y(t) + \sum_{k=1}^{\infty} \varepsilon^k x_k(t, t, y(t)) \\
&= y(t) + \sum_{k=1}^{\infty} \varepsilon^k u_t^{(k)}(y(t)) \\
&= \alpha_t^{(m)}(y(t)) + O(\varepsilon^{m+1}), \tag{4.33}
\end{aligned}$$

which gives the m -th order RG transformation if truncated at order ε^m .

In the rest of this section, we prove that a formal solution (4.24) with Eq.(4.26) obtained by the regular perturbation method includes infinitely many convergent subseries even if Eq.(2.1) (and thus Eq.(4.24)) is not analytic in ε .

Proposition 4.3. For sufficiently small $|\varepsilon t|$, the series $\sum_{l=1}^{\infty} \varepsilon^l t^l p_l^{(l+k)}(t, y)$ are convergent for $k = 0, 1, 2, \dots$.

To prove Prop.4.3, we need the next simple lemma.

Lemma 4.4. Let $h_1(t)$ and $h_2(t)$ be almost periodic functions with $\text{Mod}(h_1) \supset \text{Mod}(h_2)$ (the module of almost periodic functions is defined in Sec.2.1). If

$$\lim_{t \rightarrow \infty} (h_1(t) - h_2(t)) = 0, \tag{4.34}$$

then $h_1(t) = h_2(t)$ for all $t \in \mathbf{R}$.

Proof of Lemma 4.4. Functions h_i , $i = 1, 2$ are expanded in Fourier series as $h_i(t) = \sum_{\lambda_k \in \text{Mod}(h_1)} c_i(\lambda_k) e^{i\lambda_k t}$. Then, Eq.(4.34) is rewritten as

$$\lim_{t \rightarrow \infty} \sum_{\lambda_k \in \text{Mod}(h_1)} (c_1(\lambda_k) - c_2(\lambda_k)) e^{i\lambda_k t} = 0. \quad (4.35)$$

This proves that $c_1(\lambda_k) = c_2(\lambda_k)$ for all $\lambda_k \in \text{Mod}(h_1)$. ■

Proof of Prop.4.3. Let

$$x(t) = y_0 + \sum_{l=1}^{\infty} \varepsilon^l T^l p_l^{(l)}(y_0) + \sum_{k=1}^{\infty} \varepsilon^k \left(u_t^{(k)}(y_0) + \sum_{l=1}^{\infty} \varepsilon^l T^l p_l^{(l+k)}(t, y_0) \right) \quad (4.36)$$

be a formal solution of Eq.(2.1) constructed by the regular perturbation method and let $y = y(\varepsilon t, \varepsilon)$ be a solution of the m -th order RG equation

$$\frac{dy}{d(\varepsilon t)} = R_1(y) + \varepsilon R_2(y) + \dots + \varepsilon^{m-1} R_m(y) \quad (4.37)$$

for Eq.(2.1). By Thm.2.7, there exist positive constants C and T such that the inequality

$$\|x(t) - \alpha_t^{(m)}(y(\varepsilon t, \varepsilon))\| < C|\varepsilon|^m, \quad 0 \leq t \leq T/\varepsilon \quad (4.38)$$

holds if we choose initial values of $x(t)$ and $y(\varepsilon t, \varepsilon)$ appropriately. Putting $t = T/\varepsilon$ yields

$$\begin{aligned} \alpha_{T/\varepsilon}^{(m)}(y(T, \varepsilon)) &= y(T, \varepsilon) + \sum_{k=1}^m \varepsilon^k u_{T/\varepsilon}^{(k)}(y(T, \varepsilon)) \\ &= y_0 + \sum_{l=1}^{\infty} T^l p_l^{(l)}(y_0) + \sum_{k=1}^{m-1} \varepsilon^k \left(u_{T/\varepsilon}^{(k)}(y_0) + \sum_{l=1}^{\infty} T^l p_l^{(l+k)}(T/\varepsilon, y_0) \right) + O(\varepsilon^m). \end{aligned} \quad (4.39)$$

Note that functions $u_t^{(k)}(y)$ and $p_l^{(l+k)}(t, y)$ are almost periodic functions in t such that $\text{Mod}(u_t^{(k)})$ and $\text{Mod}(p_l^{(l+k)})$ are included in $\text{Mod}(g)$. In particular, they are bounded in $t \in \mathbf{R}$. Thus taking the limit $\varepsilon \rightarrow 0$ in Eq.(4.39) provides

$$y(T, 0) = y_0 + \sum_{l=1}^{\infty} T^l p_l^{(l)}(y_0). \quad (4.40)$$

This proves that the series $\sum_{l=1}^{\infty} T^l p_l^{(l)}(y_0)$ is convergent.

Next thing to do is to prove Prop.4.3 for $k = 1$. Using Eq.(4.40) and dividing the both sides of Eq.(4.39) by ε , we obtain

$$\sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\partial^k y}{\partial \varepsilon^k}(T, 0) + \sum_{k=1}^m \varepsilon^{k-1} u_{T/\varepsilon}^{(k)}(y(T, \varepsilon)) = \sum_{k=1}^{m-1} \varepsilon^{k-1} \left(u_{T/\varepsilon}^{(k)}(y_0) + \sum_{l=1}^{\infty} T^l p_l^{(l+k)}(T/\varepsilon, y_0) \right) + O(\varepsilon^{m-1}). \quad (4.41)$$

Taking the limit $\varepsilon \rightarrow 0$ yields

$$\frac{\partial y}{\partial \varepsilon}(T, 0) + \lim_{\varepsilon \rightarrow 0} \left(u_{T/\varepsilon}^{(1)}(y(T, 0)) - u_{T/\varepsilon}^{(1)}(y_0) - \sum_{l=1}^{\infty} T^l p_l^{(l+1)}(T/\varepsilon, y_0) \right) = 0. \quad (4.42)$$

Now Lemma 4.4 proves that

$$\sum_{l=1}^{\infty} T^l p_l^{(l+1)}(t, y_0) = \frac{\partial y}{\partial \varepsilon}(T, 0) + u_t^{(1)}(y(T, 0)) - u_t^{(1)}(y_0), \quad (4.43)$$

and the series $\sum_{l=1}^{\infty} T^l p_l^{(l+1)}(t, y_0)$ proves to be convergent.

The proof of Prop.4.3 for $k \geq 2$ is done in a similar way and omitted. ■

Though it seems that Prop.4.3 has no relationship to the RG method, it is instructive to notice the equality (4.40). Since $y(T, 0)$ in the left hand side is obtained from a solution of the first order RG equation $dy/d(\varepsilon t) = R_1(y)$, the series $\sum_{l=1}^{\infty} T^l p_l^{(l)}(y_0)$ proves to be determined by only the first order RG equation. This is remarkable because if we want to obtain $\sum_{l=1}^{\infty} T^l p_l^{(l)}(y_0)$ by using the regular perturbation method, we have to solve infinitely many ODEs (4,25). A similar argument shows that a solution of the m -th order RG equation involves all terms of the form $\varepsilon^{k+l} T^l p_l^{(l+k)}(t, y_0)$ for $k = 0, \dots, m-1$ and $l = 1, 2, \dots$ in the formal solution (4.24).

5 Infinite order RG equation

Infinite order RG equations and transformations are not convergent series in general. In this section, we give a necessary and sufficient condition for the convergence. It will be proved in Sec.5.3 that infinite order RG equations for linear systems are convergent and related to Floquet theory.

5.1 Convergence condition

Let us consider the system

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \quad x \in M \quad (5.1)$$

on a real analytic manifold M with the following assumption :

(A') The vector field g is analytic with respect to $t \in \mathbf{R}$, $x \in M$ and $\varepsilon \in I$, where $I \subset \mathbf{R}$ is an open interval containing 0. Further g is T -periodic in t .

Because of the periodicity, we can regard (5.1) as a system on $S^1 \times M$:

$$\frac{d}{dt} \begin{pmatrix} s \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon g(s, x, \varepsilon) \end{pmatrix}, \quad (5.2)$$

where S^1 is a circle with a C^ω structure.

Theorem 2.5 states that the infinite order RG transformation defined by

$$x = \alpha_t(y) := y + \varepsilon u_t^{(1)}(y) + \varepsilon^2 u_t^{(2)}(y) + \dots \quad (5.3)$$

formally brings the system (5.1) into the infinite order RG equation

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \dots, \quad (5.4)$$

where ‘‘formally’’ means that Eqs.(5.3) and (5.4) are not convergent in general. If Eq.(5.3) is convergent, so is Eq.(5.4). A necessary and sufficient condition for the convergence of Eq.(5.3) is given as follows:

Theorem 5.1. For the system (5.1) with the assumption (A’), there exist an open neighborhood $U = U_y$ of $S^1 \times \{y\} \times \{0\}$ in $S^1 \times M \times I$ for each $y \in M$ and an analytic infinite order RG transformation on U , if and only if the system (5.1) is invariant under the \mathbb{T}^1 (1-torus) action of the form

$$\mathbb{T}^1 : (t, x) \mapsto (t + k, x + \varepsilon \sigma_k(t, x, \varepsilon)), \quad k \in \mathbf{R}, \quad (5.5)$$

where $\sigma_k(t, x, \varepsilon)$ is analytic with respect to k, t, x, ε and T -periodic in k and t .

Recall that RG equations and RG transformations are not unique and not all of them are convergent even if the condition of Thm.5.1 is satisfied.

The proof of this theorem involves Lie group theory and will be given in Sec.5.2, though the idea of the proof is shown below.

Since the infinite order RG equation is an autonomous system, it is invariant under the translation of t , $(t, y) \mapsto (t + k, y)$. If an infinite order RG equation and transformation are convergent and well-defined, the system (5.1) is invariant under the action defined by pulling back the translation by the RG transformation :

$$(t, x) \mapsto (t + k, \alpha_{t+k} \circ \alpha_t^{-1}(x)). \quad (5.6)$$

Since α_t is T -periodic (Lemma 2.1 (ii)), this defines the \mathbb{T}^1 action on the space $S^1 \times M$. Conversely, if the system (5.1) is invariant under the action (5.5), then a simple extension of Bochner’s linearization theorem proves that there exists a C^ω coordinates transformation $x \mapsto y$ such that the action (5.5) is written as $(t, y) \mapsto (t + k, y)$. We can show that this

transformation is just an RG transformation. See Section 5.2 for the detail.

In the rest of this subsection, we consider an autonomous system on \mathbf{C}^n of the form

$$\begin{aligned}\dot{x} &= Fx + \varepsilon g(x, \varepsilon) \\ &= Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \quad x \in \mathbf{C}^n,\end{aligned}\tag{5.7}$$

with the assumptions (C1),(C2) (see Sec.2.1) and (C3) below :

(C3) $g(x, \varepsilon)$ is analytic with respect to $x \in \mathbf{C}^n$ and $\varepsilon \in I \subset \mathbf{R}$.

For this system, RG transformations are defined by Eq.(2.51) with Eqs.(2.46) to (2.49). The next corollary immediately follows from Thm.5.1.

Corollary 5.2. Suppose that all eigenvalues of F have pairwise rational ratios. Then, there exist an open neighborhood $U = U_y$ of $S^1 \times \{y\} \times \{0\}$ in $S^1 \times M \times I$ for each $y \in M$ and an analytic infinite order RG transformation on U , if and only if the system (5.7) is invariant under the \mathbb{T}^1 action of the form

$$\mathbb{T}^1 : x \mapsto e^{Fk}x + \varepsilon \sigma_k(t, x, \varepsilon), \quad k \in \mathbf{R},\tag{5.8}$$

where $\sigma_k(t, x, \varepsilon)$ is analytic with respect to k, t, x, ε and periodic in k and t .

Proof. By changing the coordinates as $x = e^{Ft}X$, Eq.(5.7) is rewritten as $\dot{X} = \varepsilon e^{-Ft}g(e^{Ft}X, \varepsilon)$. Since e^{Ft} is periodic because of the assumption of Corollary 5.2, we can apply Thm.5.1 to this system. Note that we do not need the assumption (C2) to prove Corollary 5.2. ■

Recall that RG equations for Eq.(5.7) are equivalent to normal forms (Sec.4.3). If there are irrational ratios among eigenvalues of F , Thm.5.1 is no longer applicable. For such a system, Zung [54] gives a necessary and sufficient condition for the convergence of normal forms of infinite order, although he supposes that $g_i(x)$ in Eq.(5.7) is a homogeneous vector field of degree i for $i = 2, 3, \cdots$. A necessary and sufficient condition for the convergence of infinite RG equations and transformations for Eq.(5.7) is given in a similar way to Zung's theorem as follows:

Remember that an RG equation for Eq.(5.7) has the property that $R_i(e^{Fk}y) = e^{Fk}R_i(y)$, $k \in \mathbf{R}$ for $i = 1, 2, \cdots$, if integral constants in Eqs.(2.47) and (2.49) are appropriately chosen (Thm.2.15). If a matrix B satisfies the equalities $FB = BF$ and $Q(e^{Bk}y) = e^{Bk}Q(y)$, $k \in \mathbf{R}$ for all polynomial vector fields Q such that $Q(e^{Fk}y) = e^{Fk}Q(y)$, $k \in \mathbf{R}$, then B is called *subordinate* to F . It is known that if F is a diagonal matrix, we can take B having the form

$B = \text{diag}(ib_1, \dots, ib_n)$, where $i = \sqrt{-1}$ and $b_j \in \mathbf{Z}$ for $j = 1, \dots, n$ (see Murdock [39], Zung [54]). Let p be the maximum number of linearly independent such matrices B_1, \dots, B_p and call it the *toric degree of F* . Then, the matrix $e^{B_1 k_1 + \dots + B_p k_p}$, $(k_1, \dots, k_p) \in \mathbf{R}^p$ induces the \mathbb{T}^p action on \mathbf{C}^n and the RG equation is invariant under this action. By a similar way to the proof of Thm.5.1, we can prove the next theorem, whose proof is omitted here.

Theorem 5.3. Let p be the toric degree of F . For the system (5.7), there exist an open neighborhood $U = U_y$ of $S^1 \times \{y\} \times \{0\}$ in $S^1 \times M \times I$ for each $y \in M$ and an analytic infinite order RG transformation on U , if and only if (5.7) is invariant under the \mathbb{T}^p action of the form

$$\mathbb{T}^p : x \mapsto e^{B_1 k_1 + \dots + B_p k_p} x + \varepsilon \sigma_{k_1, \dots, k_p}(t, x, \varepsilon), \quad (k_1, \dots, k_p) \in \mathbf{R}^p, \quad (5.9)$$

where $\sigma_{k_1, \dots, k_p}(t, x, \varepsilon)$ is analytic with respect to $k_1, \dots, k_p, t, x, \varepsilon$ and periodic in k_1, \dots, k_p and t .

5.2 Proof of Theorem 5.1

In this subsection, we give a proof of Thm.5.1. At first, we provide a few notations and facts from Lie group theory.

Let K be a compact Lie group, dk a bi-invariant measure on K , V a complete locally convex topological vector space, and π a representation of K in V . We define the *average* $\text{av}(\pi) : V \rightarrow V$ of π by

$$\text{av}(\pi)(v) = \int_K \pi(k)(v) dk. \quad (5.10)$$

Then, the next theorem holds (see Duistermaat and Kolk [15] for the proof).

Theorem 5.4. The average $\text{av}(\pi)$ is a linear projection from V onto the space $V^{\pi(K)} := \{v \in V \mid \pi(k)v = v, \forall k \in K\}$; that is, equalities

$$\pi(k) \circ \text{av}(\pi)(v) = \text{av}(\pi)(v) \quad (5.11)$$

and

$$\text{av}(\pi)(v) = v, \quad v \in V^{\pi(K)} \quad (5.12)$$

hold.

Let S^1 be a circle with a C^ω structure, M a real analytic manifold, and $I \subset \mathbf{R}$ an open interval containing 0, as Sec.5.1. The next theorem is a simple extension of Bochner's linearization theorem [5,15].

Theorem 5.5. Let $A : K \rightarrow \text{Diff}^\omega(S^1 \times M \times I)$ be a C^ω action of K on $S^1 \times M \times I$ such that $A(k)$ is expressed as

$$A(k)(t, x, \varepsilon) = (a_1(k)(t), a_2(k)(t, x, \varepsilon), \varepsilon), \quad k \in K, \quad (5.13)$$

where $a_1(k) : S^1 \rightarrow S^1$ and $a_2(k) : S^1 \times M \times I \rightarrow M$ are C^ω maps. Suppose that there exists $x_0 \in M$ such that

$$a_2(k)(t, x_0, 0) = x_0. \quad (5.14)$$

Then, there exist an open neighborhood U of $S^1 \times \{x_0\} \times \{0\}$ in $S^1 \times M \times I$ and a C^ω diffeomorphism $\bar{\varphi}$ from U into $S^1 \times T_{x_0}M \times \mathbf{R}$ such that

$$\bar{\varphi} \circ A(k) = (a_1(k) \times D_x a_2(k)(t, x_0, 0) \times id) \circ \bar{\varphi}, \quad (5.15)$$

where D_x is the derivative with respect to x . The $\bar{\varphi}$ is expressed as

$$\bar{\varphi}(t, x, \varepsilon) = (t, a_3(t, x, \varepsilon), \varepsilon) \quad (5.16)$$

with a C^ω map a_3 and satisfies

$$\bar{\varphi}(t, x_0, 0) = (t, 0, 0), \quad D_x \bar{\varphi}(t, x_0, 0) = id. \quad (5.17)$$

Further if we suppose

$$D_x^n a_2(k)(t, x_0, 0) = 0 \quad (n \geq 2), \quad (5.18)$$

then $\bar{\varphi}$ satisfies

$$D_x^n \bar{\varphi}(t, x_0, 0) = 0 \quad (n \geq 2). \quad (5.19)$$

Proof. Let W be a K -invariant open neighborhood of $S^1 \times \{x_0\} \times \{0\}$ and V a space of C^ω maps $v : W \rightarrow S^1 \times T_{x_0}M \times \mathbf{R}$ such that

$$v(S^1 \times \{x_0\} \times \{0\}) = S^1 \times \{x_0\} \times \{0\}. \quad (5.20)$$

Then V is a complete locally convex topological vector space.

Take a $\varphi \in V$ expressed as

$$\varphi(t, x, \varepsilon) = (t, \beta(t, x, \varepsilon), \varepsilon) \quad (5.21)$$

satisfying

$$D_x \beta(t, x_0, 0) = id, \quad D_x^n \beta(t, x_0, 0) = 0 \quad (n \geq 2), \quad (5.22)$$

where β is a C^ω map. Define the representation π of K in V to be

$$\pi(k)(v) = (a_1(k) \times D_x a_2(k)(t, x_0, 0) \times id) \circ v \circ A(k)^{-1}, \quad (5.23)$$

and define $\bar{\varphi} \in V$ to be

$$\bar{\varphi} = \text{av}(\pi)(\varphi) = \int_K (a_1(k) \times D_x a_2(k)(t, x_0, 0) \times id) \circ \varphi \circ A(k)^{-1} dk. \quad (5.24)$$

Then Thm.5.4 implies that $\bar{\varphi} = \pi(k)(\bar{\varphi})$ and this proves Eq.(5.15). Eqs.(5.16),(5.17) and (5.19) immediately follow from the definition (5.24) of $\bar{\varphi}$ with Eqs.(5.14),(5.18),(5.20),(5.21) and (5.22). Since $D\bar{\varphi}(t, x_0, 0) = id$, $\bar{\varphi}$ is a C^ω diffeomorphism on a neighborhood of $S^1 \times \{x_0\} \times \{0\}$ by virtue of the inverse mapping theorem. \blacksquare

Proof of Thm.5.1. Suppose that an infinite order RG transformation $\alpha_t(y)$ is analytic in U . Then, so is an infinite order RG equation. Since the RG equation is invariant under the translation of t , Eq.(5.1) is invariant under the action defined by pulling back the translation by the RG transformation:

$$\begin{aligned} (t, x) &\mapsto (t + k, \alpha_{t+k} \circ \alpha_t^{-1}(x)) \\ &= (t + k, x + \varepsilon(u_{t+k}^{(1)}(x) - u_t^{(1)}(x)) + O(\varepsilon^2)), \quad k \in \mathbf{R}. \end{aligned} \quad (5.25)$$

Since α_t is T -periodic, this defines the \mathbb{T}^1 action on $S^1 \times M$ of the form of (5.5).

Conversely, suppose that Eq.(5.1) is invariant under the \mathbb{T}^1 action (5.5). Let us rewrite Eq.(5.1) as

$$\frac{d}{dt} \begin{pmatrix} s \\ x \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon g(s, x, \varepsilon) \\ 0 \end{pmatrix}. \quad (5.26)$$

Then, the action (5.5) induces the action on $S^1 \times M \times I$ of the form

$$A(k) : (s, x, \varepsilon) \mapsto (s + k, a_2(k)(s, x, \varepsilon), \varepsilon), \quad k \in \mathbf{R}, \quad (5.27)$$

where $a_2(k)(s, x, \varepsilon) = x + \varepsilon \sigma_k(s, x, \varepsilon)$. Since $a_2(k)$ satisfies Eqs.(5.14) and (5.18) for any $x \in M$, Thm.5.5 applies to show that there exist an open neighborhood $U = U_{x_0}$ of $S^1 \times \{x_0\} \times \{0\}$ and a C^ω diffeomorphism $\bar{\varphi}$ satisfying Eqs.(5.15) to (5.17) and (5.19) for each $x_0 \in M$.

By taking a local coordinate near x_0 , we put $x_0 = 0$ and identify a neighborhood of x_0 with a neighborhood of 0 in $T_{x_0}M$. Then, Eqs.(5.16), (5.17) and (5.19) prove that $\bar{\varphi}$ is expressed as

$$\bar{\varphi}(t, x, \varepsilon) = (t, x + \varepsilon \hat{a}_3(t, x, \varepsilon), \varepsilon), \quad (5.28)$$

where \hat{a}_3 is a C^ω map. Thus $\bar{\varphi}$ defines the C^ω transformation ψ_t by

$$x = \psi_t(y) = y + \varepsilon \hat{a}_3(t, y, \varepsilon). \quad (5.29)$$

Now Eq.(5.15) proves that if we transform Eq.(5.1) by $x = \psi_t(y, \varepsilon)$, then the resultant equation is invariant under the action

$$a_1(k) \times D_x a_2(k)(t, x_0, 0) \times id : (t, x, \varepsilon) \mapsto (t + k, x, \varepsilon). \quad (5.30)$$

Since this is the translation of t , the resultant equation has to be an autonomous system of the form

$$\frac{dy}{dt} = \varepsilon \mathcal{R}(y, \varepsilon), \quad (5.31)$$

where \mathcal{R} is analytic in y and ε .

Finally, we show that there exist an RG transformation and an RG equation which coincide with Eqs.(5.29) and (5.31), respectively. Let Eqs.(5.3) and (5.4) be one of the RG transformations and the RG equations for Eq.(5.1), respectively. Since both of Eqs.(5.31) and (5.4) are obtained from Eq.(5.1) by C^∞ transformations ψ_t and α_t , respectively, there exists an C^∞ transformation ϕ which brings Eq.(5.31) into Eq.(5.4):

$$\alpha_t \circ \phi(y, \varepsilon) = \psi_t(y, \varepsilon). \quad (5.32)$$

Because of Claim 2.16, there exists an RG transformation $\tilde{\alpha}_t$ other than α_t such that $\tilde{\alpha}_t = \alpha_t \circ \phi$. This proves that there exists an analytic RG transformation $\tilde{\alpha}_t = \psi_t$. ■

5.3 Infinite order RG equation for linear systems

In this subsection, we consider an n -dimensional linear system of the form

$$\begin{aligned} \dot{x} &= \varepsilon A(t, \varepsilon)x, \\ &= \varepsilon A_1(t)x + \varepsilon^2 A_2(t)x + \cdots, \quad x \in \mathbf{C}^n, \end{aligned} \quad (5.33)$$

where $A(t, \varepsilon)$ is an $n \times n$ matrix. If this system satisfies the assumption (A), the RG method is applicable. Since the RG equation and the RG transformation are linear in $y \in \mathbf{C}^n$, it is convenient to define matrices R_i and $u_t^{(i)}$ to be

$$R_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \int^t A_1(s) ds, \quad (5.34)$$

$$u_t^{(1)} = \int^t (A_1(s) - R_1) ds, \quad (5.35)$$

and

$$R_i = \lim_{t \rightarrow \infty} \frac{1}{t} \int \left(\sum_{k=1}^{i-1} A_{i-k}(s) u_s^{(k)} + A_i(s) - \sum_{k=1}^{i-1} u_s^{(k)} R_{i-k} \right) ds, \quad (5.36)$$

$$u_t^{(i)} = \int \left(\sum_{k=1}^{i-1} A_{i-k}(s) u_s^{(k)} + A_i(s) - \sum_{k=1}^{i-1} u_s^{(k)} R_{i-k} - R_i \right) ds, \quad (5.37)$$

for $i = 2, 3, \dots$, respectively. With these matrices, the m -th order RG equation and the m -th order RG transformation for Eq.(5.33) are defined by

$$\dot{y} = \varepsilon R_1 y + \varepsilon^2 R_2 y + \dots + \varepsilon^m R_m y, \quad y \in \mathbf{C}^n, \quad (5.38)$$

$$\alpha_t^{(m)} y = y + \varepsilon u_t^{(1)} y + \dots + \varepsilon^m u_t^{(m)} y, \quad y \in \mathbf{C}^n, \quad (5.39)$$

respectively. Since the RG equation is a linear system with a constant coefficient, we can easily determine the stability of the trivial solution $x = 0$ of Eq.(5.33) by using the RG equation. See Chiba [11] for the detail.

In what follows, we suppose the following assumption :

(L) The matrix $A(t, \varepsilon)$ is T -periodic in t and analytic with respect to $\varepsilon \in D$, where $D \subset \mathbf{C}$ is an open neighborhood of the origin.

Let us consider the infinite order RG equation $\dot{y} = \sum_{k=1}^{\infty} \varepsilon^k R_k y := \mathcal{R}(\varepsilon) y$ and the infinite order RG transformation $\alpha_t y = y + \sum_{k=1}^{\infty} \varepsilon^k u_t^{(k)} y$. Then, the fundamental matrix $X(t, \varepsilon)$ of Eq.(5.33) is given by

$$X(t, \varepsilon) = \alpha_t \cdot e^{\mathcal{R}(\varepsilon)t}. \quad (5.40)$$

An initial value is given by $X(0, \varepsilon) = \alpha_0 = id + \sum_{k=1}^{\infty} \varepsilon^k u_0^{(k)}$ and it can be taken arbitrarily by choosing integral constants in Eqs.(5.35) and (5.37) appropriately. We suppose that the integral constants are chosen so that an initial value α_0 is analytic in $\varepsilon \in D$. Then, $X(t, \varepsilon)$ is analytic in $\varepsilon \in D$. Now the question arises whether α_t and $\mathcal{R}(\varepsilon)$ are analytic in ε .

Theorem 5.6. Suppose that Eq.(5.33) satisfies the assumption (L) and α_0 is chosen to be analytic in $\varepsilon \in D$. Then, there exists a positive number r_0 such that $\mathcal{R}(\varepsilon)$, $e^{\mathcal{R}(\varepsilon)t}$ and α_t are analytic on the disk $|\varepsilon| < r_0$.

Proof. Since α_t is T -periodic (Lemma 2.1 (ii)), the equality

$$\begin{aligned} X(t+T, \varepsilon) &= \alpha_{t+T} \cdot e^{\mathcal{R}(\varepsilon)(t+T)} \\ &= \alpha_t \cdot e^{\mathcal{R}(\varepsilon)t} \cdot e^{\mathcal{R}(\varepsilon)T} \\ &= X(t, \varepsilon) \cdot e^{\mathcal{R}(\varepsilon)T} \end{aligned} \quad (5.41)$$

holds. Putting $t = 0$ yields

$$e^{\mathcal{R}(\varepsilon)T} = X(0, \varepsilon)^{-1} \cdot X(T, \varepsilon). \quad (5.42)$$

Since $X(t, \varepsilon)$ is nonsingular and analytic in $\varepsilon \in D$ for all $t \in \mathbf{R}$, $e^{\mathcal{R}(\varepsilon)T}$ is also analytic in $\varepsilon \in D$. Since $\det e^{\mathcal{R}(\varepsilon)T} \neq 0$, the theory of analytic matrix-functions concludes that $\mathcal{R}(\varepsilon)T$ is analytic on some disk $\{\varepsilon \in \mathbf{C} \mid |\varepsilon| < r_0\} \subset D$ (see Yakubovich and Starzhinskii [52], Erugin [17]). This proves that $\mathcal{R}(\varepsilon)$, $e^{\mathcal{R}(\varepsilon)t}$ and $\alpha_t = X(t, \varepsilon)e^{-\mathcal{R}(\varepsilon)t}$ are also analytic on the disk. ■

A few remarks are in order. Thm.5.6 with Thm.5.1 shows that Eq.(5.33) with the assumption (L) admits a Lie group action other than the scalar multiple : $x \mapsto kx$, $k \in \mathbf{C}$.

In general, analyticity of $e^{\mathcal{R}(\varepsilon)T}$ on D does not conclude analyticity of $e^{\mathcal{R}(\varepsilon)t}$ on D for all $t \in \mathbf{R}$. For example, consider the case $\mathcal{R}(\varepsilon) = \log(1 + \varepsilon)$, $T = 1$.

The convergence radius r_0 of $\mathcal{R}(\varepsilon)$ is given as follows: Fix $\varepsilon_0 \in D$ and a pass l in D from the origin to ε_0 . Let $\lambda_1(\varepsilon), \dots, \lambda_n(\varepsilon)$ be eigenvalues of $e^{\mathcal{R}(\varepsilon)T}$. Suppose that there are $i \neq j$ such that $\lambda_i(\varepsilon_0) = \lambda_j(\varepsilon_0)$ and passes of $\lambda_i(\varepsilon)$ and $\lambda_j(\varepsilon)$ along l surround the origin (see Fig.8). Then, $\log \lambda_i(\varepsilon_0)$ and $\log \lambda_j(\varepsilon_0)$ are located in different sheets of the Riemann surface. The smallest absolute value $r_0 = |\varepsilon_0|$ among such ε_0 's gives the convergence radius.

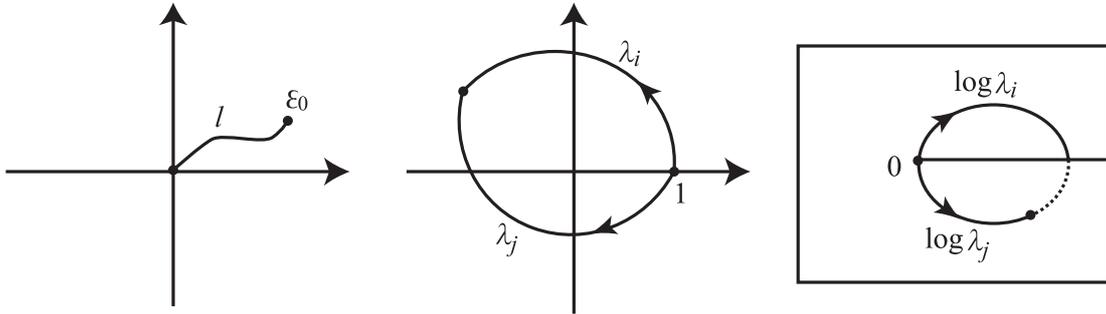


Fig. 8 Passes of eigenvalues λ_i, λ_j on \mathbf{C} and passes of $\log \lambda_i, \log \lambda_j$ on the Riemann surface.

Floquet theorem states that for a given linear system with a periodic coefficient, there exist a periodic matrix $Q(t)$ and a constant matrix B such that the fundamental matrix $X(t)$ is written as $X(t) = Q(t)e^{Bt}$. The RG method just gives the matrix B ; $B = \mathcal{R}(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k R_k$. Since $Q(t) = \alpha_t$ is periodic, the stability of the trivial solution $x = 0$ is determined by eigenvalues of $B = \mathcal{R}(\varepsilon)$, called *Floquet exponents*.

Let $T = 2\pi$ for simplicity. The matrix $A(t, \varepsilon)$ is expanded in a Fourier series as $A(t, \varepsilon) =$

$\sum_{-\infty}^{+\infty} c_n(\varepsilon)e^{int}$. By changing the independent variables as $z = e^{int}$, Eq.(5.33) is written as

$$\frac{dx}{dz} = -i\varepsilon \sum_{-\infty}^{+\infty} c_n(\varepsilon)z^{n-1}x. \quad (5.43)$$

This is a linear system on a complex domain having the singularity at the origin. It is known that the fundamental matrix of this system is expressed as

$$X(z) = S(z) \cdot e^{M \cdot \log z}, \quad (5.44)$$

where $S(z)$ is a single-valued matrix and $M = M(\varepsilon)$ is a constant matrix called *monodromy matrix*. It is easy to verify that $\mathcal{R}(\varepsilon) = iM$ and the RG method provides the monodromy matrix.

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