Lie equations for asymptotic solutions of perturbation problems of ordinary differential equations

Department of Applied Mathematics and Physics
Kyoto University, Kyoto, 606-8501, Japan
Hayato CHIBA [1]
Department of Physics,
Nagoya University, Nagoya, 464-8602, Japan
Masatomo IWASA [2]

January 19 2009; Revised February 11 2009

Abstract

Lie theory is applied to perturbation problems of ordinary differential equations to construct approximate solutions and invariant manifolds according to Iwasa and Nozaki's RG approach [Iwasa, Nozaki, Progr. Theoret. Phys. 116 (2006)]. It is proved that asymptotic behavior of solutions are obtained from the Lie equations even if original equations have no symmetries. Normal forms of the Lie equations are introduced to investigate existence of invariant manifolds.

1 Introduction

Methods for studying differential equations by means of symmetries have been well developed and known as Lie theory. If a differential equation is invariant under the action of a Lie group, a family of solutions are obtained from a special solution [12,15,16]. However, if one's purpose is to construct approximate solutions not exact solutions, a given equation need not to be exactly invariant under the action of a Lie group.

Baikov, Gazizov and Ibragimov [18] introduced approximate symmetries to obtain approximate solutions of differential equations. Cicogna and Gaeta [10] investigated the relation between approximate symmetries and normal forms of vector fields. Gaeta *et al.* [11,12] also proposed asymptotic symmetries which provide asymptotic behaviors of solutions. Iwasa and Nozaki [13] proposed the Lie equation to construct group invariant solutions of a perturbation problem of the form $dx/dt = f(x) + \varepsilon g(x)$, where ε is a small parameter. What is remarkable in their paper is that the parameter ε is also moved by an approximate Lie group action to obtain approximate solutions of the problem from an exact solution of the unperturbed problem $dx/dt = f(x)$. Though many perturbation techniques

[1]E mail address : chiba@amp.i.kyoto-u.ac.jp
[2]E mail address : miwasa@allegro.phys.nagoya-u.ac.jp

for ordinary differential equations have been developed so far [7], a main advantage of methods based on Lie theory is that they are easily extended to methods for partial differential equations [11,12,18] and difference equations [14].

The purpose of this article is to give the mathematical basis of Iwasa and Nozaki's method [13,20]. The Lie equation is defined by means of Lie theory and it is proved that a solution of the Lie equation approximates an exact solution of a given equation of the form $dx/dt = f(x) + \varepsilon g(x, \varepsilon)$. A normal form of the Lie equation is also introduced to transform the Lie equation into a simple form. If the unperturbed term $f(x)$ is linear, it turns out that a normal form of the Lie equation can be calculated systematically. Further, it will be proved that a normal form of the Lie equation provides invariant manifolds of a given equation as well as approximate solutions under appropriate assumptions. While the theory of normal forms for vector fields have been well developed [9, 19, 21], a normal form of the Lie equation provides a new approach to perturbation problems because the independent variable of the Lie equation is ε, not time t.

A few papers called methods based on Lie theory the renormalization group (RG) method [13,15] because of some analogy with the RG method in quantum field theory. We avoid using such a terminology because it may be confused with the Chen Goldenfeld and Oono's RG method (CGO RG method) [2,3,4,7], which is also one of the perturbation methods for differential equations. Nevertheless some relation between our method and the CGO RG method is shown in Section 3.4. In this article, Iwasa and Nozaki's method is called the *perturbative Lie theory*.

This article is organized as follows: In Sec.2, we demonstrate our idea on the perturbative Lie theory. The Lie equation is defined and it will be proved that it provides approximate solutions for a given equation. In Sec.3, we consider perturbed linear systems. A normal form of the Lie equation will be introduced and a main theorem of this article on existence of invariant manifolds will be proved through the CGO RG method. Sec.4 presents a few examples.

2 Perturbative Lie theory

In this section, the perturbative Lie theory is developed according to Iwasa and Nozaki [13].

Let us consider a system of differential equations on $\mathbf{R}^n$ of the form

$$\frac{dx}{dt} = \dot{x} = f(x) + \varepsilon g(x, \varepsilon), \quad x \in \mathbf{R}^n, \tag{2.1}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter and f and g are C^∞ vector fields on $\mathbf{R}^n$. We suppose that $g(x, \varepsilon)$ is C^∞ in ε and expanded in a formal Taylor series as

$$\dot{x} = f(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \varepsilon^3 g_3(x) + \cdots . \tag{2.2}$$

Let φ_t be the flow of the unperturbed system $\dot{x} = f(x)$; that is, $\varphi_t(x_0)$ is a solution of the system $\dot{x} = f(x)$ through x_0 at $t = 0$. For Eq.(2.1), we consider the associated system on

$\mathbf{R}^n \times \mathbf{R}$ of the form

$$\begin{cases} \dot{x} = f(x) + \xi g(x, \xi), \\ \dot{\xi} = 0. \end{cases} \tag{2.3}$$

Note that special solutions of Eq.(2.3) satisfying $\xi = 0$ and $\xi = \varepsilon$ are, respectively, solutions of the unperturbed system and Eq.(2.1).

Suppose that a one-parameter group $H = \{h_\tau \,|\, \tau \in \mathbf{R}\}$ acts on the (t, x, ξ) space as

$$h_\tau(t, x, \xi) = (v_\tau(t, x, \xi),\ u_\tau(t, x, \xi),\ \xi + \tau), \quad \tau \in \mathbf{R}, \tag{2.4}$$

where $v_\tau : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $u_\tau : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ are C^∞ maps. Let

$$X = \frac{\partial}{\partial \xi} + \psi(t, x, \xi)\frac{\partial}{\partial t} + \sum_{j=1}^{n} \phi_j(t, x, \xi)\frac{\partial}{\partial x_j}, \quad x = (x_1, \cdots, x_n) \tag{2.5}$$

be the infinitesimal generator of the action of H. Then, $v = v_\tau(t, x, \xi)$ and $u = u_\tau(t, x, \xi)$ are solutions of the initial value problem

$$\frac{d}{d\tau}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \psi(v, u, \xi + \tau) \\ \phi(v, u, \xi + \tau) \end{pmatrix}, \quad v|_{\tau=0} = t,\ u|_{\tau=0} = x, \tag{2.6}$$

where $\phi = (\phi_1, \cdots, \phi_n)$.

Now we suppose that Eq.(2.3) is invariant under the action of H (this assumption will be removed later). Then, the action h_τ transforms a solution of Eq.(2.3) into another solution of Eq.(2.3) (see Olver [16]). In particular, the graph $(t, \varphi_t(x_0), 0)$ of a solution of Eq.(2.3) is transformed as

$$h_\tau(t, \varphi_t(x_0), 0) = (v_\tau(t, \varphi_t(x_0), 0),\ u_\tau(t, \varphi_t(x_0), 0),\ \tau). \tag{2.7}$$

Putting $\tau = \varepsilon$ yields

$$h_\varepsilon(t, \varphi_t(x_0), 0) = (v_\varepsilon(t, \varphi_t(x_0), 0),\ u_\varepsilon(t, \varphi_t(x_0), 0),\ \varepsilon). \tag{2.8}$$

Since the right hand side of Eq.(2.8) is a graph of a solution of Eq.(2.1), we can obtain a solution of Eq.(2.1) if we know v_ε, u_ε and φ_t. Since $v = v_\varepsilon(t, x, 0)$ and $u = u_\varepsilon(t, x, 0)$ satisfy the initial value problem

$$\frac{d}{d\varepsilon}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \psi(v, u, \varepsilon) \\ \phi(v, u, \varepsilon) \end{pmatrix}, \quad v|_{\varepsilon=0} = t,\ u|_{\varepsilon=0} = x, \tag{2.9}$$

we expect that qualitative properties of the solution (2.8) are obtained from those of the vector field (ψ, ϕ) on the (v, u) space. The system (2.9) is called the *Lie equation*.

In what follows, we consider constructing ψ and ϕ. According to Eq.(2.5), the first prolongation of the infinitesimal generator X becomes

$$P^{(1)}X = \frac{\partial}{\partial \xi} + \psi(t, x, \varepsilon)\frac{\partial}{\partial t} + \sum_{j=1}^{n} \phi_j(t, x, \xi)\frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \phi_j^{(1)}(t, x, \dot{x}, \xi)\frac{\partial}{\partial \dot{x}_j}, \tag{2.10}$$

where $\phi_j^{(1)}$ is defined as

$$\phi_j^{(1)}(t, x, \dot{x}, \xi) = \frac{\partial \phi_j}{\partial t} + \sum_{k=1}^{n} \frac{\partial \phi_j}{\partial x_k}\dot{x}_k - \left(\frac{\partial \psi}{\partial t} + \sum_{k=1}^{n} \frac{\partial \psi}{\partial x_k}\dot{x}_k\right)\dot{x}_j. \tag{2.11}$$

Then, Eq.(2.3) is invariant under the action of H if and only if it satisfies

$$P^{(1)}X(\dot{x} - f(x) - \xi g(x, \xi))\Big|_{\dot{x}=f(x)+\xi g(x,\xi)} = 0, \tag{2.12}$$

see Olver [16] for the proof. This equality provides

$$\begin{aligned} &-g(x,\xi) - \xi\frac{\partial g}{\partial \xi}(x,\xi) - \left(\frac{\partial f}{\partial x}(x) + \xi\frac{\partial g}{\partial x}(x,\xi)\right)\phi(t,x,\xi) + \frac{\partial \phi}{\partial x}(t,x,\xi)(f(x) + \xi g(x,\xi)) \\ &+\frac{\partial \phi}{\partial t}(t,x,\xi) - \left(\frac{\partial \psi}{\partial t}(t,x,\xi) + \frac{\partial \psi}{\partial x}(t,x,\xi)(f(x) + \xi g(x,\xi))\right)(f(x) + \xi g(x,\xi)) = 0. \end{aligned} \tag{2.13}$$

Let us expand ψ and ϕ as

$$\psi(t, x, \xi) = \psi^{(0)}(t, x) + \xi\psi^{(1)}(t, x) + \xi^2\psi^{(2)}(t, x) + \cdots , \tag{2.14}$$
$$\phi(t, x, \xi) = \phi^{(0)}(t, x) + \xi\phi^{(1)}(t, x) + \xi^2\phi^{(2)}(t, x) + \cdots . \tag{2.15}$$

Substituting them into Eq.(2.13) and equating the coefficient of each ξ^k, we obtain the system of partial differential equations (PDEs) :

$$\left\{\begin{aligned} \frac{\partial \phi^{(0)}}{\partial t} &= \frac{\partial f}{\partial x}(x)\phi^{(0)} - \frac{\partial \phi^{(0)}}{\partial x}f(x) + g_1(x) + \left(\frac{\partial \psi^{(0)}}{\partial t} + \frac{\partial \psi^{(0)}}{\partial x}f(x)\right)f(x), \\ \frac{\partial \phi^{(1)}}{\partial t} &= \frac{\partial f}{\partial x}(x)\phi^{(1)} - \frac{\partial \phi^{(1)}}{\partial x}f(x) + \frac{\partial g_1}{\partial x}(x)\phi^{(0)} - \frac{\partial \phi^{(0)}}{\partial x}g_1(x) + 2g_2(x) \\ &\quad + \left(\frac{\partial \psi^{(1)}}{\partial t} + \frac{\partial \psi^{(1)}}{\partial x}f(x)\right)f(x) + \frac{\partial \psi^{(0)}}{\partial t}g_1(x) + \frac{\partial \psi^{(0)}}{\partial x}g_1(x)\cdot f(x) + \frac{\partial \psi^{(0)}}{\partial x}f(x)\cdot g_1(x), \\ &\ \vdots \end{aligned}\right. \tag{2.16}$$

This system can be solved with respect to $\phi^{(k)}$'s for arbitrarily given $\psi^{(0)}, \psi^{(1)}, \cdots$.

If $g(x, \xi)$ is C^ω with respect to ξ and invariant under the C^ω action of the group H, the series (2.15) obtained by solving (2.16) converges for appropriate choices of $\psi^{(0)}, \psi^{(1)}, \cdots$. However, we do not need such an assumption because our purpose in this article is to construct approximate solutions of Eq.(2.1). For this purpose, it is sufficient to calculate Eq.(2.15) up to some finite order of ξ and we need *not* assume that Eq.(2.3) is invariant under the action of H.

In what follows, we suppose that $\psi = 0$ for simplicity. This implies that $v_\tau(t, x, \xi) = t$ and the action of H does not change the time t. In this case, Eq.(2.13) provides a system of linear PDEs of $\phi^{(k)}$ of the form

$$\frac{\partial \phi^{(k)}}{\partial t} = \frac{\partial f}{\partial x}(x)\phi^{(k)} - \frac{\partial \phi^{(k)}}{\partial x}f(x) + H_k(t, x), \tag{2.17}$$

where H_k are defined to be

$$\begin{cases} H_0(t,x) = g_1(x), \\ H_1(t,x) = \dfrac{\partial g_1}{\partial x}(x)\phi^{(0)}(t,x) - \dfrac{\partial \phi^{(0)}}{\partial x}(t,x)g_1(x) + 2g_2(x), \\ \qquad \vdots \\ H_k(t,x) = \displaystyle\sum_{j=1}^{k}\left(\frac{\partial g_j}{\partial x}(x)\phi^{(k-j)}(t,x) - \frac{\partial \phi^{(k-j)}}{\partial x}(t,x)g_j(x)\right) + (k+1)g_{k+1}(x), \\ \qquad \vdots \end{cases} \tag{2.18}$$

A solution of Eq.(2.17) satisfying the initial condition $\phi^{(k)}(0,x) = h^{(k)}(x)$ is given by

$$\phi^{(k)}(t,x) = \left(\frac{\partial \varphi_{-t}}{\partial x}(x)\right)^{-1} h^{(k)}(\varphi_{-t}(x)) + \left(\frac{\partial \varphi_{-t}}{\partial x}(x)\right)^{-1}\int_0^t \left(\frac{\partial \varphi_s}{\partial x}(\varphi_{-t}(x))\right)^{-1} H_k(s, \varphi_{s-t}(x))ds, \tag{2.19}$$

where φ_t is the flow of the vector field f as was mentioned. With these $\phi^{(k)}$'s, we solve the Lie equation

$$\frac{\partial u}{\partial \varepsilon} = \phi^{(0)}(t,u) + \varepsilon\phi^{(1)}(t,u) + \cdots + \varepsilon^m \phi^{(m)}(t,u), \tag{2.20}$$

truncated at an ε^m-order term with the initial condition $u|_{\varepsilon=0} = \varphi_t(x_0)$. Then a solution $u = u(t,\varepsilon)$ gives an approximate solution of Eq.(2.1). We call Eq.(2.20) the *m-th order Lie equation*. Note that functions $h^{(k)}(x)$ in Eq.(2.19) are arbitrarily fixed to prove Thm.2.1 below. Such non-uniqueness of reduced systems in perturbation theory also arises in normal forms [19,21-23], RG equations [5] and other singular perturbation methods [24]. In the next section, $h^{(k)}$ will be chosen so that the right hand side of the Lie equation Eq.(2.20) becomes a polynomial in t when g_i's are polynomial vector fields.

Theorem 2.1. Let $u = u(t,\varepsilon)$ be a solution of the m-th order Lie equation (2.20) with $u(t,0) = \varphi_t(x_0)$ and $x(t)$ a solution of Eq.(2.1) with $x(0) = u(0,\varepsilon)$. Then, there exist positive numbers C and $T = T(\varepsilon)$ such that the inequality

$$||x(t) - u(t,\varepsilon)|| < C\varepsilon^{m+1} \tag{2.21}$$

holds for $0 \leq t \leq T(\varepsilon)$.

The function $T(\varepsilon)$ depends on problems but is larger than $O(1)$ in general. In Sec.3.4, we will show that if the unperturbed term $f(x)$ is linear and written as $f(x) = Ax$ with the matrix A all of whose eigenvalues lie on the imaginary axis, then $T(\varepsilon) \sim O(1/\varepsilon)$.

Proof. Eq.(2.20) is expressed as

$$\frac{\partial u}{\partial \varepsilon} = \phi(t,u,\varepsilon) + \varepsilon^{m+1} r(t,u,\varepsilon), \tag{2.22}$$

with some C^∞ function $r(t,u,\varepsilon)$. Differentiating the both sides of the above with respect to t, we obtain

$$\frac{\partial^2 u}{\partial \varepsilon \partial t} = \frac{\partial \phi}{\partial t}(t,u,\varepsilon) + \frac{\partial \phi}{\partial u}(t,u,\varepsilon)\frac{\partial u}{\partial t} + \varepsilon^{m+1}\frac{dr}{dt}(t,u(t,\varepsilon),\varepsilon). \tag{2.23}$$

Substituting Eq.(2.13) yields

$$\begin{aligned}\frac{\partial^2 u}{\partial\varepsilon\partial t} &= g(u,\varepsilon)+\varepsilon\frac{\partial g}{\partial\varepsilon}(u,\varepsilon)+\left(\frac{\partial f}{\partial u}(u)+\varepsilon\frac{\partial g}{\partial u}(u,\varepsilon)\right)\phi(t,u,\varepsilon)\\ &\quad -\frac{\partial\phi}{\partial u}(t,u,\varepsilon)\left(f(u)+\varepsilon g(u,\varepsilon)\right)+\frac{\partial\phi}{\partial u}(t,u,\varepsilon)\frac{\partial u}{\partial t}+\varepsilon^{m+1}\frac{dr}{dt}(t,u(t,\varepsilon),\varepsilon)\\ &= \frac{\partial}{\partial\varepsilon}\left(f(u(t,\varepsilon))+\varepsilon g(u(t,\varepsilon),\varepsilon)\right)+\frac{\partial\phi}{\partial u}(t,u,\varepsilon)\left(\frac{\partial u}{\partial t}-f(u)-\varepsilon g(u,\varepsilon)\right)\\ &\quad -\varepsilon^{m+1}\left(\frac{\partial f}{\partial u}(u)+\varepsilon\frac{\partial g}{\partial u}(u,\varepsilon)\right)r(t,u,\varepsilon)+\varepsilon^{m+1}\frac{dr}{dt}(t,u(t,\varepsilon),\varepsilon). \end{aligned} \tag{2.24}$$

Let us put

$$U(t,\varepsilon)=\frac{\partial u}{\partial t}(t,\varepsilon)-f(u(t,\varepsilon))-\varepsilon g(u(t,\varepsilon),\varepsilon). \tag{2.25}$$

Since $u(t,0)=\varphi_t(x_0)$, $U(t,0)=0$. Then Eq.(2.24) is rewritten as

$$\frac{\partial U}{\partial\varepsilon}=\frac{\partial\phi}{\partial u}(t,u,\varepsilon)U+\varepsilon^{m+1}\tilde{r}(t,\varepsilon), \tag{2.26}$$

where $\tilde{r}(t,\varepsilon)$ is a C^∞ function determined by the last two terms in Eq.(2.24). Let $X(t,\varepsilon)$ be the fundamental matrix for the linear system $\partial U/\partial\varepsilon=\partial\phi/\partial u\cdot U$ such that $X(t,0)=id$. Then, Eq.(2.26) is solved as

$$U(t,\varepsilon)=X(t,\varepsilon)\int_0^\varepsilon X(t,\eta)^{-1}\eta^{m+1}\tilde{r}(t,\eta)d\eta. \tag{2.27}$$

This proves that there exists a C^∞ function $w(t,\varepsilon)$ such that

$$U(t,\varepsilon)=\varepsilon^{m+2}w(t,\varepsilon). \tag{2.28}$$

Now we obtain the system

$$\frac{\partial u}{\partial t}=f(u)+\varepsilon g(u,\varepsilon)+\varepsilon^{m+2}w(t,\varepsilon). \tag{2.29}$$

By changing the coordinates as $x=\varphi_t(\hat{x})$ and $u=\varphi_t(\hat{u})$, Eqs.(2.1) and (2.29) are reduced to the systems

$$\frac{\partial\hat{x}}{\partial t}=\varepsilon\left(\frac{\partial\varphi_t}{\partial x}(\hat{x})\right)^{-1}g(\varphi_t(\hat{x}),\varepsilon), \tag{2.30}$$

and

$$\frac{\partial\hat{u}}{\partial t}=\varepsilon\left(\frac{\partial\varphi_t}{\partial x}(\hat{u})\right)^{-1}g(\varphi_t(\hat{u}),\varepsilon)+\varepsilon^{m+2}\left(\frac{\partial\varphi_t}{\partial x}(\hat{u})\right)^{-1}w(t,\varepsilon), \tag{2.31}$$

respectively. Let $L_1>0$ be an ε-independent Lipschitz constant of the function $(\partial\varphi_t(x)/\partial x)^{-1}g(\varphi_t(x),\varepsilon)$ on the domain $0\leq t\leq T_1(\varepsilon)$ and $x\in K$, where $K\subset\mathbf{R}^n$ is a sufficiently large compact subset. Let $L_2>0$ be an ε-independent constant such that $\|(\partial\varphi_t(x)/\partial x)^{-1}w(t,\varepsilon)\|<L_2$ for $0\leq t\leq T_2(\varepsilon)$. Then Eqs.(2.30) and (2.31) provide

$$\|\hat{x}(t)-\hat{u}(t,\varepsilon)\|<\varepsilon L_1\int_0^t\|\hat{x}(s)-\hat{u}(s,\varepsilon)\|ds+\varepsilon^{m+2}L_2 t, \tag{2.32}$$

for $0 \leq t \leq \min\{T_1, T_2\}$. Now the Gronwall lemma proves the inequality

$$||\hat{x}(t) - \hat{u}(t,\varepsilon)|| < \frac{L_2}{L_1}\varepsilon^{m+1}(e^{\varepsilon L_1 t} - 1), \tag{2.33}$$

which implies that $||\hat{x}(t) - \hat{u}(t,\varepsilon)|| < \hat{C}\varepsilon^{m+1}$ for $0 \leq t \leq \min\{T_1, T_2, 1/\varepsilon\}$ with some positive constant $\hat{C}$. Suppose that φ_t is bounded for $0 \leq t \leq T_3$. Then, we obtain the inequality $||x(t) - u(t,\varepsilon)|| < C\varepsilon^{m+1}$ for $0 \leq t \leq \min\{T_1, T_2, T_3, 1/\varepsilon\}$. This proves Theorem 2.1. ■

If the unperturbed system $\dot{x} = f(x)$ is nonlinear, to calculate Eq.(2.19) is difficult in general. In the next section, we consider the case that $f(x)$ is linear, which enables us to investigate properties of the Lie equation (2.20) in detail.

3 Lie equations for perturbed linear systems

In this section, we suppose that the unperturbed term in Eq.(2.1) is linear and written as $f(x) = Ax$, where A is an $n \times n$ constant matrix. In Sec.3.1, we introduce a decomposition of the space of C^∞ vector fields and $\mathcal{P}$-$\mathcal{Q}$ operators to simplify $\phi^{(k)}$ given in Eq.(2.19). In Sec.3.2, we calculate $\phi^{(0)}$ for the case that A is not diagonalizable and define a zeroth order normal form of the Lie equation. In Sec.3.3, we assume that A is a diagonal matrix. In this case, a normal form of the Lie equation up to all order will be obtained. In Sec 3.4, we investigate the special case that all eigenvalues of A lie on the imaginary axis. In this case, it will be proved that invariant manifolds of Eq.(2.1) are obtained from those of a normal form of the Lie equation.

3.1 Decomposition of the space of C^∞ vector fields

We consider a system of the form

$$\dot{x} = Ax + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \quad x \in \mathbf{R}^n, \tag{3.1}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter, A is an $n \times n$ constant matrix and $g_i(x)$, $i = 1, 2, \cdots$ are C^∞ vector fields. We assume that A is of the Jordan form for simplicity.

In this case, Eqs.(2.17) and (2.19) are written as

$$\frac{\partial \phi^{(k)}}{\partial t} = A\phi^{(k)} - \frac{\partial \phi^{(k)}}{\partial x}Ax + H_k(t,x), \tag{3.2}$$

$$\phi^{(k)}(t,x) = e^{At}h^{(k)}(e^{-At}x) + e^{At}\int_0^t e^{-As}H_k(s, e^{A(s-t)}x)ds, \tag{3.3}$$

respectively. In particular, $\phi^{(0)}$ is given by

$$\phi^{(0)}(t,x) = e^{At}h^{(0)}(e^{-At}x) + e^{At}\int_0^t e^{-As}g_1(e^{A(s-t)}x)ds. \tag{3.4}$$

Let us choose the undetermined function $h^{(0)}$ so that $\phi^{(0)}$ is polynomial in t when g_1 is a polynomial vector field. For this purpose, we define the operators $\mathcal{P}_I, \mathcal{P}_K$ and $\mathcal{Q}$ as follows:

Let $P_0(\mathbf{R}^n)$ be the set of polynomial vector fields on $\mathbf{R}^n$ whose degrees are equal to or larger than one. Define the linear map $\mathcal{L}_A$ on $P_0(\mathbf{R}^n)$ to be

$$\mathcal{L}_A(F)(x) = \frac{\partial F}{\partial x}(x)Ax - AF(x). \tag{3.5}$$

Then, the direct sum decomposition

$$P_0(\mathbf{R}^n) = \mathrm{Im}\,\mathcal{L}_A \oplus \mathrm{Ker}\,\mathcal{L}_{A^*} \tag{3.6}$$

holds, where A^* is the conjugate transpose of A. In particular if $A = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix, $\mathrm{Im}\,\mathcal{L}_A$ and $\mathrm{Ker}\,\mathcal{L}_{A^*}$ are given by

$$\begin{aligned} \mathrm{Im}\,\mathcal{L}_A &= \mathrm{span}\{\, x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n} \boldsymbol{e}_j \mid \sum_{k=1}^{n} \lambda_k q_k \neq \lambda_j \}, &(3.7)\\ \mathrm{Ker}\,\mathcal{L}_{A^*} &= \{F \in P_0(\mathbf{R}^n) \,|\, F(e^{At}x) = e^{At}F(x)\} \\ &= \mathrm{span}\{\, x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n} \boldsymbol{e}_j \mid \sum_{k=1}^{n} \lambda_k q_k = \lambda_j \}, &(3.8)\end{aligned}$$

respectively, where $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ are the canonical basis of $\mathbf{R}^n$ (see Chow, Li and Wang [9] for the proof). Here we note that the equality

$$\frac{\partial F}{\partial x}(x)Ax - AF(x) = 0 \tag{3.9}$$

is equivalent to the equality $F(e^{At}x) = e^{At}F(x)$.

By the completion, the direct sum decomposition (3.6) is extended to that of the set of C^∞ vector fields vanishing at the origin.

Proposition 3.1. Let $K \subset \mathbf{R}^n$ be an open set including the origin whose closure $\bar{K}$ is compact. Let $\mathcal{X}_0^\infty(K)$ be the set of C^∞ vector fields f on K satisfying $f(0) = 0$. Define the linear map $\mathcal{L}_A : \mathcal{X}_0^\infty(K) \to \mathcal{X}_0^\infty(K)$ as Eq.(3.5). Then, the direct sum decomposition

$$\mathcal{X}_0^\infty(K) = V_I \oplus V_K \tag{3.10}$$

holds, where

$$V_I := \mathrm{Im}\,\mathcal{L}_A, \tag{3.11}$$
$$V_K := \{f \in \mathcal{X}_0^\infty(K) \,|\, f(e^{A^*t}x) = e^{A^*t}f(x)\}. \tag{3.12}$$

This proposition immediately follows from the facts that the set of polynomials is dense in $\mathcal{X}_0^\infty(K)$ with respect to the C^∞ topology (see Hirsch [17]) and that the projections $\mathcal{P}_I : P_0(\mathbf{R}^n) \to \mathrm{Im}\,\mathcal{L}_A$ and $\mathcal{P}_K : P_0(\mathbf{R}^n) \to \mathrm{Ker}\,\mathcal{L}_{A^*}$ are continuous.

We define the projections $\mathcal{P}_I : \mathcal{X}_0^\infty(K) \to V_I$ and $\mathcal{P}_K : \mathcal{X}_0^\infty(K) \to V_K$. For $g \in V_I$, there exists a vector field $F \in \mathcal{X}_0^\infty(K)$ such that

$$\frac{\partial F}{\partial x}(x)Ax - AF(x) = g(x). \tag{3.13}$$

Such $F(x)$ is not unique because if F satisfies the above equality, then $F + h$ with $h \in V_K$ also satisfies it. We write $F = \mathcal{Q}(g)$ if F satisfies Eq.(3.13) and $\mathcal{P}_K(F) = 0$. Then $\mathcal{Q}$ defines the linear map from V_I to V_I.

We show a few equalities which are convenient when calculating $\phi^{(k)}$'s.

Proposition 3.2. The following equalities hold for any $g \in V_I$.

$$\text{(i)} \quad \mathcal{P}_K \circ \mathcal{Q}(g) = 0, \tag{3.14}$$

$$\text{(ii)} \quad \mathcal{Q}(Dg \cdot \mathcal{Q}(g) + D\mathcal{Q}(g) \cdot g) = \mathcal{P}_I(D\mathcal{Q}(g) \cdot \mathcal{Q}(g)), \tag{3.15}$$

$$\text{(iii)} \quad e^{-As} g(e^{As}x) = \frac{\partial}{\partial s}\left(e^{-As}\mathcal{Q}(g)(e^{As}x)\right), \quad s \in \mathbf{R}, \tag{3.16}$$

where D denotes the derivative with respect to x.

Proof. Part (i) of Prop.3.2 follows from the definition of $\mathcal{Q}$. To prove (ii) of Prop.3.2, we write $F = \mathcal{Q}(g)$. By using Eq.(3.13), it is easy to verify the equality

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}(x)F(x)\right)Ax - A\left(\frac{\partial F}{\partial x}(x)F(x)\right) = \frac{\partial g}{\partial x}(x)F(x) + \frac{\partial F}{\partial x}(x)g(x), \tag{3.17}$$

which implies (ii) of Prop.3.2. Part (iii) of Prop.3.2 is proved by a straightforward calculation. ■

3.2 Non-diagonal case

Now we turn back to $\phi^{(0)}$ in Eq.(3.4). By Prop.3.1, g_1 is decomposed as $g_1 = g_{1I} + g_{1K}$, where $g_{1I} = \mathcal{P}_I(g_1)$ and $g_{1K} = \mathcal{P}_K(g_1)$. Then, part (iii) of Prop.3.2 is used to yield

$$\begin{aligned}
\phi^{(0)}(t,x) &= e^{At}h^{(0)}(e^{-At}x) + \int_0^t e^{-A(s-t)}g_{1I}(e^{A(s-t)}x)ds + \int_0^t e^{-A(s-t)}g_{1K}(e^{A(s-t)}x)ds \\
&= e^{At}h^{(0)}(e^{-At}x) + \int_0^t \frac{\partial}{\partial s}\left(e^{-A(s-t)}\mathcal{Q}(g_{1I})(e^{A(s-t)}x)\right)ds + \int_0^t e^{-A(s-t)}g_{1K}(e^{A(s-t)}x)ds \\
&= e^{At}h^{(0)}(e^{-At}x) + \mathcal{Q}(g_{1I})(x) - e^{At}\mathcal{Q}(g_{1I})(e^{-At}x) + \int_0^t e^{-A(s-t)}g_{1K}(e^{A(s-t)}x)ds.
\end{aligned} \tag{3.18}$$

Recall that our purpose is to determine $h^{(0)}$ so that $\phi^{(0)}$ becomes a polynomial in t. For this purpose, putting $h^{(0)} = \mathcal{Q}(g_{1I})$, we obtain

$$\phi^{(0)}(t,x) = \mathcal{Q}(g_{1I})(x) + \int_0^t e^{-A(s-t)}g_{1K}(e^{A(s-t)}x)ds. \tag{3.19}$$

Next thing to do is to calculate the second term in the right hand side of the above. Let $A = \Lambda + N$ be the Jordan decomposition of A, where $\Lambda = \Lambda^*$ is a diagonal matrix and N is a nilpotent matrix. Since g_{1K} satisfies $g_{1K}(e^{A^* t}x) = e^{A^* t} g_{1K}(x)$, Eq.(3.19) is calculated as

$$\begin{aligned}
\phi^{(0)}(t,x) &= Q(g_{1I})(x) + \int_0^t e^{-N(s-t)} e^{-\Lambda(s-t)} g_{1K}(e^{\Lambda(s-t)} e^{N(s-t)} x) ds \\
&= Q(g_{1I})(x) + \int_0^t e^{-N(s-t)} e^{N^*(s-t)} e^{-A^*(s-t)} g_{1K}(e^{A^*(s-t)} e^{-N^*(s-t)} e^{N(s-t)} x) ds \\
&= Q(g_{1I})(x) + \int_0^t e^{-N(s-t)} e^{N^*(s-t)} g_{1K}(e^{-N^*(s-t)} e^{N(s-t)} x) ds. \qquad (3.20)
\end{aligned}$$

This provides a desired form of $\phi^{(0)}$. Thus the Lie equation (2.20) is given by

$$\frac{du}{d\varepsilon} = Q(g_{1I})(u) + \int_0^t e^{-N(s-t)} e^{N^*(s-t)} g_{1K}(e^{-N^*(s-t)} e^{N(s-t)} u) ds + O(\varepsilon). \qquad (3.21)$$

Further, it is easy to verify that if we change variables as

$$u = y + \varepsilon Q(g_{1I})(y), \qquad (3.22)$$

then Eq.(3.21) is transformed into the system

$$\frac{dy}{d\varepsilon} = \int_0^t e^{-N(s-t)} e^{N^*(s-t)} g_{1K}(e^{-N^*(s-t)} e^{N(s-t)} y) ds + O(\varepsilon). \qquad (3.23)$$

This system is called the *zeroth order normal form of the Lie equation* if the $O(\varepsilon)$-term is truncated. Note that since $Q(g_{1I})(0) = 0$, the transformation $u \mapsto y$ defined as Eq.(3.22) gives a diffeomorphism, which is called the near identity transformation, near the origin if $|\varepsilon|$ is sufficiently small.

Higher order terms $\phi^{(1)}, \phi^{(2)}, \cdots$ are calculated in a similar manner. In particular, we can show by induction that if $g_i(x)$, $i = 1, 2, \cdots$ are polynomials, all $\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \cdots$ are polynomials in t if we choose undetermined functions $h^{(i)}$'s appropriately as above (see also Sec.3.3). Although to derive explicit forms of $\phi^{(1)}, \phi^{(2)}, \cdots$ involves hard calculation, if the matrix A is diagonal, we can obtain all of them as is shown in the next subsection. Note that for the purpose to make $\phi^{(i)}$'s to be polynomials, there are many possibilities of choices of $h^{(i)}$'s. Actually, for any polynomial vector field $\widetilde{h} \in V_K$, put $h^{(0)} = Q(g_{1I}) + \widetilde{h}$ in Eq.(3.18). Then, the resultant Lie equation (3.21) is again polynomial in t, although it is slightly modified. Such non-uniqueness is well known in normal forms theory and have been studied by many authors for further reduction of normal forms, see Chen, Della Dora [22], Gaeta [23] and references therein. In this article, we choose $h^{(i)}$'s so that $h^{(i)} \in V_I$ for simplicity.

3.3 Diagonal case

In this subsection, we suppose that the matrix A in Eq.(3.1) is a diagonal matrix. Then, Eq.(3.20) takes the form

$$\phi^{(0)}(t,x) = Q(g_{1I})(x) + g_{1K}(x)t. \qquad (3.24)$$

Calculated in the same way as the previous subsection, $\phi^{(1)}$ proves to be of the form

$$\phi^{(1)}(t,x) = 2Q\mathcal{P}_I(R_2)(x) - \frac{\partial Q(g_{1I})}{\partial x}(x)Q(g_{1I})(x) + 2\mathcal{P}_K(R_2)(x)t + Q[g_{1I}, g_{1K}](x)t, \quad (3.25)$$

where $Q\mathcal{P}_I = Q \circ \mathcal{P}_I$, the function R_2 is defined as

$$R_2(x) = \frac{\partial g_1}{\partial x}(x)Q(g_{1I})(x) + g_2(x) - \frac{\partial Q(g_{1I})}{\partial x}(x)g_{1K}(x), \quad (3.26)$$

and where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator of vector fields defined as

$$[f,g](x) = \frac{\partial f}{\partial x}(x)g(x) - \frac{\partial g}{\partial x}(x)f(x). \quad (3.27)$$

Eq.(3.25) is proved in Appendix. Note that $\phi^{(1)}(t,x)$ is a linear function in t as well as $\phi^{(0)}(t,x)$, while $\phi^{(k)}(t,x),\ k \geq 2$ shown below is a polynomial of degree k in t. Thus the first order Lie equation is given by

$$\begin{aligned} \frac{du}{d\varepsilon} &= Q(g_{1I})(x) + g_{1K}(x)t \\ &+ \varepsilon\left(2Q\mathcal{P}_I(R_2)(x) - \frac{\partial Q(g_{1I})}{\partial x}(x)Q(g_{1I})(x) + 2\mathcal{P}_K(R_2)(x)t + Q[g_{1I}, g_{1K}](x)t\right). \end{aligned} \quad (3.28)$$

By changing the variables as

$$u = y + \varepsilon Q(g_{1I})(y) + \varepsilon^2 Q\mathcal{P}_I(R_2)(y), \quad (3.29)$$

it turns out that Eq.(3.28) is transformed into a system of the form

$$\frac{dy}{d\varepsilon} = g_{1K}(x)t + 2\varepsilon\mathcal{P}_K(R_2)(x)t + O(\varepsilon^2). \quad (3.30)$$

If the $O(\varepsilon^2)$-term is truncated, we call it the *first order normal form of the Lie equation*. This procedure is done for more higher order terms of ε. To write down them, at first, define the functions G_k through the equality

$$\sum_{k=1}^{\infty} \varepsilon^k g_k(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) = \sum_{k=1}^{\infty} \varepsilon^k G_k(y_0, y_1, \cdots, y_{k-1}). \quad (3.31)$$

For example, G_1, G_2 and G_3 are given by

$$G_1(y_0) = g_1(y_0), \quad (3.32)$$

$$G_2(y_0, y_1) = \frac{\partial g_1}{\partial y}(y_0)y_1 + g_2(y_0), \quad (3.33)$$

$$G_3(y_0, y_1, y_2) = \frac{1}{2}\frac{\partial^2 g_1}{\partial y^2}(y_0)y_1^2 + \frac{\partial g_1}{\partial y}(y_0)y_2 + \frac{\partial g_2}{\partial y}(y_0)y_1 + g_3(y_0), \quad (3.34)$$

respectively.

Theorem 3.3. Let us define functions $R_k : \mathbf{R}^n \to \mathbf{R}^n,\ k = 1, 2, \cdots$ to be

$$R_1(y) = g_1(x), \tag{3.35}$$

and

$$\begin{aligned} R_k(y) &= G_k(y, Q\mathcal{P}_I(R_1)(y), Q\mathcal{P}_I(R_2)(y), \cdots, Q\mathcal{P}_I(R_{k-1})(y)) \\ &\quad - \sum_{j=1}^{k-1} \frac{\partial Q\mathcal{P}_I(R_j)}{\partial y}(y)\mathcal{P}_K(R_{k-j})(y), \end{aligned} \tag{3.36}$$

for $k = 2, 3, \cdots$. With these R_k's, define $\mathcal{R}_k(y, t)$'s to be

$$\mathcal{R}_1(y, t) = \mathcal{P}_K(R_1)(y)t = g_{1K}(y)t, \tag{3.37}$$

$$\mathcal{R}_2(y, t) = 2\mathcal{P}_K(R_2)(y)t, \tag{3.38}$$

$$\mathcal{R}_3(y, t) = 3\mathcal{P}_K(R_3)(y)t + \sum_{j=1}^{2} \int_0^t [\mathcal{P}_K(R_j), \mathcal{R}_{3-j}](y, s)ds, \tag{3.39}$$

$$\vdots$$

$$\mathcal{R}_k(y, t) = k\mathcal{P}_K(R_k)(y)t + \sum_{j=1}^{k-1} \int_0^t [\mathcal{P}_K(R_j), \mathcal{R}_{k-j}](y, s)ds, \tag{3.40}$$

$$\vdots$$

Then, by the coordinate transformation defined to be

$$u = y + \varepsilon Q\mathcal{P}_I(R_1)(y) + \varepsilon^2 Q\mathcal{P}_I(R_2)(y) + \cdots + \varepsilon^{m+1} Q\mathcal{P}_I(R_{m+1})(y), \tag{3.41}$$

the Lie equation (2.20) is transformed into the system of the form

$$\frac{dy}{d\varepsilon} = \mathcal{R}_1(y, t) + \varepsilon\mathcal{R}_2(y, t) + \cdots + \varepsilon^m \mathcal{R}_{m+1}(y, t) + O(\varepsilon^{m+1}). \tag{3.42}$$

If the $O(\varepsilon^{m+1})$-term is truncated, we call it the *m-th order normal form of the Lie equation* for Eq.(3.1).

Note that $\mathcal{R}_1(y, t) \sim O(t)$, $\mathcal{R}_2(y, t) \sim O(t)$ and $\mathcal{R}_k(y, t) \sim O(t^{k-1})$ if $k \geq 3$. This fact is also proved by Iwasa [20]. For example, $\mathcal{R}_3(y, t)$ is rewritten as

$$\mathcal{R}_3(y, t) = 3\mathcal{P}_K(R_3)(y)t + \frac{1}{2}[\mathcal{P}_K(R_1), \mathcal{P}_K(R_2)](y)\, t^2. \tag{3.43}$$

This theorem for $m = 0$ and $m = 1$ is already proved. More higher order case is proved by a similar calculation as above, although we omit it here. It is also proved by transforming Eq.(3.42) into the RG equation (see Sec.3.4) and using Thm.A.6 of Chiba [4].

3.4 Non-hyperbolic case

If the matrix A in Eq.(3.1) is hyperbolic, which means that no eigenvalues of A lie on the imaginary axis, then the flow of Eq.(3.1) near the origin is topologically conjugate to the linear system $\dot{x} = Ax$ and the stability of the origin is easily determined. If A has eigenvalues on the imaginary axis, Eq.(3.1) has a center manifold at the origin and nontrivial phenomena, such as bifurcations, may occur on the center manifold. We consider such a situation in this subsection. By using the center manifold reduction [1,6], we can assume that all eigenvalues of A lie on the imaginary axis without loss of generality. We also suppose that A is diagonalizable. In this case, the operators $\mathcal{P}_K$ and $Q\mathcal{P}_I$ are calculated as follows:

Recall that if $A = A^*$, the equality

$$\begin{aligned}\int_0^t e^{-A(s-t)}g(e^{A(s-t)}x)ds &= \int_0^t e^{-A(s-t)}\mathcal{P}_I(g)(e^{A(s-t)}x)ds + \int_0^t e^{-A(s-t)}\mathcal{P}_K(g)(e^{A(s-t)}x)ds \\ &= Q\mathcal{P}_I(g)(x) - e^{At}Q\mathcal{P}_I(g)(e^{-At}x) + \mathcal{P}_K(g)(x)t \end{aligned} \tag{3.44}$$

holds. We have to calculate $Q\mathcal{P}_I(g)$ and $\mathcal{P}_K(g)$ to obtain the normal form of the Lie equation (3.42). Since $e^{-As}g(e^{As}x)$ is an almost periodic function with respect to s, it is expanded in a Fourier series as $e^{-As}g(e^{As}x) = \sum_{\lambda_i\in\Lambda} c(\lambda_i, x)e^{\sqrt{-1}\lambda_i s}$, where Λ is the set of Fourier exponents and $c(\lambda_i, x) \in \mathbf{R}^n$ is a Fourier coefficient. In particular, the Fourier coefficient $c(0, x)$ associated with the zero Fourier exponent is the average of $e^{-As}g(e^{As}x)$:

$$c(0, x) = \lim_{t\to\infty} \frac{1}{t}\int^t e^{-As}g(e^{As}x)ds. \tag{3.45}$$

Thus we obtain

$$\begin{aligned}\int_0^t e^{-A(s-t)}g(e^{A(s-t)}x)ds &= \int_0^t \sum_{\lambda_i\in\Lambda} c(\lambda_i, x)e^{\sqrt{-1}\lambda_i(s-t)}ds \\ &= \sum_{\lambda_i\neq 0} \frac{1}{\sqrt{-1}\lambda_i} c(\lambda_i, x)(1 - e^{-\sqrt{-1}\lambda_i t}) + c(0, x)t. \end{aligned} \tag{3.46}$$

Comparing it with Eq.(3.44), we obtain

$$\mathcal{P}_K(g)(x) = c(0, x) = \lim_{t\to\infty} \frac{1}{t}\int^t e^{-As}g(e^{As}x)ds, \tag{3.47}$$

$$Q\mathcal{P}_I(g)(x) = \sum_{\lambda_i\neq 0} \frac{1}{\sqrt{-1}\lambda_i} c(\lambda_i, x). \tag{3.48}$$

These formulas for $\mathcal{P}_K$ and $Q\mathcal{P}_I$ allow one to calculate the normal forms of the Lie equations systematically.

In our situation, Theorem 2.1 for approximate solutions is refined as follows:

Theorem 3.4. Suppose that all eigenvalues of the diagonalizable matrix A lie on the imaginary axis. Let $u = u(t, \varepsilon)$ be a solution of the m-th order Lie equation (2.20) with

$u(t,0) = e^{At}x_0$ and $x(t)$ a solution of Eq.(3.1) with $x(0) = u(0,\varepsilon)$. Then, there exist positive constants C and T such that the inequality

$$||x(t) - u(t,\varepsilon)|| < C\varepsilon^{m+1} \tag{3.49}$$

holds for $0 \leq t \leq T/\varepsilon$.

Indeed, we can show that φ_t and the error function $w(t,\varepsilon)$ in Eq.(2.29) are almost periodic functions with respect to t and thus they are bounded for all $t \in \mathbf{R}$ (see Chiba[4] for the detail). Then the numbers T_1, T_2 and T_3 in the proof of Thm.2.1 are taken to be arbitrarily large and this proves Thm.3.4.

Now we suppose that the normal form of the Lie equation for Eq.(3.1) satisfies $\mathcal{R}_1 = \cdots = \mathcal{R}_{m-1} = 0$ for some integer $m \geq 1$. Then Eq.(3.42) is reduced to

$$\begin{aligned}\frac{dy}{d\varepsilon} &= \varepsilon^{m-1}\mathcal{R}_m(y,t)\\ &= m\varepsilon^{m-1}\mathcal{P}_K(R_m)(y)t\\ &= m\varepsilon^{m-1}t\lim_{t\to\infty}\frac{1}{t}\int^t e^{-As}R_m(e^{As}y)ds.\end{aligned} \tag{3.50}$$

To change the independent variable ε to t, we integrate the above as

$$\begin{aligned} y &= y(\varepsilon=0) + \int_0^\varepsilon m\eta^{m-1}\mathcal{P}_K(R_m)(y(\eta))td\eta\\ &= y(\varepsilon=0) + \varepsilon^m\mathcal{P}_K(R_m)(y(\varepsilon))t - \int_0^\varepsilon \eta^m\frac{\partial\mathcal{P}_K(R_m)}{\partial y}(y(\eta))\frac{dy}{d\eta}td\eta\\ &= y(\varepsilon=0) + \varepsilon^m\mathcal{P}_K(R_m)(y(\varepsilon))t + O(\varepsilon^{m+1}).\end{aligned} \tag{3.51}$$

Differentiating with respect to t, we obtain

$$\begin{aligned}\frac{dy}{dt} &= \varepsilon^m\mathcal{P}_K(R_m)(y) + \varepsilon^m\frac{\partial\mathcal{P}_K(R_m)}{\partial y}(y)\frac{dy}{dt}t + O(\varepsilon^{m+1})\\ &= \varepsilon^m\mathcal{P}_K(R_m)(y) + O(\varepsilon^{m+1}).\end{aligned} \tag{3.52}$$

This equation is just the same as the m-th order RG equation in the CGO RG method [4,7]. It is known that topological properties of the original equation (3.1) are well understood by using the RG equation. In particular, the next theorem holds.

Theorem 3.5 (Chiba [4,7]). Suppose that all eigenvalues of the diagonalizable matrix A lie on the imaginary axis and that the normal form of the Lie equation for Eq.(3.1) satisfies $\mathcal{R}_1 = \cdots = \mathcal{R}_{m-1} = 0$ for some integer $m \geq 1$. If the system $dy/dt = \varepsilon^m\mathcal{P}_K(R_m)(y)$ has a normally hyperbolic invariant manifold N, then for sufficiently small $|\varepsilon|$, the system (3.1) has an invariant manifold N_ε, which is diffeomorphic to N. In particular the stability of N_ε coincides with that of N.

This theorem is used to investigate existence of invariant manifolds and bifurcations [4-8]. It is remarkable that while our idea based on Lie theory is quite different from the

CGO RG approach based on the renormalization group in quantum field theory, the Lie equation yields the same result as the CGO RG if the independent variable ε is changed to time t. The mathematical basis of the CGO RG method is well studied in [4,7]. In particular the relationship between the CGO RG method and other perturbation methods, such as normal forms [5], center manifold reduction [6], the multiple scale method *etc.* [7], is well investigated. Nevertheless, it should be emphasized that the approach based on Lie theory is easily extended to difference equations [14], and applications to a much wider range of problems remain as future works.

4 Examples

In this section, we give a few examples to verify our theorems.

Example 4.1. Consider the perturbed harmonic oscillator

$$\ddot{x} + x + 2\varepsilon \sin x = 0, \quad x \in \mathbf{R}. \tag{4.1}$$

It is convenient to introduce the complex variable z through $x = i(z - \bar{z})$, $\dot{x} = -(z + \bar{z})$, where $i = \sqrt{-1}$. Then, the above equation is rewritten as

$$\begin{cases} \dot{z} = iz + \varepsilon \sin i(z - \bar{z}), \\ \dot{\bar{z}} = -i\bar{z} + \varepsilon \sin i(z - \bar{z}). \end{cases} \tag{4.2}$$

Thus the zeroth order normal form of the Lie equation is obtained by using Eq.(3.47) as

$$\frac{d}{d\varepsilon}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathcal{P}_K(g)(y)t = t \cdot \lim_{t\to\infty} \frac{1}{t}\int_0^t \begin{pmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{pmatrix}\begin{pmatrix} \sin i(e^{is}y_1 - e^{-is}y_2) \\ \sin i(e^{is}y_1 - e^{-is}y_2) \end{pmatrix} ds, \tag{4.3}$$

where $\bar{y}_2 = y_1$. Putting $y_1 = re^{i\theta}$, $y_2 = re^{-i\theta}$ yields

$$\begin{cases} r'(\varepsilon) = -\dfrac{t}{2\pi}\displaystyle\int_0^{2\pi} \cos s \cdot \sin(2r \sin s) ds = 0, \\ \theta'(\varepsilon) = \dfrac{t}{2\pi r}\displaystyle\int_0^{2\pi} \sin s \cdot \sin(2r \sin s) ds = \dfrac{t}{r} J_1(2r), \end{cases} \tag{4.4}$$

where $J_n(r)$ is the Bessel function of the first kind defined as the solution of the equation $r^2 x'' + rx' + (r^2 - n^2)x = 0$. This system with the initial condition $y_1(0) = ce^{it}$, $y_2(0) = ce^{-it}$, $c \in \mathbf{R}$ (*i.e.* $r(0) = c$, $\theta(0) = t$) is easily solved to yield

$$r(\varepsilon) = 0, \quad \theta(\varepsilon) = t + \frac{\varepsilon t}{c} J_1(2c). \tag{4.5}$$

Thus approximate solutions of Eq.(4.2) are given by

$$z(t) = c \exp i\left(t + \frac{\varepsilon t}{c} J_1(2c)\right). \tag{4.6}$$

Finally, approximate solutions of Eq.(4.1) are given by

$$x(t) = i(z(t) - \overline{z(t)}) = -2c \sin\left(t + \frac{\varepsilon t}{c} J_1(2c)\right). \tag{4.7}$$

A numerical solution of Eq.(4.1) and the approximate solution (4.7) are presented as Fig.1 for comparison. The dashed curve is the approximate solution (4.7) for $\varepsilon = 0.1, c = 1/2$ (in this case, $x(0) = 0$ and $\dot{x}(0) = -1 - 0.2 \cdot J_1(1)$). The solid curve denotes a numerical solution of Eq.(4.1) for $\varepsilon = 0.1$ with $x(0) = 0, \dot{x}(0) = -1 - 0.2 \cdot J_1(1)$. The dotted line denotes an exact solution of the unperturbed system $\ddot{x} + x = 0$. The figure shows that the phase lag caused by the perturbation is correctly captured by the approximate solution (4.7).

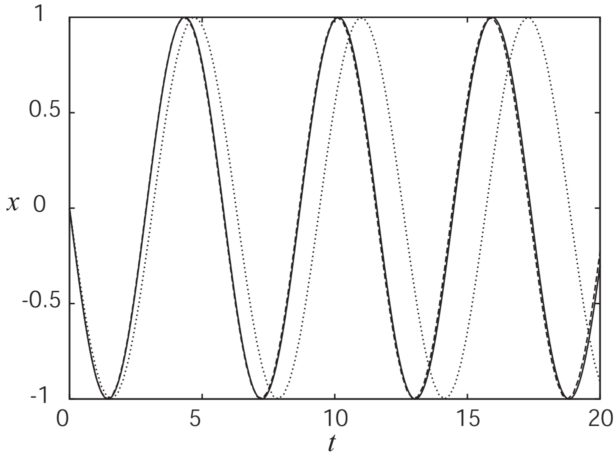

Fig. 1: The solid line denotes a numerical solution of Eq.(4.1). The dashed line denotes the approximate solution (4.7). They almost overlap with one another. The dotted line denotes an exact solution for the unperturbed system $\ddot{x} + x = 0$.

Example 4.2. Consider the system on $\mathbf{R}^2$

$$\begin{cases} \dot{x}_1 = x_2 + x_2^2 - \varepsilon^2 x_1, \\ \dot{x}_2 = -x_1 + 2\varepsilon^2 x_2 - x_1 x_2 + c x_2^2, \end{cases} \tag{4.8}$$

where $c > 0$ is a constant. Changing the coordinates by $(x_1, x_2) = (\varepsilon X_1, \varepsilon X_2)$ yields

$$\begin{cases} \dot{X}_1 = X_2 + \varepsilon X_2^2 - \varepsilon^2 X_1, \\ \dot{X}_2 = -X_1 + \varepsilon(c X_2^2 - X_1 X_2) + 2\varepsilon^2 X_2. \end{cases} \tag{4.9}$$

We introduce the complex variable z by $X_1 = z + \bar{z}$, $X_2 = i(z - \bar{z})$. Then, the above system is rewritten as

$$\begin{cases} \dot{z} = iz + \dfrac{\varepsilon}{2}\left(ic(z-\bar{z})^2 - 2z^2 + 2z\bar{z}\right) + \dfrac{\varepsilon^2}{2}(z - 3\bar{z}), \\ \dot{\bar{z}} = -i\bar{z} + \dfrac{\varepsilon}{2}\left(-ic(z-\bar{z})^2 - 2\bar{z}^2 + 2z\bar{z}\right) - \dfrac{\varepsilon^2}{2}(3z - \bar{z}). \end{cases} \tag{4.10}$$

For this system, it is easy to verify that $\mathcal{R}_1 = \mathcal{P}_K(g)t$ vanishes and the first order normal form of the Lie equation is given by

$$\frac{d}{d\varepsilon}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \varepsilon\mathcal{R}_2(y,t) = \begin{pmatrix} \dfrac{\varepsilon}{6}(3y_1 - y_1^2 y_2(9c + 12i + 4ic^2))t \\ \dfrac{\varepsilon}{6}(3y_2 - y_1 y_2^2(9c - 12i - 4ic^2))t \end{pmatrix}. \tag{4.11}$$

Putting $y_1 = re^{i\theta}$, $y_2 = re^{-i\theta}$ results in

$$\begin{cases} dr/d\varepsilon = \dfrac{1}{2}\varepsilon r(1 - 3cr^2)t, \\ d\theta/d\varepsilon = -\dfrac{2}{3}\varepsilon r^2(3 + c^2)t. \end{cases} \tag{4.12}$$

It is easy to show that this system has a stable (resp. unstable) periodic orbit $r = \sqrt{1/3c}$ if $\varepsilon > 0$ (resp. $\varepsilon < 0$). Thus Thm.3.5 implies that the original system (4.8) also has a stable (resp. unstable) periodic orbit if $\varepsilon > 0$ (resp. $\varepsilon < 0$). This type of bifurcation, periodic orbits exist for both of $\varepsilon > 0$ and $\varepsilon < 0$, is known as the degenerate Hopf bifurcation.

A Appendix

In this appendix, we prove Eq.(3.25). At first, we prove the next proposition.

Proposition A. Suppose that $A = A^*$. For $g \in V_I$ and $h \in V_K$, the following equalities hold:

$$\text{(i)} \quad \frac{\partial g}{\partial x}h \in V_I, \quad Q\left(\frac{\partial g}{\partial x}h\right) = \frac{\partial Q(g)}{\partial x}h, \tag{A.1}$$

$$\text{(ii)} \quad \frac{\partial h}{\partial x}g \in V_I, \quad Q\left(\frac{\partial h}{\partial x}g\right) = \frac{\partial h}{\partial x}Q(g), \tag{A.2}$$

$$\text{(iii)} \quad [g,h] \in V_I, \quad Q([g,h]) = [Q(g),h]. \tag{A.3}$$

Proof. Put $F = Q(g)$. Note that g and h satisfy the equalities

$$\frac{\partial F}{\partial x}(x)Ax - AF(x) = g(x), \tag{A.4}$$

$$\frac{\partial h}{\partial x}(x)Ax - Ah(x) = 0. \tag{A.5}$$

By using these equalities, we can prove the following equalities

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}(x)h(x)\right)Ax - A\left(\frac{\partial F}{\partial x}(x)h(x)\right) = \frac{\partial g}{\partial x}(x)h(x), \tag{A.6}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}(x)F(x)\right)Ax - A\left(\frac{\partial h}{\partial x}(x)F(x)\right) = \frac{\partial h}{\partial x}(x)g(x), \tag{A.7}$$

which imply that $\partial g/\partial x \cdot h \in V_I$ and $\partial h/\partial x \cdot g \in V_I$. The same calculation also shows that $\partial F/\partial x \cdot h \in V_I$ and $\partial h/\partial x \cdot F \in V_I$. They prove (i) and (ii) of Proposition A. Part (iii) of

Proposition A immediately follows from (i) and (ii). ■

Proof of Eq.(3.25). Eqs.(3.3) and (3.24) are put together to yield

$$\begin{aligned}
\phi^{(1)}(t,x) &= e^{At}h^{(1)}(e^{-At}x) + \int_0^t e^{-A(s-t)}\Big(\frac{\partial g_1}{\partial x}(e^{A(s-t)}x)\phi^{(0)}(s, e^{A(s-t)}x) \\
&\qquad -\frac{\partial \phi^{(0)}}{\partial x}(s, e^{A(s-t)}x)g_1(e^{A(s-t)}x) + 2g_2(e^{A(s-t)}x)\Big)ds \\
&= e^{At}h^{(1)}(e^{-At}x) + \int_0^t e^{-A(s-t)}\left([g_1, Q(g_{1I})] + 2g_2\right)(e^{A(s-t)}x)ds \\
&\qquad + \int_0^t e^{-A(s-t)}[g_1, g_{1K}](e^{A(s-t)}x)sds \\
&= e^{At}h^{(1)}(e^{-At}x) + \int_0^t e^{-A(s-t)}\left([g_1, Q(g_{1I})] + 2g_2\right)(e^{A(s-t)}x)ds \\
&\qquad + \int_0^t e^{-A(s-t)}[g_1, g_{1K}](e^{A(s-t)}x)ds \cdot t \\
&\qquad - \int_0^t ds \int_0^s e^{-A(s'-t)}[g_1, g_{1K}](e^{A(s'-t)}x)ds'. \qquad \text{(A.8)}
\end{aligned}$$

Since $[g_1, g_{1K}] = [g_{1I} + g_{1K}, g_{1K}] = [g_{1I}, g_{1K}]$, the above is rewritten as

$$\begin{aligned}
\phi^{(1)}(t,x) &= e^{At}h^{(1)}(e^{-At}x) + \int_0^t e^{-A(s-t)}\left([g_1, Q(g_{1I})] + 2g_2\right)(e^{A(s-t)}x)ds \\
&\qquad + \int_0^t e^{-A(s-t)}[g_{1I}, g_{1K}](e^{A(s-t)}x)ds \cdot t \\
&\qquad - \int_0^t ds \int_0^s e^{-A(s'-t)}[g_{1I}, g_{1K}](e^{A(s'-t)}x)ds'. \qquad \text{(A.9)}
\end{aligned}$$

Prop.A (iii) and Prop.3.2 (iii) are used to yield

$$\begin{aligned}
\phi^{(1)}(t,x) &= e^{At}h^{(1)}(e^{-At}x) + \int_0^t e^{-A(s-t)}\left([g_1, Q(g_{1I})] + 2g_2\right)(e^{A(s-t)}x)ds \\
&\qquad + Q[g_{1I}, g_{1K}](x)t - e^{At}Q[g_{1I}, g_{1K}](e^{-At}x)t \\
&\qquad - \int_0^t \Big(e^{-A(s-t)}Q[g_{1I}, g_{1K}](e^{A(s-t)}x) - e^{At}Q[g_{1I}, g_{1K}](e^{-At}x)\Big)ds \\
&= e^{At}h^{(1)}(e^{-At}x) + Q[g_{1I}, g_{1K}](x)t \\
&\qquad + \int_0^t e^{-A(s-t)}\left([g_1, Q(g_{1I})] - [Q(g_{1I}), g_{1K}] + 2g_2\right)(e^{A(s-t)}x)ds \\
&= e^{At}h^{(1)}(e^{-At}x) + Q[g_{1I}, g_{1K}](x)t \\
&\qquad + \int_0^t e^{-A(s-t)}\left(2\frac{\partial g_1}{\partial x}Q(g_{1I}) + 2g_2 - 2\frac{\partial Q(g_{1I})}{\partial x}g_{1K}\right)(e^{A(s-t)}x)ds \\
&\qquad - \int_0^t e^{-A(s-t)}\left(\frac{\partial g_{1I}}{\partial x}Q(g_{1I}) + \frac{\partial Q(g_{1I})}{\partial x}g_{1I}\right)(e^{A(s-t)}x)ds. \qquad \text{(A.10)}
\end{aligned}$$

Finally, Prop.3.2 (ii) and (iii) show that

$$
\begin{aligned}
\phi^{(1)}(t,x) &= e^{At}h^{(1)}(e^{-At}x) + Q[g_{1I}, g_{1K}](x)t \\
&\quad +2\int_0^t e^{-A(s-t)}\left(\frac{\partial g_1}{\partial x}Q(g_{1I}) + g_2 - \frac{\partial Q(g_{1I})}{\partial x}g_{1K}\right)(e^{A(s-t)}x)ds \\
&\quad -\int_0^t \frac{\partial}{\partial s}\left(e^{-A(s-t)}\frac{\partial Q(g_{1I})}{\partial x}(e^{A(s-t)}x)Q(g_{1I})(e^{A(s-t)}x)\right)ds \\
&= e^{At}h^{(1)}(e^{-At}x) + Q[g_{1I}, g_{1K}](x)t + 2\int_0^t e^{-A(s-t)}R_2(e^{A(s-t)}x)ds \\
&\quad -\frac{\partial Q(g_{1I})}{\partial x}(x)Q(g_{1I})(x) + e^{At}\frac{\partial Q(g_{1I})}{\partial x}(e^{-At}x)Q(g_{1I})(e^{-At}x) \\
&= e^{At}h^{(1)}(e^{-At}x) - 2e^{At}Q\mathcal{P}_I(R_2)(e^{-At}x) + e^{At}\frac{\partial Q(g_{1I})}{\partial x}(e^{-At}x)Q(g_{1I})(e^{-At}x) \\
&+ 2Q\mathcal{P}_I(R_2)(x) - \frac{\partial Q(g_{1I})}{\partial x}(x)Q(g_{1I})(x) + 2\mathcal{P}_K(R_2)(x)t + Q[g_{1I}, g_{1K}](x)t.
\end{aligned}
\tag{A.11}
$$

Choosing $h^{(1)}$ appropriately, we obtain Eq.(3.25). ■

Acknowledgements

The authors would like to thank Professor G. Gaeta for critical reading of the manuscript and for useful comments.

References

[1] J. Carr, Applications of Centre Manifold Theory, Springer-Verlag, 1981

[2] L. Y. Chen, N. Goldenfeld, Y. Oono, Renormalization group theory for global asymptotic analysis, Phys. Rev. Lett. 73 (1994), no. 10, 1311-15

[3] L. Y. Chen, N .Goldenfeld, Y. Oono, Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory, Phys. Rev. E 54, (1996), 376-394

[4] H. Chiba, C^1 approximation of vector fields based on the renormalization group method, SIAM J. Appl. Dyn. Syst. Vol.7, 3 (2008), pp. 895-932

[5] H. Chiba, Simplified Renormalization Group Method for Ordinary Differential Equations, J. Diff. Equ. 246, (2009), pp.1991-2019

[6] H. Chiba, Approximation of Center Manifolds on the Renormalization Group Method, J. Math. Phys. Vol.49, 102703 (2008).

[7] H. Chiba, Extension and Unification of Singular Perturbation Methods for ODEs Based on the Renormalization Group Method, (submitted)

[8] H. Chiba, D.Pazo, Stability of an $[N/2]$-dimensional invariant torus in the Kuramoto model at small coupling, Physica D, (in press)

[9] S. N. Chow, C. Li, D. Wang, Normal forms and bifurcation of planar vector fields, Cambridge University Press, 1994

[10] G. Cicogna, G. Gaeta, Approximate symmetries in dynamical systems, Nuovo Cimento B (11) 109 (1994), no. 9, 989–1008

[11] G. Gaeta, Asymptotic symmetries and asymptotically symmetric solutions of partial differential equations, J. Phys. A 27 (1994), no. 2, 437–451

[12] G. Gaeta, R. Mancinelli, Asymptotic scaling symmetries for nonlinear PDEs, Int. J. Geom. Methods Mod. Phys. 2 (2005), no. 6, 1081–1114

[13] M. Iwasa, K. Nozaki, A method to construct asymptotic solutions invariant under the renormalization group, Progr. Theoret. Phys. 116 (2006), no. 4, 605–613

[14] M. Iwasa, K. Nozaki, Renormalization group in difference systems, J. Phys. A, 41 085204, (2008)

[15] V. F. Kovalev, V. V. Pustovalov, D. V. Shirkov, Group analysis and renormgroup symmetries, J. Math. Phys. 39 (1998), no. 2, 1170–1188

[16] P. J. Olver, Applications of Lie groups to differential equations, Springer-Verlag, New York, 1993

[17] M. Hirsch, Differential topology, Springer-Verlag, New York-Heidelberg, 1976

[18] V. A. Baikov, R. K. Gazizov, N. H. Ibragimov, Perturbation methods in group analysis, J. Sov. Math. 55 (1991), no. 1, 1450-1490

[19] J. Murdock, Normal forms and unfoldings for local dynamical systems, Springer-Verlag, New York, (2003)

[20] M. Iwasa, Solution of reduced equations derived with singular perturbation methods, Physical Review E, 78, 066213 (2008)

[21] V. I. Arnold, Geometrical methods in the theory of ordinary differential equations, Springer-Verlag, New York, 1988

[22] G. Chen, J. Della Dora, Further reductions of normal forms for dynamical systems, J. Differential Equations 166 (2000), no. 1, 79–106

[23] G. Gaeta, Poincare normal and renormalized forms. Symmetry and perturbation theory, Acta Appl. Math. 70 (2002), no. 1-3, 113–131

[24] J. A. Murdock, Perturbations. Theory and methods, John Wiley & Sons, Inc., New York, 1991