On the degeneration of Kovalevskaya exponents of Laurent series solutions of quasi-homogeneous vector fields

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Abstract

A structure of families of Laurent series solutions of a quasi-homogeneous vector field is studied, where a given vector field is assumed to have a commutable vector field. For an m dimensional vector field, a family of Laurent series solutions is called principle if it includes m arbitrary parameters, and called non-principle if the number is smaller than m. Starting from a principle Laurent series solutions, a systematic method to obtain a non-principle Laurent series solutions is given. In particular, from the Kovalevskaya exponents of the principle Laurent series solutions, which is one of the invariants of quasi-homogeneous vector fields, the Kovalevskaya exponents of the non-principle Laurent series solutions are obtained by using the commutable vector field.

1 Introduction

A differential equation defined on a complex region is said to have the Painlevé property if any movable singularity (a singularity of a solution which depends on an initial condition) of any solution is a pole. Among them, an important class is the equations that all solutions are meromorphic.

Painlevé and his group classified second order ODEs having the Painlevé property and found new six differential equations called the Painlevé equations $\mathrm{P_I}, \cdots, \mathrm{P_{VI}}$. Nowadays, it is known that they are written in Hamiltonian forms

$$(\mathrm{P_J}) : \frac{dq}{dz} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H_J}{\partial q}, \quad J = \mathrm{I}, \cdots, \mathrm{VI}.$$

Among six Painlevé equations, the Hamiltonian functions of the first, second and fourth Painlevé equations are polynomials in both of the independent variable z and

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the dependent variables (q,p). They are given by

$$\begin{aligned} H_{\rm I}(q,p) &= \frac{1}{2}p^2 - 2q^3 - zq, \\ H_{\rm II}(q,p) &= \frac{1}{2}p^2 - \frac{1}{2}q^4 - \frac{1}{2}zq^2 - \alpha q, \\ H_{\rm IV}(q,p) &= -pq^2 + p^2q - 2pqz - \alpha p + \beta q, \end{aligned} \tag{1.1}$$

respectively, where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters. Since they satisfy Painlevé property and the right hand sides are polynomials, any solutions are meromorphic.

In general, a polynomial $H(x_1, \cdots, x_m)$ is called a quasi-homogeneous polynomial if there are a tuple of positive integers $(a_1, \cdots, a_m)$, called the weight, and h, called the weighted degree, such that

$$H(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_m) = \lambda^h H(x_1, \cdots, x_m) \tag{1.2}$$

for any $\lambda \in \mathbb{C}$. The above Hamiltonians $H_{\rm I}, H_{\rm II}, H_{\rm IV}$ are quasi-homogeneous with respect to the weights given in Table 1, if we ignore terms including parameters α, β. In Chiba [6], possible weights arising from Hamiltonian systems are classified from a view point of singularity theory and it is shown that they are related to Painlevé equations.

	$\deg(q,p,z)$	$\deg(H)$	κ
$\mathrm{P_I}$	$(2,3,4)$	6	6
$\mathrm{P_{II}}$	$(1,2,2)$	4	4
$\mathrm{P_{IV}}$	$(1,1,1)$	3	3

Table 1: $\deg(H)$ denotes the weighted degree of the Hamiltonian function with respect to the weight $\deg(q,p,z)$. κ denotes the Kovalevskaya exponent defined in Section 2.

For a general vector field $F(x_1, \cdots, x_m)$, $F = (f_1, \cdots, f_m)$ on $\mathbb{C}^m$, its weight $(a_1, \cdots, a_m)$ is defined in a similar manner as

$$f_i(\lambda^{a_1}x_1, \cdots, \lambda^{a_m}x_m) = \lambda^{a_i+\gamma} f_i(x_1, \cdots, x_m), \quad i = 1, \cdots, m, \quad \gamma \in \mathbb{N}, \tag{1.3}$$

if it exists. Once a weight is given with $\gamma = 1$, to construct a Laurent series solution of the equation $dx_i/dz = f_i$ is straightforward. It is expressed in the form

$$x_i(z) = c_i(z-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(z-\alpha_0)^{-a_i+j},$$

where coefficients c_i and $d_{i,j}$ are determined by substituting this expression into the equation. Important features are that a position of a pole α_0 can take arbitrary number depending on an initial condition, that is called the movable singularity, and that the order a_i of the pole is the same as the weight of x_i.

From Laurent series solutions, we define the Kovalevskaya exponents as follows: As an example, we consider the first Painlevé equation (1.1). Written in the second order equation, it is expressed as $q'' = 6q^2 + z$. Its Laurent series solution is given by

$$q(z) = (z-\alpha_0)^{-2} - \frac{\alpha_0}{10}(z-\alpha_0)^2 - \frac{1}{6}(z-\alpha_0)^3 + \alpha_1(z-\alpha_0)^4 + O((z-\alpha_0)^5), \quad (1.4)$$

where α_1 is an arbitrary parameter called a free parameter. Counting from the lowest order -2, α_1 is included in the 6th place and this 6 is called the Kovalevskaya exponent. In general, for a Laurent series solution $x_i(z)$ of a quasi-homogeneous system, if an arbitrary parameter is included in the coefficient of $(z-\alpha_0)^{-a_i+j}$, j is called the Kovalevskaya exponent.

As an another example, let us consider the autonomous limit of the fourth order first Painlevé equation. It is a four dimensional system defined by the following Hamiltonian

$$H_1(q_1, p_1, q_2, p_2) = 2p_1p_2 + 3p_2^2q_1 + q_1^4 - q_1^2q_2 - q_2^2. \quad (1.5)$$

The weight is given by $\deg(q_1, p_1, q_2, p_2) = (2, 5, 4, 3)$. The Hamiltonian vector field has two types of families of Laurent series solutions. The one is starting from $q_1(z) = (z-\alpha_0)^{-2} + \cdots$, and it includes three arbitrary parameters at 2nd, 5th, 8th coefficients. Thus the Kovalevskaya exponents are $2, 5, 8$. The other family starting from $q_1(z) = 3(z-\alpha_0)^{-2} + \cdots$ includes only two arbitrary parameters at 8th and 10th coefficients. Thus the Kovalevskaya exponents are 8 and 10.

In general, for a given m-dimensional vector field F, if any solution is meromorphic, a general solution should include m arbitrary parameters determined by an initial condition. In the above example, they are the pole α_0 and three parameters in coefficients. Such a family of Laurent series solutions is called the *principle* Laurent series that constructs an m dimensional manifold $\mathcal{M}_m$. A boundary of the manifold, if it exists, may be an $m-1$ dimensional manifold $\mathcal{M}_{m-1}$. It is occupied by a different family of Laurent series solutions, that includes only $m-1$ arbitrary parameters. Such a family is called *non-principle* or *lower*. The latter family in the above example is this case.

Then, a natural question arises: can we construct a lower Laurent series solutions from the principle one? The purpose in this article is to consider this problem and give a systematic way to obtain the lower one from the principle one.

For this purpose, we assume that for a given quasi-homogeneous vector field F, there exists a quasi-homogeneous vector field G that commutes with F. Let us take an initial point on $\mathcal{M}_m$ and consider the solution $x(z)$ of $dx/dz = F$. Then, $x(z) \in \mathcal{M}_m$ for any $z \in \mathbb{C}$. However, if we change the "route" in the sense that $x(z)$ is governed by the flow of G from some point, a solution may reach at $\mathcal{M}_{m-1}$ along the orbit of G and a principle Laurent series solutions degenerates to a lower one. Indeed, for the above example, there is another Hamiltonian H_2 that commutes with H_1 in Poisson bracket, see Example 4.13 for H_2. Based on this idea, we will obtain a lower Laurent series solutions from the principle one.

For the list of weights, types of Laurent series solutions and their Kovalevskaya exponents of four dimensional polynomial Painlevé equations, the reader can refer to [6]. The method developed in this article is applicable to all of them.

2 Settings and the Kovalevskaya exponents

Let $F = (f_1, \cdots, f_m)$ and $G = (g_1, \cdots, g_m)$ be quasi-homogeneous polynomial vector fields on $\mathbb{C}^m$. We consider the following partial differential equations

$$\frac{\partial x_i}{\partial z_1} = f_i(x), \quad \frac{\partial x_i}{\partial z_2} = g_i(x), \quad i = 1, \cdots, m, \tag{2.1}$$

where $x = (x_1, \cdots, x_m) \in \mathbb{C}^m$ and $z_1, z_2 \in \mathbb{C}$. We suppose the following.

(A1) F and G are quasi-homogeneous: there exists a tuple of positive integers $(a_1, \cdots, a_m) \in \mathbb{N}^m$ and $\gamma \in \mathbb{N}$ such that

$$\begin{cases} f_i(\lambda^{a_1} x_1, \cdots, \lambda^{a_m} x_m) = \lambda^{a_i+1} f_i(x_1, \cdots, x_m) \\ g_i(\lambda^{a_1} x_1, \cdots, \lambda^{a_m} x_m) = \lambda^{a_i+\gamma} g_i(x_1, \cdots, x_m), \end{cases} \tag{2.2}$$

for any $\lambda \in \mathbb{C}$. We call γ the degree of G with respect to the weight $(a_1, \cdots, a_m)$. The degree of F is assumed to be 1.

(A2) F and G commute with each other with respect to the Lie bracket: $[F, G] = 0$. This is equivalent to

$$\sum_{j=1}^{m} \left(f_j(x) \frac{\partial g_i}{\partial x_j}(x) - g_j(x) \frac{\partial f_i}{\partial x_j}(x) \right) = 0, \quad i = 1, \cdots, m. \tag{2.3}$$

(A3) $F(x) = 0$ only when $x = 0$.

In this section, we consider only the flow of F and z_1 is denoted by z for simplicity. Let us consider the *formal* series solution of $dx_i/dz = f_i(x)$ of the form

$$x_i(z) = c_i(z - \alpha_0)^{-q_i} + \sum_{j=1}^{\infty} d_{i,j}(z - \alpha_0)^{-q_i+j}, \tag{2.4}$$

where $q_i \in \mathbb{N}$, α_0 is a possible singularity and $c_i, d_{i,j} \in \mathbb{C}$ are coefficients.

Theorem 2.1([3], Thm.2.9). Under the assumptions (A1) and (A3), any *formal* Laurent series solution (2.4) is a *convergent* Laurent series solution of the form

$$x_i(z) = c_i(z - \alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(z - \alpha_0)^{-a_i+j}, \quad (c_1, \cdots, c_m) \neq (0, \cdots, 0), \tag{2.5}$$

where the exponents $(a_1, \cdots, a_m)$ are the weight of F.

This theorem means that there are no Laurent series solutions whose orders of poles are larger than $(a_1, \cdots, a_m)$. Further, we can show that if $(c_1, \cdots, c_m) = (0, \cdots, 0)$, then $d_{i,j} = 0$ for $j = 1, \cdots, a_i - 1$. This means that (2.5) is a local holomorphic solution. To prove it, the assumption (A3) is essentially used, while for the convergence of the series, (A1) is enough. By substituting (2.5) into the equation $dx_i/dz = f_i(x)$ and comparing the coefficients of $(z - \alpha_0)^j$ in both sides, it turns out that c_i and $d_{i,j}$ are given as follows.

Definition 2.2. A root $c = (c_1, \cdots, c_m) \in \mathbb{C}^m,\ c \neq (0, \cdots, 0)$ of the equation

$$-a_i c_i = f_i(c_1, \cdots, c_m), \quad i = 1, \cdots, m \tag{2.6}$$

is called the indicial locus. For a fixed indicial locus c, $d_j = (d_{1,j}, \cdots, d_{m,j})$ is iteratively determined as a solution of the equation [2]

$$(K(c) - j \cdot I) d_j = (\text{ polynomial of } c \text{ and } d_k \text{ for } k = 1, \cdots, j-1), \tag{2.7}$$

where I is the identity matrix and $K(c)$ is defined by

$$K(c) = \left\{ \frac{\partial f_i}{\partial x_j}(c_1, \cdots, c_m) + a_i \delta_{i,j} \right\}_{i,j=1}^{m}, \tag{2.8}$$

that is called the Kovalevskaya matrix (K-matrix). Its eigenvalues $\kappa_0, \kappa_1, \cdots, \kappa_{m-1}$ are called the Kovalevskaya exponents (K-exponents) associated with c.

The following results are well known, see [1, 10].

Proposition 2.3.
(i) -1 is always a Kovalevskaya exponent. One of its eigenvectors is given by $(a_1 c_1, \cdots, a_m c_m)$. In what follows, we set $\kappa_0 = -1$.

(ii) $\kappa = 0$ is a Kovalevskaya exponent associated with c if and only if c is not an isolated root of the equation (2.6).

From (2.7), it follows that if a positive integer j is not an eigenvalue of $K(c)$, d_j is uniquely determined. If a positive integer j is an eigenvalue of $K(c)$ and (2.7) has a solution d_j, then $d_j + v$ is also a solution for any eigenvectors v. This implies that the Laurent series solution (2.5) includes an arbitrary parameter (called a free parameter) in $d_j = (d_{1,j}, \cdots, d_{m,j})$. Therefore, if (2.5) represents a k-parameter family of Laurent series solutions which includes $k-1$ free parameters other than α_0, at least $k-1$ Kovalevskaya exponents have to be nonnegative integers and we need $k-1$ independent eigenvectors associated with them. Hence, the classical Painlevé test [10] for the necessary condition for the Painlevé property is stated as follows;

Classical Painlevé test. If the system (2.1) satisfying (A1) and (A3) has a m-parameter family of Laurent series solutions of the form (2.5), there exists an indicial

[2] In this article, a column vector is often written as a row vector.

locus $c = (c_1, \cdots, c_m)$ such that all Kovalevskaya exponents except for $\kappa_0 = -1$ are nonnegative integers, and the Kovalevskaya matrix is semisimple.

In Chiba [3], the necessary and sufficient condition for the system (2.1) to have an m-parameter family of Laurent series solutions is given, that is called the extended Painlevé test. The gap of them is that even if the necessary condition for the classical Painlevé test is satisfied, (2.7) may not have a solution d_j. In this case, the series solution (2.5) is modified as a combination of powers of $(z - \alpha_0)$ and $\log(z - \alpha_0)$.

Definition 2.4. An indicial locus $c = (c_1, \cdots, c_m)$ is called *principle* if the associated Laurent series solution (2.5) exists and includes m free parameters. If the number of free parameters is smaller than m, the locus is called a *lower* indicial locus.

In the rest of this article, we assume that there exists an *isolated* principle indicial locus c of the vector field F; that is, its all K-exponents are positive integers other than $\kappa_0 = -1$ and the series solution (2.5) includes m free parameters (one of which is α_0). In this case, for each κ_j $(j \neq 0)$, we can take d_{i,κ_j} as a free parameter for some i. We denote it as $d_{i,\kappa_j} = \alpha_j$. Then, all coefficients $d_{i,j}$ of (2.5) are polynomials of $\alpha_1, \cdots, \alpha_{m-1}$ and the solution is expressed as

$$
\begin{aligned}
x_i(z) &= c_i(z-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(z-\alpha_0)^{-a_i+j} \\
&= x_i(z; \alpha_0, \alpha_1, \cdots, \alpha_{m-1}).
\end{aligned}
$$

The initial value of $x_i(z; \alpha_0, \alpha_1, \cdots, \alpha_{m-1})$ is denoted by

$$
\begin{aligned}
&x_i(0; \alpha_0, \alpha_1, \cdots, \alpha_{m-1}) = \Phi_i(\alpha_0, \alpha_1, \cdots, \alpha_{m-1}) = \Phi_i(A), \\
&\Phi(A) = (\Phi_1(A), \cdots, \Phi_m(A)), \quad A = (\alpha_0, \alpha_1, \cdots, \alpha_{m-1}),
\end{aligned}
$$

which is well-defined for small $|\alpha_0| \neq 0$ and $\Phi(A)$ is a locally biholomorphic map into $\mathbb{C}^m$. In what follows, $d_{i,j}$ is denoted by $d_{i,j}(A)$ as a polynomial of $\alpha_0, \cdots, \alpha_{m-1}$, though it does not depend on α_0 by the construction.

Proposition 2.5. Put

$$\lambda \cdot A := (\lambda^{-1}\alpha_0, \lambda^{\kappa_1}\alpha_1, \cdots, \lambda^{\kappa_{m-1}}\alpha_{m-1}).$$

Functions $d_{i,j}$ and Φ_i are quasi-homogeneous satisfying

$$d_{i,j}(\lambda \cdot A) = \lambda^j d_{i,j}(A), \quad \Phi_i(\lambda \cdot A) = \lambda^{a_i}\Phi_i(A), \quad \lambda \in \mathbb{C}. \tag{2.9}$$

Proof. Put $\widetilde{z} = \lambda^{-1}z$ and $\widetilde{x}_i = \lambda^{a_i}x_i$. Then, $\widetilde{x}(\widetilde{z})$ satisfies the same equation as $x(z)$ because of (A1). Let

$$x_i = c_i(z-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(A)(z-\alpha_0)^{-a_i+j} \tag{2.10}$$

be a Laurent series solution with free parameters $\alpha_0, \cdots, \alpha_{m-1}$. Then

$$\Phi_i(A) = c_i(-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(A)(-\alpha_0)^{-a_i+j}, \tag{2.11}$$

$$\Phi_i(\lambda \cdot A) = \lambda^{a_i} \left(c_i(-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} \lambda^{-j} d_{i,j}(\lambda \cdot A)(-\alpha_0)^{-a_i+j} \right). \tag{2.12}$$

Similarly, consider the Laurent series solution of $\widetilde{x}$, whose locus c is the same as that of (2.11):

$$\widetilde{x}_i = c_i(\widetilde{z} - \widetilde{\alpha}_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(\widetilde{A})(\widetilde{z} - \widetilde{\alpha}_0)^{-a_i+j}, \quad \widetilde{A} = (\widetilde{\alpha}_0, \cdots, \widetilde{\alpha}_{m-1}). \tag{2.13}$$

Since $x(z)$ and $\widetilde{x}(\widetilde{z})$ satisfy the same equation, $d_{i,j}$ in (2.10) and (2.13) are the common function of A, though we can choose different values of free parameters. If we put $\widetilde{\alpha}_0 = \lambda^{-1}\alpha_0$, (2.13) is rewritten as

$$\lambda^{a_i} x_i = c_i \lambda^{a_i}(z - \alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(\widetilde{A})\lambda^{a_i - j}(z - \alpha_0)^{-a_i+j},$$

$$\Rightarrow \quad \Phi_i(A) = c_i(-\alpha_0)^{-a_i} + \sum_{j=1}^{\infty} d_{i,j}(\widetilde{A})\lambda^{-j}(-\alpha_0)^{-a_i+j}. \tag{2.14}$$

It follows from (2.11) and (2.14) that $d_{i,j}(A) = \lambda^{-j} d_{i,j}(\widetilde{A})$. When κ_j is one of the K-exponents, $d_{i,\kappa_j}(A) = \alpha_j$ for some i by the definition. Hence, we have $\alpha_j = \lambda^{-\kappa_j}\widetilde{\alpha}_j$. This shows $\widetilde{A} = \lambda \cdot A$ and $d_{i,j}(A) = \lambda^{-j} d_{i,j}(\lambda \cdot A)$. Therefore, (2.11) and (2.12) prove the desired result. □

Corollary 2.6. $d_{i,j}(A) \neq 0$ only when there exists a tuple of integers $(n_1, \cdots, n_{m-1}) \neq (0, \cdots, 0)$ such that $n_1\kappa_1 + \cdots + n_{m-1}\kappa_{m-1} = j$.

Proof. Assume that a monomial $\alpha_1^{n_1}\alpha_2^{n_2} \cdot \cdots \cdot \alpha_{m-1}^{n_{m-1}}$ is included in $d_{i,j}(A)$. Substituting it into (2.9) proves the desired result. □

For a quasi-homogeneous vector field $dx/dz = F(x)$ satisfying (A1), let $x = \varphi(y_1, \cdots, y_m)$, $\varphi = (\varphi_1, \cdots, \varphi_m)$ be a (locally) holomorphic coordinate transformation satisfying

$$\varphi_i(\lambda^{q_1} y_1, \cdots, \lambda^{q_m} y_m) = \lambda^{a_i}\varphi_i(y_1, \cdots, y_m), \quad i = 1, \cdots, m,$$

where $(q_1, \cdots, q_m) \in \mathbb{Z}^m$ is an arbitrary tuple of integers and $(a_1, \cdots, a_m)$ is the same as in (A1). Remark that (2.9) is just in this case. By the transformation, $dx/dz = F(x)$ is transformed into the system

$$\frac{dy}{dz} = (D\varphi)^{-1} F(\varphi(y)) := \widetilde{F}(y),$$

where $D\varphi$ is the Jacobi matrix.

Theorem 2.7 ([3], Thm.2.5).
$\widetilde{F}(y)$ is quasi-homogeneous with respect to the weight $(q_1, \cdots, q_m)$ whose degree is the same as that of F. If c is an indicial locus of F, $\widetilde{c} = \varphi^{-1}(c)$ is an indicial locus of $\widetilde{F}$. The K-exponents of $\widetilde{F}$ at $\widetilde{c}$ are the same as those of F at c.

Example 2.8. Consider the 2-dim system

$$\frac{dx}{dz} = y, \quad \frac{dy}{dz} = 6x^2.$$

This satisfies the assumptions (A1) and (A3) with the weight $(a_1, a_2) = (2, 3)$. The indicial locus is uniquely given by $(c_1, c_2) = (1, -2)$. Thus, the Laurent series solution starts from $(x, y) = (T^{-2}, -2T^{-3})$, $T = z - \alpha_0$. The K-exponent associated with the locus is $\kappa_1 = 6$. Hence, a free parameter appears in $d_{1,6}$ and/or $d_{2,6}$. Indeed, we can verify that $d_{2,6} = 4d_{1,6}$ and a solution is given by

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ -2 \end{pmatrix} T^{-3} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} T^{-2} + \begin{pmatrix} 0 \\ 4d_{1,6} \end{pmatrix} T^3 + \begin{pmatrix} d_{1,6} \\ 0 \end{pmatrix} T^4 \\ &\quad + \begin{pmatrix} 0 \\ 10d_{1,6}^2/13 \end{pmatrix} T^9 + \begin{pmatrix} d_{1,6}^2/13 \\ 0 \end{pmatrix} T^{10} + \cdots . \end{aligned} \tag{2.15}$$

In this case, we put $\alpha_1 := d_{1,6}$. All other coefficients are polynomials of α_1 and we can confirm Prop. 2.5.

3 Properties of the vector field G

To consider the vector field $G = (g_1, \cdots, g_m)$, we prepare several formulae and notations. The derivative of (A1) at $\lambda = 1$ yields

$$\sum_{j=1}^{m} a_j x_j \frac{\partial f_i}{\partial x_j}(x) = (a_i + 1) f_i(x), \quad \sum_{j=1}^{m} a_j x_j \frac{\partial g_i}{\partial x_j}(x) = (a_i + \gamma) g_i(x). \tag{3.1}$$

Putting $x = c$ to the latter one, we obtain

$$-\sum_{j=1}^{m} f_j(c) \frac{\partial g_i}{\partial x_j}(c) = (a_i + \gamma) g_i(c). \tag{3.2}$$

The derivative of (A1) by x_j gives

$$\begin{cases} \dfrac{\partial f_i}{\partial x_j}(\lambda^{a_1} x_1, \cdots, \lambda^{a_m} x_m) = \lambda^{a_i + 1 - a_j} \dfrac{\partial f_i}{\partial x_j}(x_1, \cdots, x_m), \\ \dfrac{\partial g_i}{\partial x_j}(\lambda^{a_1} x_1, \cdots, \lambda^{a_m} x_m) = \lambda^{a_i + \gamma - a_j} \dfrac{\partial g_i}{\partial x_j}(x_1, \cdots, x_m). \end{cases} \tag{3.3}$$

As before, we assume that c is a principle locus so that there is a Laurent series solution

$$x_i(z) = (z-\alpha_0)^{-a_i}\left(c_i + \sum_{k=1}^{\infty} d_{i,k}(A)(z-\alpha_0)^k\right) =: (z-\alpha_0)^{-a_i} y_i(z) \tag{3.4}$$

including m free parameters $A = (\alpha_0, \cdots, \alpha_{m-1})$. Substituting this solution into $g_i(x)$ gives

$$\begin{aligned} g_i(x(z)) &= (z-\alpha_0)^{-a_i-\gamma} g_i(y_1, \cdots, y_m) \\ &= (z-\alpha_0)^{-a_i-\gamma} \sum_{k=0}^{\infty} g_{i,k}(A)(z-\alpha_0)^k, \end{aligned} \tag{3.5}$$

where $g_{i,k}(A)$ is the coefficient of the Taylor expansion of $g_i(y_1, \cdots, y_m)$ in $z-\alpha_0$. We denote $G_k(A) := (g_{1,k}(A), \cdots, g_{m,k}(A))$. In particular $g_{i,0} = g_i(c)$ and $G_0 = G(c)$.

Proposition 3.1. For a given indicial locus c, the identity $(K(c)+\gamma)G(c) = 0$ holds. In particular, if $-\gamma$ is not a K-exponent, then $G(c) = 0$ (here, we need not assume that c is principle).

Proof. (A2) and (3.2) provide

$$0 = \sum_{j=1}^{m}\left(f_j(c)\frac{\partial g_i}{\partial x_j}(c) - g_j(c)\frac{\partial f_i}{\partial x_j}(c)\right) = -\sum_{j=1}^{m}\frac{\partial f_i}{\partial x_j}(c)g_j(c) - (a_i+\gamma)g_i(c),$$

which proves the proposition. □

Corollary 3.2. If $G(c) \neq 0$, then $-\gamma$ is a K-exponent. In particular, when $\gamma \neq 1$, there exists a lower indicial locus.

Corollary 3.3. Suppose c is a principle indicial locus.
(i) If $\gamma \geq 2$, $G(c) = 0$.
(ii) If $\gamma = 1$, $G(c) = k(a_1c_1, \cdots, a_mc_m)$ for some $k \in \mathbb{C}$.

Proof. (i) By the assumption, there are no negative K-exponents other than $\kappa_0 = -1$. (ii) When $\gamma = 1$, $G(c) = 0$ or $G(c)$ is an eigenvector of $\kappa_0 = -1$. Since $\kappa_0 = -1$ is a simple eigenvalue by the assumption, the statement (ii) follows from Prop.2.3. □

More generally, the next theorem holds.

Theorem 3.4. Let c be a principle indicial locus.

The equality $(K(c)+\gamma-k)G_k(A) = 0$ holds for $k = 0, 1, \cdots, \gamma-1$.

Corollary 3.5. When $\gamma \geq 2$, $G_0 = G_1 = \cdots = G_{\gamma-2} = 0$ and $G_{\gamma-1}$ is of the form $G_{\gamma-1} = h(A)\cdot(a_1c_1, \cdots, a_mc_m)$, where $h(A)$ is a certain polynomial of $\alpha_1, \cdots, \alpha_{m-1}$.

Proof. The case $\gamma = 1$ (i.e. $k = 0$ in the theorem) had been proved in Prop.3.1. Thus we consider $\gamma \geq 2$.

Since $x(z)$ in (3.4) is a solution of $dx/dz = f(x)$,

$$\begin{aligned} f_j(x(z)) &= -a_j c_j (z-\alpha_0)^{-a_j-1} - \sum_{k=1}^{\infty}(a_j-k)d_{j,k}(z-\alpha_0)^{-a_j+k-1} \\ &= -(z-\alpha_0)^{-1}a_j x_j(z) + \sum_{k=1}^{\infty} k d_{j,k}(z-\alpha_0)^{-a_j+k-1}. \end{aligned}$$

Eq.(3.3) shows

$$\frac{\partial f_i}{\partial x_j}(x(z)) = (z-\alpha_0)^{a_j-a_i-1}\frac{\partial f_i}{\partial x_j}(y), \quad \frac{\partial g_i}{\partial x_j}(x(z)) = (z-\alpha_0)^{a_j-a_i-\gamma}\frac{\partial g_i}{\partial x_j}(y),$$

where $y = (y_1, \cdots, y_m)$ is defined in (3.4). Substituting them into (A2) with (3.1) and (3.5), we have

$$\begin{aligned} 0 &= \sum_{j=1}^{m}\left((z-\alpha_0)^{a_j-a_i-1}\frac{\partial f_i}{\partial x_j}(y)\cdot(z-\alpha_0)^{-a_j-\gamma}\sum_{k=0}^{\infty} g_{j,k}(A)(z-\alpha_0)^k\right) \\ &\qquad + \sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(x(z))\left(a_j x_j(z)(z-\alpha_0)^{-1} - \sum_{k=1}^{\infty} k d_{j,k}(z-\alpha_0)^{-a_j+k-1}\right) \\ &= \sum_{j=1}^{m}(z-\alpha_0)^{-a_i-\gamma-1}\frac{\partial f_i}{\partial x_j}(y)\sum_{k=0}^{\infty} g_{j,k}(A)(z-\alpha_0)^k \\ &\quad + (a_i+\gamma)(z-\alpha_0)^{-a_i-\gamma-1}\sum_{k=0}^{\infty} g_{i,k}(A)(z-\alpha_0)^k \\ &\qquad - \sum_{j=1}^{m}(z-\alpha_0)^{a_j-a_i-\gamma}\frac{\partial g_i}{\partial x_j}(y)\sum_{k=1}^{\infty} k d_{j,k}(z-\alpha_0)^{-a_j+k-1}. \end{aligned}$$

Multiplied by $(z-\alpha_0)^{a_i+\gamma+1}$, this is rewritten as

$$\sum_{k=0}^{\infty}(z-\alpha_0)^k\sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(y) + (a_j+\gamma)\delta_{i,j}\right)g_{j,k}(A) - \sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y)\sum_{k=1}^{\infty} k d_{j,k}(z-\alpha_0)^k = 0. \tag{3.6}$$

To estimate the last term, we use induction. Recall that we consider $\gamma \geq 2$. Assume that $(K(c)+\gamma-k)G_k(A) = 0$ holds for $k = 0, 1, \cdots, n-1$, where $n \leq \gamma - 1$. Then $G_0 = G_1 = \cdots = G_{n-1} = 0$ holds because $-\gamma, -\gamma+1, \cdots, -\gamma+n-1\,(\leq -2)$ are not K-exponents. Hence, (3.6) gives

$$\sum_{k=n}^{\infty}(z-\alpha_0)^k\sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(y) + (a_j+\gamma)\delta_{i,j}\right)g_{j,k}(A) - \sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y)\sum_{k=1}^{\infty} k d_{j,k}(z-\alpha_0)^k = 0$$

(the first summation starts from $k = n$). Divide by $(z-\alpha_0)^n$ and consider the limit $z \to \alpha_0$:

$$\sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(c) + (a_j+\gamma)\delta_{i,j}\right)g_{j,n}(A) - \lim_{z\to\alpha_0}\sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y)\sum_{k=1}^{\infty} k d_{j,k}\frac{(z-\alpha_0)^k}{(z-\alpha_0)^n} = 0. \tag{3.7}$$

By the definition of $g_{i,k}(A)$, we have

$$g_i(y_1,\cdots,y_m)=\sum_{k=0}^{\infty}g_{i,k}(A)(z-\alpha_0)^k=\sum_{k=n}^{\infty}g_{i,k}(A)(z-\alpha_0)^k,$$
$$y_j=c_j+\sum_{k=1}^{\infty}d_{j,k}(z-\alpha_0)^k.$$

The derivation of both sides by z yields

$$\sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y_1,\cdots,y_m)\sum_{k=1}^{\infty}kd_{j,k}(z-\alpha_0)^{k-1}=\sum_{k=n}^{\infty}kg_{i,k}(A)(z-\alpha_0)^{k-1}$$
$$\Rightarrow\sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y_1,\cdots,y_m)\sum_{k=1}^{\infty}kd_{j,k}\frac{(z-\alpha_0)^{k-1}}{(z-\alpha_0)^{n-1}}=ng_{i,n}(A)+O(z-\alpha_0).$$

As $z\to\alpha_0$,

$$\lim_{z\to\alpha_0}\sum_{j=1}^{m}\frac{\partial g_i}{\partial x_j}(y_1,\cdots,y_m)\sum_{k=1}^{\infty}kd_{j,k}\frac{(z-\alpha_0)^{k}}{(z-\alpha_0)^{n}}=ng_{i,n}(A)$$

This and (3.7) gives

$$\sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial x_j}(c)+(a_j+\gamma-n)\delta_{i,j}\right)g_{j,n}(A)=0. \tag{3.8}$$

This proves that $(K(c)+\gamma-k)G_k(A)=0$ holds for $k=n$. The induction step continues up to $k=\gamma-1$. □

Even when the degree γ of a given quasi-homogeneous equation is larger than 1, the K-exponents are defined in a similar manner as follows. Let us consider $dx_i/dz_2=g_i(x)$ given in (A1) having the degree γ. We consider the Puiseux series solution of the form

$$x_i(z_2)=p_i(z_2-\beta_0)^{-a_i/\gamma}+\sum_{k=1}^{\infty}q_{i,k}(z_2-\beta_0)^{(-a_i+k)/\gamma},\quad i=1,\cdots,m, \tag{3.9}$$

where β_0 is a singularity and p_i, $q_{i,k}$ are constants to be determined. Substituting it into the equation, it turns out that an indicial locus $p=(p_1,\cdots,p_m)$ is given as a root the equation

$$-\frac{a_i}{\gamma}p_i=g_i(p_1,\cdots,p_m),\quad i=1,\cdots,m. \tag{3.10}$$

For a given p, $q_j=(q_{1,j},\cdots,q_{m,j})$ is iteratively determined as a solution of

$$(K_\gamma(p)-\frac{j}{\gamma}\cdot I)q_j=(\text{polynomial of } p \text{ and } q_k \text{ for } k=1,\cdots,j-1), \tag{3.11}$$

where the K-matrix $K_\gamma(p)$ is defined by

$$K_\gamma(p) = \left\{ \frac{\partial g_i}{\partial x_j}(p) + \frac{a_i}{\gamma}\delta_{i,j} \right\}_{i,j=1}^{m}. \tag{3.12}$$

The eigenvalues $\rho_0, \rho_1, \cdots, \rho_{m-1}$ are called the K-exponents. If j/γ is a K-exponent, q_j includes a free parameter. As in Prop.2.3, $\rho_0 = -1$ is always a K-exponent with the eigenvector $(a_1 p_1, \cdots, a_m p_m)$. When $\rho_1, \cdots, \rho_{m-1} \in \mathbb{N}/\gamma$, p is called a principle indicial locus.

4 Flow of the free parameters

Let $\varphi^F_{z_1}$ and $\varphi^G_{z_2}$ be the flow of F and G, respectively; $\varphi^F_{z_1}$ maps $x(0)$ to $x(z_1)$ along the orbit of vector field F, and similarly for $\varphi^G_{z_2}$. To obtain a solution of the system (2.1) as a function of z_1 and z_2, for a solution $x(z_1; A) = \varphi^F_{z_1} \circ \Phi(A)$ of $\partial x/\partial z_1 = F(x)$, we assume that $A = A(z_2)$ is a function of z_2. Let us substitute this x into the second equation $\partial x/\partial z_2 = G(x)$.

$$\text{(left hand side)} = \frac{\partial \varphi^F_{z_1}}{\partial x}(\Phi(A))\frac{d\Phi}{dz_2}(A) = \frac{\partial \varphi^F_{z_1}}{\partial x}(\Phi(A))\sum_{l=0}^{m-1}\frac{\partial \Phi}{\partial \alpha_l}(A)\frac{d\alpha_l}{dz_2}.$$

It is known that (A2) is equivalent to the identity $\varphi^F_t \circ \varphi^G_s = \varphi^G_s \circ \varphi^F_t$. The derivative of it at $s = 0$ gives

$$\frac{\partial \varphi^F_t}{\partial x}(x)G(x) = G(\varphi^F_t(x)).$$

Hence,

$$\text{(right hand side)} = G(x(z_1; A)) = G(\varphi^F_{z_1} \circ \Phi(A)) = \frac{\partial \varphi^F_{z_1}}{\partial x}(\Phi(A))G(\Phi(A)).$$

This proves

$$\sum_{l=0}^{m-1}\frac{\partial \Phi}{\partial \alpha_l}(A)\frac{d\alpha_l}{dz_2} = G(\Phi(A)) \quad \Rightarrow \quad \frac{dA}{dz_2} = \left(\frac{\partial \Phi}{\partial A}(A)\right)^{-1} G(\Phi(A)), \tag{4.1}$$

that gives the flow of $A(z_2)$.

Proposition 4.1. The right hand side of (4.1) is
(i) independent of α_0 and polynomial in $\alpha_1, \cdots, \alpha_{m-1}$,
(ii) quasi-homogeneous of degree γ with respect to the weight $(\kappa_0, \cdots, \kappa_{m-1})$.

Proof. (i) Denote $c_i = d_{i,0}$ for simplicity to express (2.5) as $x_i(z_1) = \sum_{j=0}^{\infty} d_{i,j}(z_1-$

$\alpha_0)^{-a_i+j}$. By substituting it to $\partial x_i/\partial z_2 = g_i(x)$, we obtain

$$\text{(left hand side)} = \sum_{j=0}^{\infty}(a_i - j)d_{i,j}(z_1 - \alpha_0)^{-a_i+j-1}\frac{d\alpha_0}{dz_2} + \sum_{j=0}^{\infty}\frac{dd_{i,j}}{dz_2}(z_1 - \alpha_0)^{-a_i+j},$$

$$\text{(right hand side)} = (z_1 - \alpha_0)^{-a_i-\gamma}g_i(\sum_{j=0}^{\infty} d_{1,j}(z_1 - \alpha_0)^j, \cdots, \sum_{j=0}^{\infty} d_{m,j}(z_1 - \alpha_0)^j),$$

$$= (z_1 - \alpha_0)^{-a_i-\gamma}\sum_{j=0}^{\infty} g_{i,j}(A)(\alpha_1, \cdots, \alpha_{m-1}) \cdot (z_1 - \alpha_0)^j,$$

where $g_{i,j}(A)$ is a j-th coefficient of the Taylor expansion of $g_i(\sum_{j=0}^{\infty} d_{1,j}(z_1-\alpha_0)^j, \cdots)$ in $z_1-\alpha_0$, that was introduced in Sec.3. This is a polynomial of $\alpha_1, \cdots, \alpha_{m-1}$ because so is $d_{i,j}$. Comparing the coefficients of $(z_1 - \alpha_0)^{-a_i-1}$ in both sides, we obtain

$$a_i c_i \frac{d\alpha_0}{dz_2} = g_{i,\gamma-1}(\alpha_1, \cdots, \alpha_{m-1}). \tag{4.2}$$

Similarly, coefficients of $(z_1 - \alpha_0)^{-a_i+\kappa_l}$ provide

$$(a_i - 1 - \kappa_l)d_{i,1+\kappa_l}\frac{d\alpha_0}{dz_2} + \frac{d\alpha_l}{dz_2} = g_{i,\kappa_l+\gamma}(\alpha_1, \cdots, \alpha_{m-1}), \quad l = 1, \cdots, m-1, \tag{4.3}$$

where i is chosen so that $\alpha_l = d_{i,\kappa_l}$. Eq.(4.2) shows that $d\alpha_0/dz_2$ is independent of α_0. Thus, (4.3) proves that $d\alpha_l/dz_2$ is also independent of α_0 and is polynomial in $\alpha_1, \cdots, \alpha_{m-1}$ for $l = 1, \cdots, m-1$.

(ii) Eq.(4.2) and (4.3) provide another expression of (4.1) as

$$\begin{cases} \dfrac{d\alpha_0}{dz_2} = \dfrac{g_{i,\gamma-1}(A)}{a_i c_i} =: \hat{g}_0(A), \\ \dfrac{d\alpha_l}{dz_2} = g_{i,\kappa_l+\gamma}(A) - (a_i - 1 - \kappa_l)d_{i,1+\kappa_l}(A)\dfrac{g_{i,\gamma-1}(A)}{a_i c_i} =: \hat{g}_l(A). \end{cases} \tag{4.4}$$

Remark that $g_{i,\gamma-1}(A)/a_i c_i$ is independent of i and well-defined even when $c_i = 0$ because of Corollary 3.2 and 3.4 ($\hat{g}_0(A)$ here is $h(A)$ there). As above, we put

$$g_i(\sum_{j=0}^{\infty} d_{1,j}(A)(z_1 - \alpha_0)^j, \cdots, \sum_{j=0}^{\infty} d_{m,j}(A)(z_1 - \alpha_0)^j) = \sum_{j=0}^{\infty} g_{i,j}(A) \cdot (z_1 - \alpha_0)^j.$$

Since $\lambda^j d_{i,j}(A) = d_{i,j}(\lambda \cdot A)$ by Prop.2.5, the left hand side above is invariant by $A \mapsto \lambda \cdot A$, $\alpha_0 \mapsto \lambda^{-1}\alpha_0$ and $z_1 \mapsto \lambda^{-1}z_1$ (recall that $\lambda \cdot A = (\lambda^{-1}\alpha_0, \lambda^{\kappa_1}\alpha_1, \cdots, \lambda^{\kappa_{m-1}}\alpha_{m-1})$). Thus, we obtain $g_{i,j}(\lambda \cdot A) = \lambda^j g_{i,j}(A)$ from the right hand side. This proves

$$\hat{g}_l(\lambda^{\kappa_1}\alpha_1, \cdots, \lambda^{\kappa_{m-1}}\alpha_{m-1}) = \lambda^{\kappa_l+\gamma}\hat{g}_l(\alpha_1, \cdots, \alpha_{m-1}) \tag{4.5}$$

for $l = 0, 1, \cdots, m-1$. □

Now we investigate the system (4.1) or equivalently (4.4);

$$\frac{dA}{dz_2} = \left(\frac{\partial \Phi}{\partial A}(A)\right)^{-1} G(\Phi(A)) \quad \Leftrightarrow \quad \begin{cases} \dfrac{d\alpha_0}{dz_2} = \hat{g}_0(A) \\ \dfrac{d\alpha_i}{dz_2} = \hat{g}_i(A), \quad i = 1, \cdots, m-1 \end{cases} \tag{4.6}$$

satisfying (4.5). The next lemma immediately follows from (4.5).

Lemma 4.2. A monomial $\alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_{m-1}^{n_{m-1}}$ can be included in $\hat{g}_l(A)$ only when a tuple of nonnegative integers $(n_1, \cdots, n_{m-1})$ satisfies $\kappa_1 n_1 + \cdots + \kappa_{m-1} n_{m-1} = \kappa_l + \gamma$.

From the lemma, it turns out that $\hat{g}_l(A)$ for $l = 1, \cdots, m-1$ does not include a constant term when c is an isolated principle indicial locus (i.e. $\kappa_l > 0$). Applying the lemma to $l = 0$ gives $\kappa_1 n_1 + \cdots + \kappa_{m-1} n_{m-1} = \gamma - 1$. When $\gamma = 1$, $\hat{g}_0$ is a constant function. When $\gamma \geq 2$, $\hat{g}_0$ does not include a constant term. For example when $\gamma = 2$, there exists a K-exponent $\kappa = 1$. More generally, there exists a K-exponent smaller than γ if $\hat{g}_0$ is not identically zero.

Let $\xi = (\xi_0, \cdots, \xi_{m-1})$ be an indicial locus of (4.6) and $\rho_0 = -1, \rho_1, \cdots, \rho_{m-1}$ its K-exponents.

Lemma 4.3. Besides $\rho_0 = -1$, (4.6) has another negative exponent $\rho_1 = -1/\gamma$.

Proof. The proof is based on the fact that the right hand side of (4.6) is independent of α_0. The K-matrix of (4.6) defined by (3.12) is given as

$$K_\gamma(\xi) = \left(\begin{array}{c|ccc} 0 & \partial\hat{g}_0/\partial\alpha_1 & \cdots & \partial\hat{g}_0/\partial\alpha_{m-1} \\ \hline 0 & & \left\{\dfrac{\partial \hat{g}_i}{\partial \alpha_j}\right\}_{i,j=1}^{m-1} & \end{array}\right) + \left(\begin{array}{c|ccc} -1/\gamma & & 0 & \\ \hline & \kappa_1/\gamma & & \\ 0 & & \ddots & \\ & & & \kappa_{m-1}/\gamma \end{array}\right). \tag{4.7}$$

Hence, its eigenvalues are $-1/\gamma$ and K-exponents of the subsystem $d\alpha_i/dz_2 = \hat{g}_i(A)$, $i = 1, \cdots, m-1$. Since the subsystem has a K-exponent -1, the proof is completed. □

Remark 4.4. The first expression of (4.6) implies that it is obtained from $dx/dz_2 = G(x)$ by the coordinate transformation $x = \Phi(A)$. For the system $dx/dz_2 = G(x)$, let $p = (p_1, \cdots, p_m)$ be an indicial locus. Then, $\xi = \Phi^{-1}(p)$ is an indicial locus of (4.6) as long as ξ is in the domain on which Φ is a diffeomorphism. In this case, the K-exponents of (4.6) at $\Phi^{-1}(p)$ coincide with those of G at p due to Prop.2.5 and Thm.2.7.

4.1 $\gamma = 1$

When $\gamma = 1$, $d\alpha_0/dz_2 = \hat{g}_0(A) = g_i(c)/(a_i c_i)$ (=: constant k_1) due to (4.2). It is solved as $\alpha_0 = k_1 z_2 + k_2$. Thus, we consider the system

$$\frac{d\alpha_i}{dz_2} = \hat{g}_i(\alpha_1, \cdots, \alpha_{m-1}), \quad i = 1, \cdots, m-1, \tag{4.8}$$

which is of degree $\gamma = 1$ with respect to the weight $(\kappa_1, \cdots, \kappa_{m-1})$. Let $\xi = (\xi_1, \cdots, \xi_{m-1})$ be an indicial locus and $\rho_0 = -1, \rho_2, \cdots, \rho_{m-1}$ its K-exponents (we skip $\rho_1 = -1/\gamma$ shown in Lemma 4.3 because it arises from the equation of α_0). The Laurent series solution is written as

$$\alpha_i(z_2) = (z_2 - \beta_0)^{-\kappa_i} \left(\xi_i + \sum_{j=1}^{\infty} \eta_{i,j} (z_2 - \beta_0)^j \right) =: (z_2 - \beta_0)^{-\kappa_i} y_i, \tag{4.9}$$

with coefficients $\eta_{i,j}$ and a pole β_0. Thus, $x(z_1, z_2)$ satisfying both equations of (2.1) is given by

$$x_i(z_1, z_2) = (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(\alpha_1, \cdots, \alpha_{m-1})(z_1 - \alpha_0)^j \right) \tag{4.10}$$

$$= (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(y_1, \cdots, y_{m-1}) \frac{(z_1 - \alpha_0)^j}{(z_2 - \beta_0)^j} \right), \tag{4.11}$$

where we used $d_{i,j}(\lambda \cdot A) = \lambda^j d_{i,j}(A)$. This gives a solution of (2.1) as a function of (z_1, z_2) as long as the right hand side converges. Assume that the series (4.9) converges when $|z_2 - \beta_0| \leq \varepsilon_2$ and (4.10) converges when $|z_1 - \alpha_0| \leq \varepsilon_1$. Let ε be a small number. Now we consider the solution restricted on the line

$$z_1 - \alpha_0 = \varepsilon(z_2 - \beta_0) \quad \Leftrightarrow \quad z_2 := q(z_1) = \frac{z_1 + \varepsilon\beta_0 - k_2}{\varepsilon + k_1}, \tag{4.12}$$

This yields

$$x_i(z_1, q(z_1)) = (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(y_1, \cdots, y_{m-1}) \cdot \varepsilon^j \right), \tag{4.13}$$

$$y_i = \xi_i + \sum_{j=1}^{\infty} \eta_{i,j} (z_1 - \alpha_0)^j / \varepsilon^j.$$

This is a convergent series when $|\varepsilon| < |\varepsilon_1|$ and $|z_1 - \alpha_0| < \varepsilon\varepsilon_2$. Expanding it gives a new Laurent series solution

$$x_i(z_1, q(z_1)) = (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(\xi_1, \cdots, \xi_{m-1}) \cdot \varepsilon^j + O(z_1 - \alpha_0) \right), \tag{4.14}$$

with the indicial locus

$$(c_1 + \sum_{j=1}^{\infty} d_{1,j}(\xi_1, \cdots, \xi_{m-1})\varepsilon^j, \cdots, c_m + \sum_{j=1}^{\infty} d_{m,j}(\xi_1, \cdots, \xi_{m-1})\varepsilon^j). \tag{4.15}$$

If there are $k-2$ nonnegative integers among $\rho_2, \cdots, \rho_{m-1}$, (4.14) includes $k-1$ free parameters (α_0 and $k-2$ parameters in $\eta_{i,j}$). Suppose ξ is a principle indicial locus of (4.8); all $\rho_2, \cdots, \rho_{m-1}$ are nonnegative integers, then (4.14) represents $m-1$ parameter family of Laurent series. This is not a solution of $dx/dz_1 = F$ but a combination of F and G;

$$\begin{aligned}\frac{d}{dz_1}x_i(z_1, q(z_1)) &= \frac{\partial}{\partial z_1}\Big|_{z_2=q(z_1)} x_i(z_1, z_2) + \frac{\partial}{\partial z_2}\Big|_{z_2=q(z_1)} x_i(z_1, z_2)\cdot\frac{dq}{dz_1} \\ &= f_i(x(z_1, q(z_1))) + \frac{1}{\varepsilon + k_1} g_i(x(z_1, q(z_1))).\end{aligned} \tag{4.16}$$

This implies that there exists a lower indicial locus of the vector field $F + G/(\varepsilon + k_1)$ whose K-exponents are given by $\rho_0 = -1, \rho_1 = -1$ and $\rho_2, \cdots, \rho_{m-1}$. Since the number of free parameters is smaller than m, $x(z_1, q(z_1))$ is a *non*-principle Laurent series solution.

Proposition 4.5. Suppose ε and $|z_1 - \alpha_0|$ are sufficiently small. The Laurent series solution (4.14) is convergent and it satisfies (4.16). In particular, there exists an indicial locus of the vector field $F + G/(\varepsilon + k_1)$ whose K-exponents are given by $\rho_0 = -1, \rho_1 = -1$ and $\rho_2, \cdots, \rho_{m-1}$ for any small $\varepsilon \neq 0$.

Although the series (4.14) does not converge when ε is large, K-exponents $\rho_0 = -1, \rho_1 = -1, \rho_2, \cdots, \rho_{m-1}$ can be defined as the eigenvalues of the K-matrix of $F + G/(\varepsilon + k_1)$. Since they analytically depend on ε and are constants in ε when ε is small, we can take $\varepsilon \to \infty$ and obtain the main theorem in this subsection.

Theorem 4.6. There exists a lower indicial locus of the vector field F whose K-exponents are given by $\rho_0 = -1, \rho_1 = -1$ and $\rho_2, \cdots, \rho_{m-1}$.

The expression of the corresponding indicial locus $c_i + \sum_{j=1}^{\infty} d_{i,j}(\xi_1, \cdots, \xi_{m-1})\varepsilon^j$ makes sense by an analytic continuation when ε is large. It is notable that F has a lower indicial locus, but G need not have. When $\gamma = 1$, the difference of assumptions for F and G are only (A3). This suggests that (A3) is essential for the existence of lower indicial loci. This is illustrated in the next example.

Example 4.7. Let us consider the two Hamiltonian functions

$$\left\{\begin{array}{l} H_F(q_1, p_1, q_2, p_2) = (p_1^2/2 - 2q_1^3) + (p_2^2/2 - 2q_2^3), \\ H_G(q_1, p_1, q_2, p_2) = p_1^2/2 - 2q_1^3, \end{array}\right. \tag{4.17}$$

and let F and G be the corresponding Hamiltonian vector fields. The vector field F is a direct product of the two-dimensional system given in Example 2.8. The weight is $(a_1, b_1, a_2, b_2) = (2, 3, 2, 3)$ and $\gamma = 1$. The only F satisfies the condition (A3). The vector field F has three indicial loci with K-exponents as

$$\begin{aligned} &(\mathrm{P}_1) : (q_1, p_1, q_2, p_2) = (1, -2, 0, 0), \quad \kappa = -1, 2, 3, 6, \\ &(\mathrm{P}_2) : (q_1, p_1, q_2, p_2) = (0, 0, 1, -2), \quad \kappa = -1, 2, 3, 6, \\ &(\mathrm{P}_3) : (q_1, p_1, q_2, p_2) = (1, -2, 1, -2), \quad \kappa = -1, -1, 6, 6. \end{aligned}$$

The only (P_3) is a lower locus. For the first one (P_1), the Laurent series solution of (q_1, p_1) is given by (2.15), which has a free parameter $d_{1,6}$ at 6-th place (counting from T^{-3}). (q_2, p_2) is a holomorphic solution. Thus, free parameters are the constant terms of the Taylor expansion those are in 2nd and 3rd places counting from T^{-2} and T^{-3}, respectively. These three parameters correspond to $\kappa = 2, 3, 6$. Similarly for (P_2). For (P_3), both of (q_1, p_1) and (q_2, p_2) are Laurent series solutions of the form (2.15). Hence, there are two free parameters at 6-th place, that correspond to $\kappa = 6, 6$.

For (P_1), the flow of the free parameters are given by

$$\alpha_0' = -1,\ \alpha_1' = -\alpha_2,\ \alpha_2' = -6\alpha_1^2,\ \alpha_3' = 0.$$

It has only one indicial locus and its K-exponents are given by $\kappa = -1, -1, 6, 6$, that confirms Theorem 4.6.

On the other hand, the vector field G has only one indicial locus whose K-exponents are given by $(-1, 2, 3, 6)$. There are no lower indicial loci. When $\gamma = 1$, we cannot distinguish F and G by assumptions (A1) and (A2). This suggests that the assumption (A3) "$F(x) = 0$ only when $x = 0$" plays a crucial role for the existence of a lower indicial locus, though the authors do not know a clear statement.

Fig. 1 represents a schematic view of the flow of F. In Chiba [3, 6], the geometry of families of Laurent series solutions of quasi-homogeneous vector fields are investigated via the weighted projective space (in this example, it is $\mathbb{C}P^4(2, 3, 2, 3)$). This is a compact manifold constructed by attaching an $m-1$ dimensional manifold, denoted by D in the figure, to the original phase space $\mathbb{C}^m$ at infinity. A given vector field on $\mathbb{C}^m$ is extended to a vector field on $\mathbb{C}^m \cup D$, and it is shown that there is a one-to-one correspondence between indicial loci and fixed points on D. Further, eigenvalues of the Jacobi matrix of F at the fixed point on D coincide with K-exponents. In the figure, (P_1) and (P_2) are stable fixed points that correspond to principle indicial loci, respectively, and (P_3) is a saddle point that corresponds to the lower indicial locus. The space is divided into two regions that are occupied by two families of principle Laurent series solutions, and their boundary is occupied by the non-principle Laurent series solutions. The red dotted orbit indicates an orbit of the vector field $F + G/(\varepsilon + k_1)$ given in Proposition 4.5.

4.2 $\gamma \geq 2$

Let us consider the case $\gamma \geq 2$. For the equation $d\alpha_0/dz_2 = \hat{g}_0(A)$, the right hand side is not a constant because of Lemma 4.2, however, it may become identically zero. For it, we can prove the next proposition. The proof is given in Appendix.

Proposition 4.8. Let c be an isolated principle indicial locus of $F = (f_1, \cdots, f_m)$ and $\kappa_1 \geq 1$ be the smallest K-exponent other than $\kappa_0 = -1$. If the vector $(d_{1,\kappa_1}, \cdots, d_{m,\kappa_1})$ is not an eigenvector associated with a zero eigenvalue of the Jacobi matrix of G at c, then $\hat{g}_0(A)$ is not identically zero.

In what follows, we assume that $\hat{g}_0(A) \not\equiv 0$, not identically zero. We rewrite (4.6) as

$$\frac{d\alpha_i}{d\alpha_0} = \frac{\hat{g}_i(A)}{\hat{g}_0(A)}, \quad i = 1, \cdots, m-1, \quad A = (\alpha_1, \cdots, \alpha_{m-1}). \tag{4.18}$$

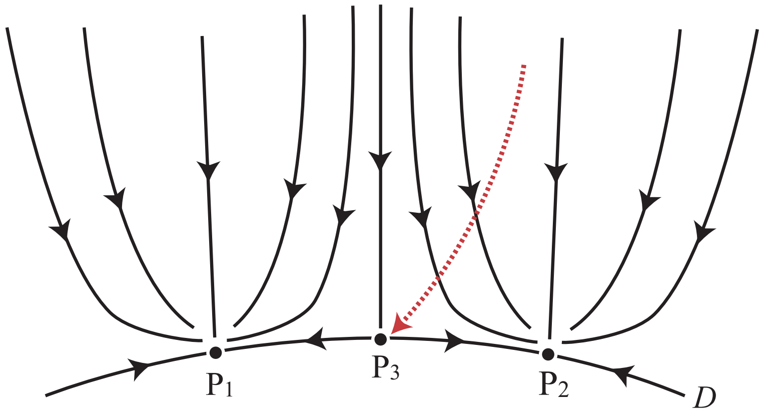

Fig. 1: A schematic view of the flow of F on the compactified space. The original phase space $\mathbb{C}^4$ is compactified by attaching D at "infinity". There are three fixed points on D that correspond to three indicial loci. The red dotted orbit indicates an orbit of the vector field $F + G/(\varepsilon + k_1)$ given in Proposition 4.5.

Because of (4.5), it has the degree 1 with respect to the weight $(\kappa_1, \cdots, \kappa_{m-1})$.

Lemma 4.9. Let $\xi = (\xi_0, \cdots, \xi_{m-1})$ be an indicial locus of (4.6). Assume that $\hat{g}_0(\xi) \neq 0$. Then, (4.18) has an indicial locus $\widetilde{\xi} = (\widetilde{\xi}_1, \cdots, \widetilde{\xi}_{m-1})$ with $\widetilde{\xi}_i = \xi_0^{\kappa_i}\xi_i$.

Proof. Since (4.6) has the degree γ with respect to the weight $\kappa_0, \cdots, \kappa_{m-1}$, the indicial locus is given by the root of the equation (see (3.10))

$$\hat{g}_0(\xi) = -\kappa_0\xi_0/\gamma = \xi_0/\gamma, \quad \hat{g}_i(\xi) = -\kappa_i\xi_i/\gamma.$$

Putting $\xi_i = \xi_0^{-\kappa_i}\widetilde{\xi}_i$ gives

$$\begin{aligned} \xi_0 \frac{\hat{g}_i(\xi)}{\hat{g}_0(\xi)} = -\kappa_i\xi_i \quad &\Rightarrow \quad \xi_0 \frac{\xi_0^{-\kappa_i-\gamma}\hat{g}_0(\widetilde{\xi})}{\xi_0^{-\kappa_0-\gamma}\hat{g}_0(\widetilde{\xi})} = -\kappa_i \xi_0^{-\kappa_i}\widetilde{\xi}_i \\ &\Rightarrow \quad \frac{\hat{g}_i(\widetilde{\xi})}{\hat{g}_0(\widetilde{\xi})} = -\kappa_i\widetilde{\xi}_i \end{aligned} \tag{4.19}$$

Hence, $\widetilde{\xi}$ satisfies the definition of an indicial locus of (4.18). □

Proposition 4.10. Let $\rho_0 = -1$, $\rho_1 = -1/\gamma, \rho_2, \cdots, \rho_{m-1}$ be K-exponents of (4.6) at a locus ξ (see Lemma 4.3). Then, the K-exponents of (4.18) at the locus $\widetilde{\xi}$ above are given by $\rho_0 = -1$ and $\gamma\rho_2, \cdots, \gamma\rho_{m-1}$.

Proof. Denote

$$\mathrm{grad}(\hat{g}_0) = \left(\frac{\partial \hat{g}_0}{\partial \alpha_1}(\xi), \cdots, \frac{\partial \hat{g}_0}{\partial \alpha_{m-1}}(\xi)\right), \quad (J\hat{g}) = \left\{\frac{\partial \hat{g}_i}{\partial \alpha_j}(\xi)\right\}_{i,j=1}^{m-1},$$

and $\mathrm{diag}(\kappa) = \mathrm{diag}(\kappa_1, \cdots, \kappa_{m-1})$. Then, the K-matrix of (4.6) given in (4.7) is written as

$$K_\gamma(\xi) = \left(\begin{array}{c|c} 0 & \mathrm{grad}(\hat{g}_0) \\ \hline 0 & (J\hat{g}) \end{array}\right) + \left(\begin{array}{c|c} -1/\gamma & 0 \\ \hline 0 & \mathrm{diag}(\kappa)/\gamma \end{array}\right). \tag{4.20}$$

Further, put $a = 1/(\gamma \hat{g}_0(\xi))$ and $v = (\kappa_1 \xi_1, \cdots, \kappa_{m-1}\xi_{m-1})^T$, which is an eigenvector associated with the K-exponent $\rho_0 = -1$ of the subsystem (4.8). Define

$$P = \left(\begin{array}{c|c} a & 0 \\ \hline -v & I \end{array}\right). \tag{4.21}$$

Then, we can verify that

$$P^{-1}K_\gamma(\xi)P = \left(\begin{array}{c|c} k_{11} & \mathrm{grad}(\hat{g}_0)/a \\ \hline k_{21} & k_{22} \end{array}\right),$$

where

$$\begin{cases} k_{11} = -\mathrm{grad}(\hat{g}_0)\cdot v/a - 1/\gamma, \\ k_{21} = -(\mathrm{grad}(\hat{g}_0)\cdot v)v/a - v/\gamma - (J\hat{g})v - \mathrm{diag}(\kappa)v/\gamma, \\ k_{22} = (J\hat{g}) + v\cdot \mathrm{grad}(\hat{g}_0)/a + \mathrm{diag}(\kappa)/\gamma. \end{cases}$$

Since v is an eigenvector of the K-exponent -1 of (4.8), we have

$$((J\hat{g}_0) + \mathrm{diag}(\kappa)/\gamma)v = -v.$$

From the derivative of (4.5) for $l = 0$ at $\lambda = 1$, we have

$$\sum_{j=1}^{m-1} \frac{\partial \hat{g}_0}{\partial \alpha_j}(\xi)\kappa_j \xi_j = (-1+\gamma)\hat{g}_0(\xi).$$

By using these two equalities, we can show that $k_{11} = -1$ and $k_{21} = 0$. Thus, the eigenvalues of k_{22} are $\rho_1 = -1/\gamma$ and $\rho_2, \cdots, \rho_{m-1}$. This implies that the eigenvalues of $\gamma\cdot k_{22}$ are -1 and $\gamma\rho_2, \cdots, \gamma\rho_{m-1}$. Hence, it is sufficient to prove that the K-matrix of (4.18) at $\widetilde{\xi}$ is conjugate to $\gamma \cdot k_{22}$.

By a straightforward calculation with the aid of (4.19) and (3.3), it is easy to see that the (i,j)-component of the K-matrix of (4.18) is given by

$$\begin{aligned} \widetilde{K}(\widetilde{\xi})_{i,j} &= \frac{1}{\hat{g}_0(\widetilde{\xi})}\left(\frac{\partial \hat{g}_i}{\partial \alpha_j}(\widetilde{\xi}) + \kappa_i \widetilde{\xi}_i \frac{\partial \hat{g}_0}{\partial \alpha_j}(\widetilde{\xi})\right) + \kappa_i \delta_{ij} \\ &= \frac{\xi_0}{\hat{g}_0(\xi)}\left(\xi_0^{\kappa_i - \kappa_j}\frac{\partial \hat{g}_i}{\partial \alpha_j}(\xi) + \xi_0^{\kappa_i-\kappa_j-1}\kappa_i\xi_i\frac{\partial \hat{g}_0}{\partial \alpha_j}(\xi)\right) + \kappa_i\delta_{ij}. \end{aligned}$$

Put $Q = \mathrm{diag}(\xi_0^{\kappa_1}, \cdots, \xi_0^{\kappa_{m-1}})$. Then, we can show that $Q^{-1}\widetilde{K}(\widetilde{\xi})Q = \gamma k_{22}$. □

Let $\xi = (\xi_0, \cdots, \xi_{m-1})$ be an indicial locus of (4.6) and assume that the associated Puiseux series solution includes $m-1$ free parameters. As is shown in the end of Sec.3, its K-exponents should be $\rho_i \in \mathbb{N}/\gamma$ for $i = 2, \cdots, m-1$. Then, $\widetilde{\xi}$ is a principle indicial locus of the system (4.18) with K-exponents $-1, \gamma\rho_2, \cdots, \gamma\rho_{m-1}$ satisfying $\gamma\rho_2, \cdots, \gamma\rho_{m-1} \in \mathbb{N}$. Since (4.18) is a rational vector field of degree 1 with respect to the weight $(\kappa_1, \cdots, \kappa_{m-1})$, it has a Laurent series solution of the form

$$\alpha_i(\alpha_0) = (\alpha_0 - \beta_0)^{-\kappa_i} \left(\widetilde{\xi}_i + \sum_{j=1}^{\infty} \widetilde{\eta}_{i,j} (\alpha_0 - \beta_0)^j \right) =: (\alpha_0 - \beta_0)^{-\kappa_i} \widetilde{y}_i, \tag{4.22}$$

for $i = 1, \cdots, m-1$. Free parameters are included in $\widetilde{\eta}_{i,j}$ for $j = \gamma\rho_2, \cdots, \gamma\rho_{m-1}$. Hence, $x(z_1, \alpha_0)$ satisfying both equations of (2.1) is given by

$$x_i(z_1, \alpha_0) = (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(\alpha_1, \cdots, \alpha_{m-1})(z_1 - \alpha_0)^j \right) \tag{4.23}$$

$$= (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(\widetilde{y}_1, \cdots, \widetilde{y}_{m-1}) \frac{(z_1 - \alpha_0)^j}{(\alpha_0 - \beta_0)^j} \right), \tag{4.24}$$

where $\alpha_0 = \alpha_0(z_2)$ is related to z_2 though $d\alpha_0/dz_2 = \hat{g}_0(A)$. Assume that the series (4.22) converges when $|\alpha_0 - \beta_0| \leq \varepsilon_2$ and (4.23) converges when $|z_1 - \alpha_0| \leq \varepsilon_1$. Let ε be a small number. Now we consider the solution restricted on the line

$$z_1 - \alpha_0 = \varepsilon(\alpha_0 - \beta_0) \quad \Leftrightarrow \quad \alpha_0 := q(z_1) = \frac{z_1 + \varepsilon\beta_0}{1 + \varepsilon}, \tag{4.25}$$

This yields

$$x_i(z_1, q(z_1)) = (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1}^{\infty} d_{i,j}(\widetilde{y}_1, \cdots, \widetilde{y}_{m-1}) \cdot \varepsilon^j \right), \tag{4.26}$$

$$\widetilde{y}_i = \widetilde{\xi}_i + \sum_{j=1}^{\infty} \widetilde{\eta}_{i,j} (z_1 - \alpha_0)^j / \varepsilon^j.$$

This is a convergent series when $|\varepsilon| < |\varepsilon_1|$ and $|z_1 - \alpha_0| < \varepsilon\varepsilon_2$. Expanding it gives a new Laurent series solution as in Section 4.1. If all $\gamma\rho_2, \cdots, \gamma\rho_{m-1}$ are positive integers, then it represents $m-1$ parameter family of Laurent series. This satisfies

$$\begin{aligned} \frac{d}{dz_1} x_i(z_1, q(z_1)) &= \frac{\partial}{\partial z_1}\Big|_{\alpha_0 = q(z_1)} x_i(z_1, \alpha_0) + \frac{\partial}{\partial \alpha_0}\Big|_{\alpha_0 = q(z_1)} x_i(z_1, \alpha_0) \cdot \frac{dq}{dz_1} \\ &= f_i(x(z_1, q(z_1))) + \frac{1}{1+\varepsilon} \frac{\partial}{\partial \alpha_0}\Big|_{\alpha_0 = q(z_1)} x_i(z_1, \alpha_0). \end{aligned}$$

On the other hand, we have

$$g_i(x) = \frac{\partial x_i}{\partial z_2} = \frac{\partial x_i}{\partial \alpha_0} \cdot \frac{d\alpha_0}{dz_2} = \frac{\partial x_i}{\partial \alpha_0} \cdot \hat{g}_0(A).$$

This shows

$$\frac{d}{dz_1} x_i(z_1, q(z_1)) = f_i(x(z_1, q(z_1))) + \frac{1}{1+\varepsilon} \frac{g_i(x(z_1, q(z_1)))}{\hat{g}_0(A)}. \tag{4.27}$$

Here, $\hat{g}_0(A)$ is regarded as a function of $x = x(z_1, q(z_1))$ through $A = \Phi^{-1}(x)$. By the same way as Section 4.1, we obtain the next results.

Proposition 4.11. Suppose ε and $|z_1 - \alpha_0|$ are sufficiently small. The Laurent series solution (4.26) is convergent and it satisfies (4.27). In particular, there exists an indicial locus of the vector field $F + G/(\hat{g}_0 \cdot (1+\varepsilon))$ whose K-exponents are given by $\rho_0 = -1, \rho_1 = -\gamma$ and $\gamma\rho_2, \cdots, \gamma\rho_{m-1}$ for any small $\varepsilon \neq 0$.

Theorem 4.12. There exists a lower indicial locus of the vector field F whose K-exponents are given by $\rho_0 = -1, \rho_1 = -\gamma$ and $\gamma\rho_2, \cdots, \gamma\rho_{m-1}$.

Example 4.13. Let us consider the two Hamiltonian functions

$$\begin{cases} H_F(q_1, p_1, q_2, p_2) = 2p_1p_2 + 3p_2^2 q_1 + q_1^4 - q_1^2 q_2 - q_2^2, \\ H_G(q_1, p_1, q_2, p_2) = p_1^2 + 2p_1p_2q_1 - q_1^5 + p_2^2 q_2 + 3q_1^3 q_2 - 2q_1 q_2^2, \end{cases} \tag{4.28}$$

and let F and G be the corresponding Hamiltonian vector fields. This is known as the autonomous version of the 4-dimensional first Painlevé equation [6].

The weight is $(a_1, b_1, a_2, b_2) = (2, 5, 4, 3)$ and $\gamma = 3$. With this weight, the weighted degrees of Hamiltonian are $\deg(H_F) = 8$ and $\deg(H_G) = 10$.

The vector field F has two indicial loci with K-exponents as

$$(\mathrm{P}_1) : (q_1, p_1, q_2, p_2) = (1, 1, 1, -1), \quad \kappa = -1, 2, 5, 8,$$
$$(\mathrm{P}_2) : (q_1, p_1, q_2, p_2) = (3, 27, 0, -3), \quad \kappa = -3, -1, 8, 10.$$

The only (P_2) is a lower locus. It is known that for Hamiltonian vector fields, the K-exponents always appear as a pair in the sense that;

Proposition 4.14 [6]. For a quasi-homogeneous Hamiltonian system F satisfying (A1), if κ is a Kovalevskaya exponent, so is μ given by $\kappa + \mu = \deg(H_F) - 1$. Further, the following formula holds

$$\kappa + \mu = \deg(H_F) - 1 = \deg(q_i) + \deg(p_i). \tag{4.29}$$

In this example, it means that

$$\kappa_i + \kappa_{4-i-1} = \deg(q_j) + \deg(p_j) = \deg(H_F) - 1 = 7, \quad i = 0, 1, \ j = 1, 2.$$

Applying the proposition to $\kappa_0 = -1$, it turns out that $\deg(H_F)$ is always a K-exponent. By Theorem 4.12, $\kappa_1 = -\gamma$ is a K-exponent for a lower indicial locus. In this example, this means $(-3) + \kappa_2 = 7$ and $\kappa_2 = 10 = \deg(H_G)$. Thus, we can obtain K-exponents of lower indicial locus from the weight of Hamiltonian functions.

For (P_1), the flow of the free parameters are given by

$$\alpha_0' = 3\alpha_1, \ \alpha_1' = -\frac{3}{2}\alpha_2, \ \alpha_2' = -54\alpha_1^4, \ \alpha_3' = 42\alpha_1^3\alpha_2.$$

It has three lower indicial loci and their K-exponents are given by $\rho = -1/3, -1, 8/3, 10/3$, that confirms Theorem 4.12. The vector field G has a lower indicial locus whose K-exponents are also $\rho = -1/3, -1, 8/3, 10/3$. As was explained in Remark 4.4, it is induced from the lower locus of the parameter flow.

A The proof of Proposition 4.8.

We again state the proposition;

Proposition A.1. Let c be an isolated principle indicial locus of $F = (f_1, \cdots, f_m)$ and $\kappa_1 \geq 1$ be the smallest K-exponent other than $\kappa_0 = -1$. If the vector $(d_{1,\kappa_1}, \cdots, d_{m,\kappa_1})$ is not an eigenvector associated with a zero eigenvalue of the Jacobi matrix of G at c, then $\hat{g}_0(A)$ is not identically zero.

Proof. The first half of the proof is similar to that of Prop.4.1. We repeat it to fix our notation. A Laurent series solution of $dx_i/dz_1 = f_i(x)$ is given by

$$
\begin{aligned}
x_i &= (z_1 - \alpha_0)^{-a_i} \left(c_i + \sum_{j=1} d_{i,j}(A)(z_1 - \alpha_0)^j \right), \\
&=: (z_1 - \alpha_0)^{-a_i} y_i(A), \quad A = (\alpha_1(z_2), \cdots, \alpha_{m-1}(z_2)).
\end{aligned}
$$

Substituting it into $dx_i/dz_2 = g_i(x)$, for the left hand side we have

$$
\begin{aligned}
\frac{dx_i}{dz_2} &= a_i c_i (z_1 - \alpha_0)^{-a_i - 1} \frac{d\alpha_0}{dz_2} + \sum_{j=1} (a_i - j) d_{i,j}(A)(z_1 - \alpha_0)^{j - a_i - 1} \frac{d\alpha_0}{dz_2} \\
&\quad + \sum_{j=1} \frac{d}{dz_2} (d_{i,j}(A)) \cdot (z_1 - \alpha_0)^{j - a_i}.
\end{aligned}
$$

For the right hand side,

$$
\begin{aligned}
g_i(x) &= (z_1 - \alpha_0)^{-a_i - \gamma} g_i(y_1, \cdots, y_m) \\
&=: (z_1 - \alpha_0)^{-a_i - \gamma} \sum_{k=0} g_{i,k}(A)(z_1 - \alpha_0)^k,
\end{aligned}
$$

where $g_{i,k}(A)$ is a coefficient of the Taylor expansion of $g_i(y_1, \cdots, y_m)$. They are given through

$$
g_i(y_1, \cdots, y_m) = g_i(c) + \sum_{l=1}^{m} \frac{\partial g_i}{\partial x_l}(c) \cdot \left(\sum_{j=1} d_{l,j}(A)(z_1 - \alpha_0)^j \right) + \cdots . \tag{A.1}
$$

Comparing coefficients of both sides of $dx_i/dz_2 = g_i(x)$, we obtain $g_{i,0} = \cdots =$

$g_{i,\gamma-2} = 0$ and

$$
\begin{aligned}
(z_1-\alpha_0)^{-a_i-1}: &\quad a_i c_i \frac{d\alpha_0}{dz_2} = g_{i,\gamma-1}(A) \\
(z_1-\alpha_0)^{-a_i}: &\quad (a_i-1)d_{i,1}(A)\frac{d\alpha_0}{dz_2} = g_{i,\gamma}(A) \\
(z_1-\alpha_0)^{-a_i+1}: &\quad (a_i-2)d_{i,2}(A)\frac{d\alpha_0}{dz_2} + \frac{d}{dz_2}d_{i,1}(A) = g_{i,\gamma+1}(A) \\
&\quad \vdots \\
(z_1-\alpha_0)^{-a_i+j}: &\quad (a_i-j-1)d_{i,j+1}(A)\frac{d\alpha_0}{dz_2} + \frac{d}{dz_2}d_{i,j}(A) = g_{i,\gamma+j}(A) \\
&\quad \vdots
\end{aligned}
$$

Now we assume that $\hat{g}_0(A) = g_{i,\gamma-1}(A)/(a_i c_i) = 0$ and we will derive a contradiction. Since $d\alpha_0/dz_2 = \hat{g}_0(A) = 0$, the above equations yield $g_{i,\gamma}(A) = 0$ and

$$
\frac{d}{dz_2}d_{i,j}(A) = g_{i,\gamma+j}(A), \quad j = 1, 2, \cdots. \tag{A.2}
$$

Let $\kappa_1 \geq 1$ be the smallest K-exponent. Then, $d_{i,j}(A) = 0$ for $j = 1, \cdots, \kappa_1 - 1$ and $i = 1, \cdots, m$, and $d_{l,\kappa_1}(A) = \alpha_1$ for some l by the definition of the free parameter α_1 (see Definition 2.4 below). Hence, (A.1) becomes

$$
g_i(y_1, \cdots, y_m) = g_i(c) + \sum_{l=1}^{m} \frac{\partial g_i}{\partial x_l}(c) \cdot (d_{l,\kappa_1}(A)(z_1-\alpha_0)^{\kappa_1} + \cdots) + O((z_1-\alpha_0)^{2\kappa_1}).
$$

This provides

$$
g_{i,\kappa_1}(A) = \sum_{l=1}^{m} \frac{\partial g_i}{\partial x_l}(c) d_{l,\kappa_1}(A).
$$

Further, $d_{i,j}(A) = 0$ for $j = 1, \cdots, \kappa_1 - 1$ with (A.2) gives $g_{i,\gamma+1} = \cdots = g_{i,\gamma+\kappa_1-1} = 0$. In particular,

$$
g_{i,\kappa_1}(A) = 0 = \sum_{l=1}^{m} \frac{\partial g_i}{\partial x_l}(c) d_{l,\kappa_1}(A).
$$

It contradicts with the assumption of the proposition. □

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