The third, fifth and sixth Painlevé equations on weighted projective spaces

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Abstract

The third, fifth and sixth Painlevé equations are studied by means of the weighted projective spaces $\mathbb{C}P^3(p,q,r,s)$ with suitable weights (p,q,r,s) determined by the Newton polyhedrons of the equations. Singular normal forms of the equations, symplectic atlases of the spaces of initial conditions, Riccati solutions and Boutroux's coordinates are systematically studied in a unified way with the aid of the orbifold structure of $\mathbb{C}P^3(p,q,r,s)$ and dynamical systems theory.
Keywords: the Painlevé equations; weighted projective space

1 Introduction

The first to sixth Painlevé equations written in Hamiltonian forms are given by

$$(\mathrm{P_J}) : \frac{dx}{dz} = -\frac{\partial H_J}{\partial y}, \quad \frac{dy}{dz} = \frac{\partial H_J}{\partial x}, \tag{1.1}$$

$J = \mathrm{I}, \mathrm{II}, \mathrm{IV}, \mathrm{III}(D_8), \mathrm{III}(D_7), \mathrm{III}(D_6), \mathrm{V}, \mathrm{VI}$, with the Hamiltonian functions defined as

$$\begin{aligned}
H_{\mathrm{I}} &= \frac{1}{2}x^2 - 2y^3 - zy, \\
H_{\mathrm{II}} &= \frac{1}{2}x^2 - \frac{1}{2}y^4 - \frac{1}{2}zy^2 - \alpha y, \\
H_{\mathrm{IV}} &= -xy^2 + x^2y - 2xyz - 2\alpha x + 2\beta y, \\
zH_{\mathrm{III}(D_8)} &= x^2y^2 - \frac{z}{2y} - \frac{1}{2}y, \\
zH_{\mathrm{III}(D_7)} &= x^2y^2 + zx + y + \alpha xy, \\
zH_{\mathrm{III}(D_6)} &= x^2y^2 - xy^2 + zx + (\alpha+\beta)xy - \alpha y, \\
zH_{\mathrm{V}} &= x(x+z)y(y-1) + \alpha_2 yz - \alpha_3 xy - \alpha_1 x(y-1), \\
z(z-1)H_{\mathrm{VI}} &= y(y-1)(y-z)x^2 + \alpha_2(\alpha_1+\alpha_2)(y-z) \\
&\quad - \left(\alpha_4(y-1)(y-z) + \alpha_3 y(y-z) + \alpha_0 y(y-1)\right)x,
\end{aligned}$$

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where $\alpha_i, \beta \in \mathbb{C}$ are arbitrary parameters. Parameters of $H_{\rm VI}$ satisfy the constraint $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 0$. The third Painlevé equation is divided into three cases ${\rm III}(D_8), {\rm III}(D_7), {\rm III}(D_6)$ due to the geometry of the spaces of initial conditions [8, 7]. See [9] for the list of Hamiltonians and Bäcklund transformations written in these coordinates, although $H_{{\rm III}(D_8)}$ here is obtained by putting $x \mapsto x - 1/(2y)$ for that in [9].

Among these Hamiltonians, only $({\rm P_I}), ({\rm P_{II}})$ and $({\rm P_{IV}})$ are polynomials with respect to both of the dependent variables x, y and the independent variable z. In Chiba [2, 3], $({\rm P_I}), ({\rm P_{II}})$ and $({\rm P_{IV}})$ are studied by means of a weighted projective space $\mathbb{C}P^3(p,q,r,s)$, whose weight (p,q,r,s) is one of the invariants of the equation determined by the Newton polyhedron. In particular, the Painlevé property, the spaces of initial conditions and Kovalevskaya exponents are investigated in detail. The purpose in this paper is to extend the previous result to the third, fifth and sixth Painlevé equations, whose Newton polyhedrons are degenerate.

According to [2, 3], let us recall the definition of the Newton diagram of a polynomial differential system. Consider the system of polynomial differential equations

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_m, z), \quad i = 1, \cdots, m. \tag{1.2}$$

The exponent of a monomial $x_1^{\mu_1} \cdots x_m^{\mu_m} z^{\eta}$ included in the right hand side f_i is defined as $(\mu_1, \cdots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \cdots, \mu_m, \eta + 1)$. Each exponent specifies a point of the integer lattice in $\mathbb{R}^{m+1}$. The Newton polyhedron of (1.2) is the convex hull of the union of the positive quadrants $\mathbb{R}_+^{m+1}$ with vertices at the exponents of the monomials which appear in the system. The Newton diagram of the system is the union of the compact faces of its Newton polyhedron.

We also consider the perturbative system

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_m, z) + g_i(x_1, \cdots, x_m, z), \quad i = 1, \cdots, m, \tag{1.3}$$

where f_i and g_i are polynomials. We suppose that

(A1) the Newton polyhedron of the truncated system (1.2) has only one compact face and all exponents of monomials included in (1.2) lie on the face.

In this case, there is a tuple of relatively prime *positive* integers $(p_1, \cdots, p_m, r, s)$ and a hyperplane $p_1 x_1 + \cdots + p_m x_m + rz = s$ in $\mathbb{R}^{m+1}$ such that all exponents lie on the plane; i.e. any monomials $x_1^{\mu_1} \cdots x_m^{\mu_m} z^{\eta}$ included in f_i satisfy

$$p_1\mu_1 + \cdots + p_i(\mu_i - 1) + \cdots + p_m\mu_m + r(\eta + 1) = s. \tag{1.4}$$

In [3], we further suppose that $s - r = 1$ though it is not so essential. To regard g_i as a perturbation, we suppose that

(A2) any monomials $x_1^{\mu_1} \cdots x_m^{\mu_m} z^{\eta}$ included in g_i satisfy

$$p_1\mu_1 + \cdots + p_i(\mu_i - 1) + \cdots + p_m\mu_m + r(\eta + 1) < s, \tag{1.5}$$

(this implies that the exponents of g_i lie on the lower side of the hyperplane).

Due to the property of the Newton diagram, it is easy to verify that the truncated system (1.2) is invariant under the $\mathbb{Z}_s$-action given by

$$(x_1, \cdots, x_m, z) \mapsto (\omega^{p_1} x_1, \cdots, \omega^{p_m} x_m, \omega^r z), \quad \omega := e^{2\pi i/s}. \tag{1.6}$$

We require the same for Eq.(1.3):

(A3) Eq.(1.3) is invariant under the $\mathbb{Z}_s$-action (1.6).

A tuple of positive integers $(p_1, \cdots, p_m, r, s)$ is called the weight of the system (1.3). It is known that there is a one-to-one correspondence between nondegenerate Newton diagrams and toric varieties. If exponents lie on the unique plane $p_1 x_1 + \cdots + p_m x_m + rx = s$ (assumption (A1)), then the associated toric variety is the weighted projective space $\mathbb{C}P^{m+1}(p_1, \cdots, p_m r, s)$, which is an $m+1$ dimensional orbifold, see Sec.2.1 for the definition.

The first, second and fourth Painlevé equations satisfy the above assumptions. For the first Painlevé equation $x' = 6y^2 + z$, $y' = x$, put $f = (6y^2 + z, x)$ and $g = (0, 0)$. The Newton diagram is determined by three points $(-1, 2, 1), (-1, 0, 2)$ and $(1, -1, 1)$. They lie on the unique plane $3x+2y+4z = 5$. For the second Painlevé equation $x' = 2y^3+yz+\alpha$, $y' = x$ with a parameter α, put $f = (2y^3+yz, x)$ and $g = (\alpha, 0)$. The Newton diagram is determined by points $(-1, 3, 1), (-1, 1, 2), (1, -1, 1)$, which lie on the plane $2x + y + 2z = 3$. For the fourth Painlevé equation, put $f = (-x^2 + 2xy + 2xz, -y^2 + 2xy - 2yz)$ and $g = (-2\beta, -2\alpha)$. The Newton diagram is given by the unique face on the plane $x + y + z = 2$ passing through the exponents $(1, 0, 1), (0, 1, 1)$ and $(0, 0, 2)$. Hence, the weighted projective spaces associated with them are given by $\mathbb{C}P^3(3, 2, 4, 5), \mathbb{C}P^3(2, 1, 2, 3)$ and $\mathbb{C}P^3(1, 1, 1, 2)$, respectively. Note that g consists of terms including arbitrary parameters α, β for these systems.

The orbifold $\mathbb{C}P^{m+1}(p_1, \cdots, p_m, r, s)$ is regarded as a compactification of the phase space $\mathbb{C}^{m+1} = \{(x_1, \cdots, x_m, z)\}$ of the system (1.3). In [2, 3], the system (1.3), in particular the first, second and fourth Painlevé equations, are studied with the aid of the geometry of $\mathbb{C}P^{m+1}(p_1, \cdots, p_m, r, s)$. In this paper, the third, fifth and sixth Painlevé equations will be investigated, for which the Newton polyhedrons are degenerate and do not satisfy (A1).

The third Painlevé equation of type D_6 is explicitly given by

$$(\mathrm{P}_{\mathrm{III}(D_6)}) : \begin{cases} zx' = -2x^2y + 2xy - (\alpha+\beta)x + \alpha, \\ zy' = 2xy^2 - y^2 + z + (\alpha+\beta)y. \end{cases} \tag{1.7}$$

Put $f = (-2x^2y+2xy, 2xy^2-y^2+z)$ and $g = (-(\alpha+\beta)x+\alpha, (\alpha+\beta)y)$. Exponents of f are given by $(1, 1, 0), (0, 1, 0), (0, -1, 1)$. The Newton polyhedron generated by them has no compact faces because the positive quadrant $\mathbb{R}^3_+$ with a vertex at $(0, 1, 0)$ includes $(1, 1, 0)$; the Newton diagram is empty. Nevertheless, these three points lie on the plane $y + 2z = 1$. Hence, we define the weight of $(\mathrm{P}_{\mathrm{III}(D_6)})$ by $(0, 1, 2, 1)$ and use the weighted projective space $\mathbb{C}P^3(0, 1, 2, 1)$, which is *not* compact because of the nonpositive weight.

The third Painlevé equation of type D_7 is given by

$$(\mathrm{P}_{\mathrm{III}(D_7)}) : \left\{ \begin{array}{l} zx' = -2x^2y - 1 - \alpha x, \\ zy' = 2xy^2 + z + \alpha y. \end{array} \right. \tag{1.8}$$

Put $f = (-2x^2y - 1, 2xy^2 + z)$ and $g = (-\alpha x, \alpha y)$. Exponents of f are given by $(1, 1, 0), (-1, 0, 0), (0, -1, 1)$. The Newton diagram is again empty, however, these three points lie on the plane $-x+2y+3z = 1$. Thus we define the weight of $(\mathrm{P}_{\mathrm{III}(D_7)})$ by $(-1, 2, 3, 1)$ and use the weighted projective space $\mathbb{C}P^3(-1, 2, 3, 1)$.

The third Painlevé equation of type D_8 is given by

$$(\mathrm{P}_{\mathrm{III}(D_8)}) : \left\{ \begin{array}{l} zx' = \dfrac{1}{2} - 2x^2y - \dfrac{z}{2y^2}, \\ zy' = 2xy^2. \end{array} \right. \tag{1.9}$$

There are no parameters and put $g = (0, 0)$. Exponents of the system are given by $(1, 1, 0), (-1, 0, 0), (-1, -2, 1)$. The Newton polyhedron is degenerate, however, these three points lie on the plane $-x + 2y + 4z = 1$. Thus we define the weight of $(\mathrm{P}_{\mathrm{III}(D_8)})$ to be $(-1, 2, 4, 1)$ and use the weighted projective space $\mathbb{C}P^3(-1, 2, 4, 1)$.

The fifth Painlevé equation is given by

$$(\mathrm{P}_{\mathrm{V}}) : \left\{ \begin{array}{l} zx' = -2x^2y + x^2 + xz - 2xyz + (\alpha_1 + \alpha_3)x - \alpha_2 z, \\ zy' = 2xy^2 - 2xy - yz + y^2z - (\alpha_1 + \alpha_3)y + \alpha_1. \end{array} \right. \tag{1.10}$$

Put $f = (-2x^2y + x^2 + xz - 2xyz, 2xy^2 - 2xy - yz + y^2z)$ and $g = ((\alpha_1 + \alpha_3)x - \alpha_2 z, -(\alpha_1 + \alpha_3)y + \alpha_1)$. Exponents of f are $(1, 1, 0), (1, 0, 0), (0, 0, 1)$ and $(0, 1, 1)$. Although there are four exponents, they lie on the unique plane $x + z = 1$. Thus we define the weight of $(\mathrm{P}_{\mathrm{V}})$ by $(1, 0, 1, 1)$ and use the weighted projective space $\mathbb{C}P^3(1, 0, 1, 1)$.

The sixth Painlevé equation is given by

$$(\mathrm{P}_{\mathrm{VI}}) : \left\{ \begin{array}{l} z(z-1)x' = -3x^2y^2 + 2x^2y + 2x^2yz - x^2z + g_1, \\ z(z-1)y' = 2xy^3 - 2xy^2 - 2xy^2z + 2xyz + g_2, \end{array} \right. \tag{1.11}$$

where (g_1, g_2) consists of terms including parameters. Exponents of the other terms are given by $(1, 2, 0), (1, 1, 0), (1, 1, 1)$ and $(1, 0, 1)$, which lie on the unique plane $x = 1$. Thus we define the weight of $(\mathrm{P}_{\mathrm{VI}})$ as $(1, 0, 0, 1)$ and use the weighted projective space $\mathbb{C}P^3(1, 0, 0, 1)$.

In this paper, the third, fifth and sixth Painlevé equations are studied with the aid of the weighted projective spaces $\mathbb{C}P^3(p, q, r, s)$ and dynamical systems theory. The weighted projective space is decomposed into the disjoint sum $\mathbb{C}P^3(p, q, r, s) \simeq \mathbb{C}P^2(p, q, r) \cup \mathbb{C}^3$. This implies that the natural phase space $\mathbb{C}^3 = \{(x, y, z)\}$ of the Painlevé equations is embedded in $\mathbb{C}P^3(p, q, r, s)$ and the two dimensional manifold $\mathbb{C}P^2(p, q, r)$ is attached at infinity. We regard the Painlevé equation as a three dimensional autonomous vector field defined on $\mathbb{C}P^3(p, q, r, s)$. Then, the vector

	(p,q,r,s)	$\deg(H)$	κ	$\lambda_1,\lambda_2,\lambda_3$	l	c
$\mathrm{P_I}$	$(3,2,4,5)$	6	6	$6,4,5$	1	2
$\mathrm{P_{II}}$	$(2,1,2,3)$	4	4	$4,2,3$	2	3
$\mathrm{P_{IV}}$	$(1,1,1,2)$	3	3	$3,1,2$	3	4
$\mathrm{P_{III(D_8)}}$	$(-1,2,4,1)$	2	2	$2,4,1$	2	3
$\mathrm{P_{III(D_7)}}$	$(-1,2,3,1)$	2	2	$2,3,1$	2	3
$\mathrm{P_{III(D_6)}}$	$(0,1,2,1)$	2	2	$2,2,1$	3	4
$\mathrm{P_V}$	$(1,0,1,1)$	2	2	$2,1,1$	4	5
$\mathrm{P_{VI}}$	$(1,0,0,1)$	2	2	$2,0,1$	5	6

Table 1: $\deg(H)$ denotes the weighted degree of the Hamiltonian function with respect to the weight $\deg(x)=p, \deg(y)=q, \deg(z)=r$. κ denotes the Kovalevskaya exponent defined in Sec.2.2. $(\lambda_1,\lambda_2,\lambda_3)$ is the weight for the weighted blow-up, which coincides with the eigenvalues of the Jacobi matrix at the singularity. In Chiba [3], it is proved that $\deg(H)=\kappa=\lambda_1$ and $r=\lambda_2, s=\lambda_3$. l gives the number of types of Laurent series solutions given in Sec.2.2, and c is the number of local charts for the space of initial conditions. We always have $c=l+1$.

field has several fixed points on the infinity set $\mathbb{C}P^2(p,q,r)$. These fixed points describe the asymptotic behavior of solutions. Some of these fixed points correspond to movable singularities, and the other correspond to the irregular singular point.

The dynamical systems theory is applied to these fixed points to investigate the Painlevé equations. The singular normal form of the Painlevé equation [2, 6], which is a local integrable system around a movable singularity, is obtained by applying the normal form theory around fixed points. The space of initial conditions and its symplectic atlas are constructed by the weighted blow-up at these fixed points. The weight for the weighted blow-up, which is also an invariant of the Painlevé equation related to the Kovalevskaya exponent [2, 3], is determined by eigenvalues of the Jacobi matrix of the vector field at the fixed points. It is known that the Painlevé equations are reduced to the Riccati equations when the parameters take certain specific values. Such Riccati solutions are characterized as a center (un)stable manifold at the fixed point on $\mathbb{C}P^3(p,q,r,s)$. Although some of these results are well known for experts, our new approach based on the weighted projective space and dynamical systems theory provides a systematic way to investigate them. From our analysis, it turns out that the weights and the Kovalevskaya exponents are important invariants of the Painlevé equations. In particular, the Painlevé equations may be classified by these invariants, which will be reported in a forthcoming paper.

Our method will be explained in detail for the third Painlevé equation of type D_6 in Section 3. Since the strategy for the other Painlevé equations ($\mathrm{P_{III(D_7)}}$), ($\mathrm{P_{III(D_8)}}$), ($\mathrm{P_V}$) and ($\mathrm{P_{VI}}$) is completely the same as that for ($\mathrm{P_{III(D_6)}}$), we only show a sketch and several formulae for them after Section 4. See [2] for ($\mathrm{P_I}$), ($\mathrm{P_{II}}$) and ($\mathrm{P_{IV}}$).

2 Settings

2.1 Weighted projective spaces

For a tuple of integers $(p_1, \cdots, p_m, r, s)$, consider the weighted $\mathbb{C}^*$-action on $\mathbb{C}^{m+2}$ defined by

$$(x_1, \cdots, x_m, z, \varepsilon) \mapsto (\lambda^{p_1} x_1, \cdots, \lambda^{p_m} x_m, \lambda^r z, \lambda^s \varepsilon), \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}. \tag{2.1}$$

The quotient space

$$\mathbb{C}P^{m+1}(p_1, \cdots, p_m, r, s) := \mathbb{C}^{m+2}\backslash\{0\}/\mathbb{C}^*$$

gives an $m+1$ dimensional orbifold called the weighted projective space. In this paper, we only use a three dimensional space. When (p, q, r, s) are positive integers, the orbifold structure of $\mathbb{C}P^3(p, q, r, s)$ is obtained as follows:

The space $\mathbb{C}P^3(p, q, r, s)$ is defined by the equivalence relation on $\mathbb{C}^4\backslash\{0\}$

$$(x, y, z, \varepsilon) \sim (\lambda^p x, \lambda^q y, \lambda^r z, \lambda^s \varepsilon).$$

(i) When $x \neq 0$,

$$(x, y, z, \varepsilon) \sim (1,\ x^{-q/p} y,\ x^{-r/p} z,\ x^{-s/p} \varepsilon) =: (1, Y_1, Z_1, \varepsilon_1).$$

Due to the choice of the branch of $x^{1/p}$, we also obtain

$$(Y_1, Z_1, \varepsilon_1) \sim (e^{-2q\pi i/p} Y_1, e^{-2r\pi i/p} Z_1, e^{-2s\pi i/p} \varepsilon_1),$$

by putting $x \mapsto e^{2\pi i} x$. This implies that the subset of $\mathbb{C}P^3(p, q, r, s)$ such that $x \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_p$, where the $\mathbb{Z}_p$-action is defined as above.

(ii) When $y \neq 0$,

$$(x, y, z, \varepsilon) \sim (y^{-p/q} x,\ 1,\ y^{-r/q} z,\ y^{-s/q} \varepsilon) =: (X_2, 1, Z_2, \varepsilon_2).$$

Because of the choice of the branch of $y^{1/q}$, we obtain

$$(X_2, Z_2, \varepsilon_2) \sim (e^{-2p\pi i/q} X_2, e^{-2r\pi i/q} Z_2, e^{-2s\pi i/q} \varepsilon_2).$$

Hence, the subset of $\mathbb{C}P^3(p, q, r, s)$ with $y \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_q$.

(iii) When $z \neq 0$,

$$(x, y, z, \varepsilon) \sim (z^{-p/r} x,\ z^{-q/r} y,\ 1,\ z^{-s/r} \varepsilon) =: (X_3, Y_3, 1, \varepsilon_3).$$

Similarly, the subset $\{z \neq 0\} \subset \mathbb{C}P^3(p, q, r, s)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_r$.

(iv) When $\varepsilon \neq 0$,

$$(x, y, z, \varepsilon) \sim (\varepsilon^{-p/s} x,\ \varepsilon^{-q/s} y,\ \varepsilon^{-r/s} z,\ 1) =: (X_4, Y_4, Z_4, 1).$$

The subset $\{\varepsilon \neq 0\} \subset \mathbb{C}P^3(p, q, r, s)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_s$.

This proves that the orbifold structure of $\mathbb{C}P^3(p,q,r,s)$ is given by

$$\mathbb{C}P^3(p,q,r,s) = \mathbb{C}^3/\mathbb{Z}_p \ \cup \ \mathbb{C}^3/\mathbb{Z}_q \ \cup \ \mathbb{C}^3/\mathbb{Z}_r \ \cup \ \mathbb{C}^3/\mathbb{Z}_s.$$

The local charts $(Y_1, Z_1, \varepsilon_1)$, $(X_2, Z_2, \varepsilon_2)$, $(X_3, Y_3, \varepsilon_3)$ and (X_4, Y_4, Z_4) defined above are called inhomogeneous coordinates as the usual projective space. Note that they give coordinates on the lift $\mathbb{C}^3$, not on the quotient $\mathbb{C}^3/\mathbb{Z}_i$ $(i = p, q, r, s)$. Therefore, any equations written in these inhomogeneous coordinates should be invariant under the corresponding $\mathbb{Z}_i$ actions.

In what follows, we use the notation (x, y, z) for the fourth local chart instead of (X_4, Y_4, Z_4) because the Painlevé equation will be given on this chart. The transformations between inhomogeneous coordinates are give by

$$\begin{cases} x = & \varepsilon_1^{-p/s} = & X_2\varepsilon_2^{-p/s} = & X_3\varepsilon_3^{-p/s} \\ y = & Y_1\varepsilon_1^{-q/s} = & \varepsilon_2^{-q/s} = & Y_3\varepsilon_3^{-q/s} \\ z = & Z_1\varepsilon_1^{-r/s} = & Z_2\varepsilon_2^{-r/s} = & \varepsilon_3^{-r/s}. \end{cases} \tag{2.2}$$

The same transformation rule holds even if p, q, r, s include negative integers. If there are 0 among them, for example if $p = 0$, then we have $\mathbb{C}P^3(0,q,r,s) \simeq \mathbb{C} \times \mathbb{C}P^2(q,r,s)$. $\mathbb{C}P^3(p,q,r,s)$ is compact if and only if all p, q, r, s are positive.

2.2 Laurent series solutions and Kovalevskaya exponents

To construct the space of initial conditions, we need the expressions of the Laurent series of solutions. Let (p, q, r, s) be the weight of a given system determined by the Newton polyhedron. Suppose that the system has a Laurent series solution of the form

$$\begin{cases} x = \displaystyle\sum_{n=0}^{\infty} A_n (z - z_0)^{-p+n}, \\ y = \displaystyle\sum_{n=0}^{\infty} B_n (z - z_0)^{-q+n}, \end{cases} \tag{2.3}$$

where $(A_0, B_0) \neq (0, 0)$ and z_0 is a movable pole. Such a Laurent series solution is called *regular*. A Laurent series solution is called *exceptional* if it is not expressed in this form; i.e. $(A_0, B_0) = (0, 0)$ or the order of a pole of either x or y is larger than p or q, respectively. If a regular Laurent series represents a general solution of the system, it includes an arbitrary parameter, which depends on initial conditions, other than z_0. The smallest integer κ such that (A_κ, B_κ) includes an arbitrary parameter is called the Kovalevskaya exponent. In [3], it is proved that the Kovalevskaya exponent of the regular Laurent series solution is invariant under a certain class of coordinates transformations including the automorphism group of $\mathbb{C}P^3(p,q,r,s)$. For the first, second and fourth Painlevé equations, all Laurent series solutions are regular because they satisfy the assumptions (A1) and (A2). The third, fifth and sixth Painlevé equations have exceptional Laurent series solutions, however, they can be converted into the regular series by the Bäcklund transformations. Hence,

the Kovalevskaya exponents κ of exceptional Laurent series solutions are well-defined and given as in Table 1. In what follows, denote $T := z - z_0$.

($\mathrm{P}_{\mathrm{III}(D_6)}$): The third Painlevé equation of type D_6 has three types of Laurent series solutions given by

$$\text{(i)} \quad \begin{cases} x = 1 + \dfrac{\beta}{z_0} \cdot T + A_2 \cdot T^2 + O(T^3), \\ y = -z_0 \cdot T^{-1} + \dfrac{1}{2}(-1 - \alpha + \beta) + B_2 \cdot T + O(T^2), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = 0 \cdot T^0 - \dfrac{\alpha}{z_0} \cdot T + A_2 \cdot T^2 + O(T^3), \\ y = z_0 \cdot T^{-1} + \dfrac{1}{2}(1 - \alpha + \beta) + B_2 \cdot T + O(T^2), \end{cases}$$

$$\text{(iii)} \quad \begin{cases} x = -z_0 \cdot T^{-2} + 0 \cdot T^{-1} + A_2 + O(T), \\ y = -T + \dfrac{-2 + \alpha + \beta}{2z_0} \cdot T^2 + B_2 \cdot T^3 + O(T^4), \end{cases}$$

where A_2 is an arbitrary constant and B_2 is a certain function of A_2. Since $(p, q) = (0, 1)$, the first two series are regular, while the last one is exceptional. The Kovalevskaya exponents of all series are $\kappa = 2$.

($\mathrm{P}_{\mathrm{III}(D_7)}$): The third Painlevé equation of type D_7 has two types of Laurent series solutions given by

$$\text{(i)} \quad \begin{cases} x = \dfrac{1}{z_0} \cdot T + \dfrac{\alpha - 1}{2z_0^2} \cdot T^2 + A_2 \cdot T^3 + O(T^4), \\ y = -z_0^2 \cdot T^{-2} - z_0 \cdot T^{-1} + B_2 + O(T), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = -z_0 \cdot T^{-2} + 0 \cdot T^{-1} + A_2 + O(T), \\ y = -T^1 + \dfrac{\alpha - 2}{2z_0} \cdot T^2 + B_2 \cdot T^3 + O(T^4), \end{cases}$$

where A_2 is an arbitrary constant and B_2 is a certain function of A_2. Since $(p, q) = (-1, 2)$, the former series is regular, while the latter one is exceptional. The Kovalevskaya exponents of both series are $\kappa = 2$.

($\mathrm{P}_{\mathrm{III}(D_8)}$): The third Painlevé equation of type D_8 has two types of Laurent series solutions given by

$$\text{(i)} \quad \begin{cases} x = -\dfrac{1}{2z_0} \cdot T + \dfrac{1}{4z_0^2} \cdot T^2 + \dfrac{2B_2 - 1}{4z_0^2} \cdot T^3 + O(T^4), \\ y = 2z_0^2 \cdot T^{-2} + 2z_0 \cdot T^{-1} + B_2 + O(T), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = 2z_0^2 \cdot T^{-3} + 2z_0 \cdot T^{-2} + 0 \cdot T^{-1} + O(1), \\ y = \dfrac{1}{2z_0} T^2 + 0 \cdot T^3 + B_2 \cdot T^4 + O(T^5), \end{cases}$$

where B_2 is an arbitrary constant. Since $(p, q) = (-1, 2)$, the former series is regular, while the latter one is exceptional. The Kovalevskaya exponents of both series are $\kappa = 2$.

(P_V): The fifth Painlevé equation has four types of Laurent series solutions given by

$$\text{(i)} \quad \begin{cases} x = z_0 \cdot T^{-1} + \dfrac{1}{2}(1 - z_0 + \alpha_1 - \alpha_3) + A_2 \cdot T + O(T^2), \\ y = 1 + \dfrac{\alpha_3}{z_0} \cdot T + B_2 \cdot T^2 + O(T^3), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = -z_0 \cdot T^{-1} + \dfrac{1}{2}(-1 - z_0 + \alpha_1 - \alpha_3) + A_2 \cdot T + O(T^2), \\ y = 0 \cdot T^0 - \dfrac{\alpha_1}{z_0} \cdot T + B_2 \cdot T^2 + O(T^3), \end{cases}$$

$$\text{(iii)} \quad \begin{cases} x = -z_0 + (\alpha_1 + \alpha_2 + \alpha_3 - 2) \cdot T + A_2 \cdot T^2 + O(T^3), \\ y = T^{-1} + \dfrac{z_0 + \alpha_1 + 2\alpha_2 + \alpha_3 - 2}{2z_0} + B_2 \cdot T + O(T^2), \end{cases}$$

$$\text{(iv)} \quad \begin{cases} x = 0 \cdot T^0 + \alpha_2 \cdot T + A_2 \cdot T^2 + O(T^3), \\ y = -T^{-1} + \dfrac{z_0 + \alpha_1 + 2\alpha_2 + \alpha_3}{2z_0} + B_2 \cdot T + O(T^2), \end{cases}$$

where A_2 is an arbitrary constant and B_2 is a certain function of A_2. Since $(p, q) = (1, 0)$, (i) and (ii) are regular, while (iii) and (iv) are exceptional. The Kovalevskaya exponents of all series are $\kappa = 2$.

(P_{VI}): The sixth Painlevé equation has five types of Laurent series solutions given by

$$\text{(i)} \quad \begin{cases} x = T^{-1} + \dfrac{2 - 4z_0 + \alpha_0 - 2z_0\alpha_0 + z_0\alpha_3 - \alpha_4 + z_0\alpha_4}{2z_0(z_0 - 1)} + A_2 \cdot T + O(T^2), \\ y = z_0 + (2 + \alpha_0) \cdot T + B_2 \cdot T^2 + O(T^3), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} x = -z_0 \cdot T^{-1} + \dfrac{1 - z_0 - \alpha_0 + 2\alpha_3 - z_0\alpha_3 - \alpha_4 + z_0\alpha_4}{2(z_0 - 1)} + A_2 \cdot T + O(T^2), \\ y = 1 - \dfrac{\alpha_3}{z_0} \cdot T + B_2 \cdot T^2 + O(T^3), \end{cases}$$

$$\text{(iii)} \quad \begin{cases} x = (z_0 - 1) \cdot T^{-1} + \dfrac{z_0 - \alpha_0 - z_0\alpha_3 + \alpha_4 + z_0\alpha_4}{2z_0} + A_2 \cdot T + O(T^2), \\ y = 0 \cdot T^0 + \dfrac{\alpha_4}{z_0 - 1} \cdot T + B_2 \cdot T^2 + O(T^3), \end{cases}$$

$$\text{(iv)} \quad \begin{cases} x = -\dfrac{\alpha_1(\alpha_1 + \alpha_2)}{z_0(z_0 - 1)} \cdot T + O(T^2), \\ y = \dfrac{z_0(z_0 - 1)}{\alpha_1} \cdot T^{-1} + O(1), \end{cases}$$

$$\text{(v)} \quad \begin{cases} x = \dfrac{\alpha_1\alpha_2}{z_0(z_0 - 1)} \cdot T + O(T^2), \\ y = -\dfrac{z_0(z_0 - 1)}{\alpha_1} \cdot T^{-1} + O(1), \end{cases}$$

where A_2 is an arbitrary constant and B_2 is a certain function of A_2. Since $(p, q) = (1, 0)$, (i), (ii) and (iii) are regular, while (iv) and (v) are exceptional. The Kovalevskaya exponents of all series are $\kappa = 2$.

For all Painlevé equations, the number of types of Laurent series solutions is smaller than the number of local charts of the space of initial conditions by one, see Table 1.

3 The third Painlevé equation of type D_6

3.1 $\mathbf{P_{III(D_6)}}$ on $\mathbb{C}P^3(0,1,2,1)$

The orbifold structure of $\mathbb{C}P^3(0,1,2,1)$ is given by

$$\begin{aligned}\mathbb{C}P^3(0,1,2,1) &= \mathbb{C}\times\mathbb{C}P^2(1,2,1)\\ &= \mathbb{C}\times\left(\mathbb{C}^2\cup\mathbb{C}^2/\mathbb{Z}_2\cup\mathbb{C}^2\right)\\ &= \mathbb{C}^3\cup(\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2)\cup\mathbb{C}^3.\end{aligned}$$

Thus, the space is covered by three inhomogeneous coordinates (X_2,Z_2,ε_2), (X_3,Y_3,ε_3) and (x,y,z) related as

$$\begin{cases} x = X_2 = X_3,\\ y = \varepsilon_2^{-1} = Y_3\varepsilon_3^{-1},\\ z = Z_2\varepsilon_2^{-2} = \varepsilon_3^{-2},\end{cases} \tag{3.1}$$

(the first local chart (Y_1,Z_1,ε_1) does not appear because $p=0$).

We give the third Painlevé equation of type D_6 on the local chart (x,y,z). On the other local charts, $(\mathrm{P}_{\mathrm{III}(D_6)})$ is expressed as

$$\begin{cases} \dfrac{dX_2}{d\varepsilon_2} = \dfrac{2X_2-2X_2^2+\varepsilon_2(\alpha-\alpha X_2-\beta X_2)}{\varepsilon_2(1-2X_2-Z_2-\alpha\varepsilon_2-\beta\varepsilon_2)},\\ \dfrac{dZ_2}{d\varepsilon_2} = \dfrac{2Z_2-4X_2Z_2-2Z_2^2+\varepsilon_2Z_2(1-2\alpha-2\beta)}{\varepsilon_2(1-2X_2-Z_2-\alpha\varepsilon_2-\beta\varepsilon_2)},\end{cases} \tag{3.2}$$

$$\begin{cases} \dfrac{dX_3}{d\varepsilon_3} = -\dfrac{2}{\varepsilon_3^2}\left(2X_3Y_3-2X_3^2Y_3+\varepsilon_3(\alpha-\alpha X_3-\beta X_3)\right),\\ \dfrac{dY_3}{d\varepsilon_3} = -\dfrac{2}{\varepsilon_3^2}\left(1-Y_3^2+2X_3Y_3^2+\varepsilon_3Y_3(-\dfrac{1}{2}+\alpha+\beta)\right).\end{cases} \tag{3.3}$$

In order to apply dynamical systems theory later, it is convenient to rewrite them as 3-dim autonomous vector fields of the form

$$\begin{cases} \dot{X}_2 = 2X_2-2X_2^2+\varepsilon_2(\alpha-\alpha X_2-\beta X_2),\\ \dot{Z}_2 = 2Z_2-4X_2Z_2-2Z_2^2+\varepsilon_2Z_2(1-2\alpha-2\beta),\\ \dot{\varepsilon}_2 = \varepsilon_2(1-2X_2-Z_2-\alpha\varepsilon_2-\beta\varepsilon_2),\end{cases} \tag{3.4}$$

and

$$\begin{cases} \dot{X}_3 = 2X_3Y_3-2X_3^2Y_3+\varepsilon_3(\alpha-\alpha X_3-\beta X_3),\\ \dot{Y}_3 = 1-Y_3^2+2X_3Y_3^2+\varepsilon_3Y_3(-\dfrac{1}{2}+\alpha+\beta),\\ \dot{\varepsilon}_3 = -\dfrac{\varepsilon_3^2}{2},\end{cases} \tag{3.5}$$

where $(\dot{}) = d/dt$ and $t \in \mathbb{C}$ parameterizes each integral curve.

We also have the decomposition

$$\mathbb{C}P^3(0,1,2,1) \;=\; \mathbb{C}P^2(0,1,2) \cup \mathbb{C}^3 \quad \text{(disjoint)}.$$

Since $\mathbb{C}P^2(0,1,2) = \mathbb{C} \times \mathbb{C}P^1(1,2)$ and $\mathbb{C}P^1(1,2)$ is the usual projective line,

$$\mathbb{C}P^3(0,1,2,1) \;=\; (\mathbb{C} \times \mathbb{C}P^1) \cup \mathbb{C}^3, \tag{3.6}$$

where $\mathbb{C}^3 = \{(x,y,z)\}$ and $\mathbb{C} \times \mathbb{C}P^1 = \{(X_2, Z_2, 0)\} \cup \{(X_3, Y_3, 0)\}$ in coordinates. This implies that the set $\mathbb{C} \times \mathbb{C}P^1$ is attached at infinity of the natural phase space $\mathbb{C}^3 = \{(x,y,z)\}$ of $(\mathrm{P}_{\mathrm{III}(D_6)})$, and the asymptotic behavior of solutions can be studied by the limit $\varepsilon_2 \to 0$ or $\varepsilon_3 \to 0$.

The vector fields (3.4) and (3.5) have exactly four fixed points on the infinity set $\{\varepsilon_2 = 0\} \cup \{\varepsilon_3 = 0\}$ given by

$$(X_2, Z_2, \varepsilon_2) = (0,0,0), \quad (1,0,0), \quad (0,1,0), \quad (1,-1,0). \tag{3.7}$$

The Jacobi matrices of (3.4) at the fixed points are

$$\begin{pmatrix} 2 & 0 & \alpha \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, -\begin{pmatrix} 2 & 0 & \beta \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & \alpha \\ -4 & -2 & \gamma \\ 0 & 0 & 0 \end{pmatrix}, -\begin{pmatrix} 2 & 0 & \beta \\ -4 & -2 & \gamma \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

respectively, where $\gamma = 1 - 2\alpha - 2\beta$. The latter two points, for which $Z_2 \neq 0 \Rightarrow z = \infty$, correspond to the irregular singular point. Since the Jacobi matrix has a zero eigenvalue, there exists a one dimensional center manifold at each fixed point. The asymptotic expansion of the center manifold [5] yields the asymptotic expansion of $(x(z), y(z))$ as $|z| \to \infty$.

On the other hand, the former two fixed points correspond to movable poles. Indeed, it is easy to verify by using (3.1) that the regular Laurent series solutions (ii) and (i) given in Sec.2.2 converge to the points $(X_2, Z_2, \varepsilon_2) = (0,0,0)$ and $(1,0,0)$, respectively, as $z \to z_0$. The exceptional Laurent series solution (iii) does not converge to some point on $\mathbb{C}P^3(0,1,2,1)$ as $z \to z_0$; the space $\mathbb{C}P^3(0,1,2,1)$ is not compact.

To treat the Laurent series (iii), we use the Bäcklund transformations. See [9] for the complete list of the Bäcklund transformations of two dimensional Painlevé equations. It is known that the transformation groups of the Painlevé equations are isomorphic to the extended affine Weyl groups. For a classical root system R, the affine Weyl group and the extended affine Weyl group are denoted by $W(R^{(1)})$ and $\widetilde{W}(R^{(1)})$, respectively. Let $G = \mathrm{Aut}(R^{(1)})$ be the Dynkin automorphism group of the extended Dynkin diagram. We have $\widetilde{W}(R^{(1)}) \cong G \ltimes W(R^{(1)})$. For the third Painlevé equation of type D_6, $R = 2A_1$ and

$$\begin{aligned} &\widetilde{W}((2A_1)^{(1)}) \cong G \ltimes W((2A_1)^{(1)}), \\ &W((2A_1)^{(1)}) = \langle s_0, s_1, s_0', s_1' \rangle, \\ &G = \mathrm{Aut}(D_6^{(1)}) = \mathrm{Aut}((2A_1)^{(1)}) = \langle \pi_1, \pi_2, \sigma_1 \rangle. \end{aligned}$$

	α	β	y	x	z
s_0	$2-\alpha$	β	$y+\dfrac{1-\alpha}{f_0-1}$	$x-\dfrac{(1-\alpha)(f_3-2y)}{f_1}-\dfrac{(1-\alpha)^2 z}{f_1^2}$	z
s_1	$-\alpha$	β	$y+\dfrac{\alpha}{x}$	x	z
s_0'	α	$2-\beta$	$y+\dfrac{1-\beta}{f_0}$	$x-\dfrac{(1-\beta)f_3}{f_2}-\dfrac{(1-\beta)^2 z}{f_2^2}$	z
s_1'	α	$-\beta$	$y+\dfrac{\beta}{x-1}$	x	z
π_1	$1-\alpha$	β	$-\dfrac{z}{y}$	$\dfrac{y}{z}(xy-y+\beta)+1$	z
π_2	α	$1-\beta$	$\dfrac{z}{y}$	$-\dfrac{y}{z}(xy+\alpha)$	z
σ_1	β	α	$-y$	$1-x$	$-z$

Table 2: The action of the extended affine Weyl group for $(\mathrm{P}_{\mathrm{III}(D_6)})$.

The action of each element is given in Table 2, where $f_0 = x + (\alpha+\beta-1)/y + z/y^2$, $f_1 = \beta y + (x-1)y^2 + z$, $f_2 = \alpha y + xy^2 + z$ and $f_3 = 2xy + \alpha + \beta - 1$.

Let us consider another space $\mathbb{C}P^3(0,1,2,1)$ with inhomogeneous coordinates $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2), (\widetilde{X}_3, \widetilde{Y}_3, \widetilde{\varepsilon}_3)$ and $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ satisfying the same formula as (3.1). We glue two copies of $\mathbb{C}P^3(0,1,2,1)$ by the transformation π_2

$$\begin{aligned}(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\alpha}, \widetilde{\beta}) &= \pi_2(x, y, z, \alpha, \beta) \\ &= \left(-\frac{y}{z}(xy+\alpha),\ \frac{z}{y},\ z,\ \alpha,\ 1-\beta\right), \end{aligned} \tag{3.9}$$

which defines a manifold denoted by $\mathcal{M}_1$. Since $(\mathrm{P}_{\mathrm{III}(D_6)})$ is invariant under the action of π_2, the system written in the $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2)$-chart also satisfies Eq.(3.4), in which (α, β) is replaced by $(\widetilde{\alpha}, \widetilde{\beta}) = (\alpha, 1-\beta)$. By the formula (3.9) or

$$\begin{cases} \widetilde{X}_2 = -(xy^2 + \alpha y)/z, \\ \widetilde{Z}_2 = y^2/z, \\ \widetilde{\varepsilon}_2 = y/z, \end{cases} \tag{3.10}$$

it is easy to show that the exceptional Laurent series (iii) is converted to the regular series (i) in the $(\widetilde{x}, \widetilde{y}, \widetilde{z})$-chart, and it approaches to the fixed point $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2) = (1,0,0)$ as $z \to z_0$.

We have proved that Laurent series solutions (i), (ii) and (iii) of $(\mathrm{P}_{\mathrm{III}(D_6)})$ converge to the fixed points $(X_2, Z_2, \varepsilon_2) = (1,0,0), (0,0,0)$ and $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2) = (1,0,0)$ on $\mathcal{M}_1$, respectively, as $z \to z_0$. Hence, the study of movable poles are reduced to the

study of the fixed points and dynamical systems theory is applicable to investigate them.

As a simple application, we can prove the next theorem.

Theorem 3.1. There exists a local analytic transformation $(X_2, Z_2, \varepsilon_2) \mapsto (u, v, w)$ defined near $(X_2, Z_2, \varepsilon_2) = (0, 0, 0)$ such that Eq.(3.4) is transformed into the linearized system

$$\begin{cases} \dot{u} = 2u + \alpha w, \\ \dot{v} = 2v, \\ \dot{w} = w. \end{cases} \tag{3.11}$$

Similarly, the vector field is locally linearized around $(X_2, Z_2, \varepsilon_2) = (1, 0, 0)$ and $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2) = (1, 0, 0)$.

This result implies that $(\mathrm{P}_{\mathrm{III}(D_6)})$ is locally transformed to the integrable system around each movable singularity. A proof is a straightforward application of Poincaré's linearization theorem [5] in dynamical systems theory. See [2] for the detail, in which a similar result is proved for the first, second and fourth Painlevé equations, and also [3], in which it is proved that any differential equations having the Painlevé property is locally linearizable. Such a local integrable system for the Painlevé equation is called the singular normal form in [2, 6].

3.2 The space of initial conditions

A purpose in this section is to construct the space of initial conditions for $(\mathrm{P}_{\mathrm{III}(D_6)})$. On the manifold $\mathcal{M}_1$, there are three singularities of the foliation of integral curves; $(X_2, Z_2, \varepsilon_2) = (1, 0, 0), (0, 0, 0)$ and $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2) = (1, 0, 0)$. They correspond to movable poles of the Laurent series solutions (i), (ii) and (iii), respectively. We will resolve these singularities by weighted blow-ups. On the blow-up space, $(\mathrm{P}_{\mathrm{III}(D_6)})$ is again written in a Hamiltonian system, whose Hamiltonian function is polynomial in dependent variables. This implies that singularities of the foliation are resolved and the space of initial conditions is obtained by three times blow-ups of $\mathcal{M}_1$. This strategy is applicable to the other Painlevé equations, even for higher dimensional Painlevé equations.

(i) blow-up at $(X_2, Z_2, \varepsilon_2) = (0, 0, 0)$.
For the vector field (3.4), put

$$\begin{cases} u = X_2 + \alpha\varepsilon_2, \\ v = Z_2, \\ w = \varepsilon_2. \end{cases}$$

Then, the linear part is diagonalized and we obtain

$$\begin{cases} \dot{u} = 2u - 2u^2 + w(\alpha u - \alpha v - \beta u), \\ \dot{v} = 2v + v(w - 4u - 2v + 2\alpha w - 2\beta w), \\ \dot{w} = w - w(2u + v - \alpha w + \beta w). \end{cases}$$

The origin $(u, v, w) = (0, 0, 0)$ is a singularity of the foliation of integral curves. To resolve it, we introduce the weighted blow-up defined by

$$\left\{\begin{array}{llll} u & = u_1^2 & = v_2^2 u_2 & = w_3^2 u_3, \\ v & = u_1^2 v_1 & = v_2^2 & = w_3^2 v_3, \\ w & = u_1 w_1 & = v_2 w_2 & = w_3. \end{array}\right. \tag{3.12}$$

The weight $(2, 2, 1)$, the exponents in the right hand sides, is taken from the eigenvalues of the Jacobi matrix at the singularity. The exceptional divisor $\{u_1 = 0\} \cup \{v_2 = 0\} \cup \{w_3 = 0\}$ is the weighted projective space $\mathbb{C}P^2(2, 2, 1)$. The relation between the original coordinates (x, y, z) and the new coordinates (u_3, v_3, w_3) is given by

$$\left\{\begin{array}{l} x = u_3 w_3^2 - \alpha w_3, \\ y = 1/w_3, \\ z = v_3, \end{array}\right. \qquad \left\{\begin{array}{l} u_3 = xy^2 + \alpha y, \\ w_3 = 1/y, \\ v_3 = z. \end{array}\right. \tag{3.13}$$

Note that eventually the independent variable z is not changed and (3.13) defines a fiber bundle over z-space, whose fiber is a (x, y)-space and (u_3, w_3)-space glued by the above relation. In the new coordinates, $(\mathrm{P}_{\mathrm{III}(D_6)})$ is transformed into the Hamiltonian system

$$\frac{dw_3}{dz} = -\frac{\partial H_1}{\partial u_3}, \quad \frac{du_3}{dz} = \frac{\partial H_1}{\partial w_3},$$

with the Hamiltonian function

$$zH_1 = -u + u^2 w^2 + uw^2 z - \alpha wz - (\alpha - \beta)uw, \tag{3.14}$$

(here, the subscript is omitted for simplicity). Since H_1 is polynomial in (u_3, w_3), there are no singularities of the foliation in this chart; the singularity associated with the Laurent series solution (ii) is resolved. Furthermore, we can verify that

$$dx \wedge dy = dw_3 \wedge du_3, \ dx \wedge dy + dH \wedge dz = dw_3 \wedge du_3 + dH_1 \wedge dz. \tag{3.15}$$

(ii) blow-up at $(X_2, Z_2, \varepsilon_2) = (1, 0, 0)$.

Putting $X_2 - 1 = \hat{X}_2$ and $u = \hat{X}_2 + \beta\varepsilon_2, v = Z_2, w = \varepsilon_2$ for the vector field (3.4) results in

$$\left\{\begin{array}{l} \dot{u} = -2u - 2u^2 - w(\alpha u - \beta u + \beta v), \\ \dot{v} = -2v + v(w - 4u - 2v - 2\alpha w + 2\beta w), \\ \dot{w} = -w - w(2u + v + \alpha w - \beta w). \end{array}\right.$$

To resolve the singularity at $(u, v, w) = (0, 0, 0)$, we introduce the weighted blow-up with the weight $(2, 2, 1)$

$$\left\{\begin{array}{llll} u & = u_4^2 & = v_5^2 u_5 & = w_6^2 u_6, \\ v & = u_4^2 v_4 & = v_5^2 & = w_6^2 v_6, \\ w & = u_4 w_4 & = v_5 w_5 & = w_6. \end{array}\right. \tag{3.16}$$

The relation between the original coordinates (x, y, z) and the new coordinates (u_6, v_6, w_6) is given by

$$\begin{cases} x = u_6 w_6^2 - \beta w_6 + 1, \\ y = 1/w_6, \\ z = v_6, \end{cases} \qquad \begin{cases} u_6 = xy^2 + \beta y - y^2, \\ w_6 = 1/y, \\ v_6 = z, \end{cases} \tag{3.17}$$

which defines a fiber bundle over z-space. In the new coordinates, $(\mathrm{P}_{\mathrm{III}(D_6)})$ is written as the Hamiltonian system

$$\frac{dw_6}{dz} = -\frac{\partial H_2}{\partial u_6}, \quad \frac{du_6}{dz} = \frac{\partial H_2}{\partial w_6},$$

with the Hamiltonian function

$$zH_2 = u + u^2 w^2 + uw^2 z - \beta wz + (\alpha - \beta)uw, \tag{3.18}$$

(here, the subscript is omitted for simplicity). Since H_2 is polynomial in (u_6, w_6), the singularity associated with the Laurent series solution (i) is resolved. As before, we can verify the symplectic relation (3.15).

(iii) blow-up at $(\widetilde{X}_2, \widetilde{Z}_2, \widetilde{\varepsilon}_2) = (1, 0, 0)$.

This singularity is resolved by the same way as above by the weighted blow-up with the weight $(2, 2, 1)$. The final result is easily obtained by the Bäcklund transformation (3.9) as follows. Define the new coordinates (u_9, v_9, w_9) by

$$\begin{cases} u_9 = \widetilde{x}\widetilde{y}^2 + \widetilde{\beta}\widetilde{y} - \widetilde{y}^2, \\ w_9 = 1/\widetilde{y}, \\ v_9 = \widetilde{z}, \end{cases} \tag{3.19}$$

which is the same formula as (3.17). Substituting (3.9) yields

$$\begin{cases} u_9 = -z^2/y^2 - xz - (\alpha + \beta - 1)z/y, \\ w_9 = y/z, \\ v_9 = z. \end{cases} \tag{3.20}$$

By this coordinates change, $(\mathrm{P}_{\mathrm{III}(D_6)})$ is transformed into the Hamiltonian system of (u_9, w_9), whose Hamiltonian function has the same form as (3.18), though (α, β) is replaced by $(\widetilde{\alpha}, \widetilde{\beta}) = (\alpha, 1 - \beta)$.

In this manner, all singularities are resolved and we have

Theorem 3.2. The space of initial conditions $E(z)$ of $(\mathrm{P}_{\mathrm{III}(D_6)})$ is given by $\mathbb{C}^2_{(x,y)} \cup \mathbb{C}^2_{(u_3,w_3)} \cup \mathbb{C}^2_{(u_6,w_6)} \cup \mathbb{C}^2_{(u_9,w_9)}$ glued by the symplectic transformations (3.13), (3.17) and (3.20). The space $E(z)$ is a nonsingular symplectic surface parameterized by $z \in \mathbb{C}\backslash\{0\}$, on which $(\mathrm{P}_{\mathrm{III}(D_6)})$ is expressed as a polynomial Hamiltonian system.

3.3 The Riccati solutions

It is known that when the parameter α is an integer or β is an integer, there exists a one-parameter family of solutions of $(\mathrm{P}_{\mathrm{III}(D_6)})$ satisfying the Riccati equation, which is equivalent to the Bessel equation. For example, when $\alpha = 0$ (resp. $\beta = 0$), the Riccati equation is given by

$$z\frac{dy}{dz} = -y^2 + \beta y + z, \quad (\text{resp.} \quad z\frac{dy}{dz} = y^2 + \alpha y + z). \tag{3.21}$$

The Bäcklund transformations yield the Riccati equations for a general case $\alpha \in \mathbb{Z}$ or $\beta \in \mathbb{Z}$. Let us prove this fact from a view point of dynamical systems theory.

Recall that there are two fixed points $(X_2, Z_2, \varepsilon_2) = (0, 1, 0), (1, -1, 0)$ of the vector field (3.4) corresponding to the irregular singular point. The Jacobi matrices at these points are shown in (3.8), and eigenvalues of them are $2, -2, 0$. Hence, there exist a center-stable manifold and a center-unstable manifold at these points. Let us calculate them explicitly.

For the point $(X_2, Z_2, \varepsilon_2) = (0, 1, 0)$, put $\hat{Z}_2 = Z_2 - 1$ and

$$\begin{cases} u = X_2 + \hat{Z}_2 + \dfrac{1}{2}\varepsilon_2(\alpha + 2\beta - 1), \\ v = -X_2 - \dfrac{1}{2}\alpha\varepsilon_2, \\ w = \varepsilon_2. \end{cases}$$

Then, the linear part of Eq.(3.4) is diagonalized around $(0, 1, 0)$ and we obtain

$$\begin{cases} \dot{u} = -2u + u\left(-2u - \dfrac{1}{2}w - \dfrac{\alpha}{2}w + \beta w\right) + (1-\alpha)\left(\dfrac{1}{2}vw + \dfrac{1}{4}w^2 - \dfrac{\beta}{2}w^2\right), \\ \dot{v} = 2v + v\left(2v + \dfrac{\alpha}{2}w - \beta w\right) + \dfrac{1}{2}\alpha\left(uw + \dfrac{1}{2}w^2 - \beta w^2\right), \\ \dot{w} = w\left(v - u - \dfrac{1}{2}w\right). \end{cases} \tag{3.22}$$

The stable, unstable, center subspaces are u, v, w-directions, respectively. In particular, the center-unstable manifold (resp. the center-stable manifold) is expressed as the graph of some function $v = \phi(u, w)$ (resp. $u = \varphi(v, w)$).

It is obvious that when $\alpha = 0$, the center-unstable manifold is exactly given by $v = 0$. This implies $X_2 = x = 0$. When $\alpha = 0$ and $x = 0$, Eq.(1.7) is reduced to the Riccati equation (3.21). Similarly, when $\alpha = 1$, the center-stable manifold is exactly given by $u = 0$. This implies

$$0 = X_2 + \hat{Z}_2 + \beta\varepsilon_2 = x + z/y^2 + \beta/y - 1.$$

Substituting this relation to Eq.(1.7) yields the Riccati equation for $\alpha = 1$, though it is also obtained by the Bäcklund transformation to that for $\alpha = 0$. The same argument at the point $(X_2, Z_2, \varepsilon_2) = (1, -1, 0)$ provides the Riccati equation for

$\beta = 0$.

Theorem 3.3. When $\alpha \in \mathbb{Z}$ or $\beta \in \mathbb{Z}$, there exists a one-parameter family of solutions of $(\mathrm{P}_{\mathrm{III}(D_6)})$ governed by the Riccati equation of Bessel type. The one-parameter family of solutions forms a center-(un)stable manifold of (3.4) in $\mathbb{C}P^3(0,1,2,1)$.

The same argument is applicable to the other Painlevé equations, even for higher dimensional Painlevé equations. For two dimensional equations, it is well known that when parameters take certain specific values, the Painlevé equations are reduced to Riccati-type equations except for $(\mathrm{P_I}), (\mathrm{P}_{\mathrm{III}(D_7)})$ and $(\mathrm{P}_{\mathrm{III}(D_8)})$. For such specific values of parameters, center-(un)stable manifolds in $\mathbb{C}P^3(p,q,r,s)$ are exactly calculated and the Riccati equations are obtained by restricting the equations to the center-(un)stable manifolds.

The result is summarized in Table 3. The third column shows one of the parameters for which equations are reduced to the Riccati equations. The other possible parameters are obtained by the Bäcklund transformations. The fourth column denotes the name of the Riccati equation when it is written as a second order linear equation. Note that the weight of the Riccati equation is also defined through the Newton polyhedron. For example, the Riccati equation of Airy type is defined by

$$\frac{dy}{dz} = y^2 + z. \tag{3.23}$$

The exponents of monomials in the right hand side are $(1,1)$ and $(-1,2)$. Since they are on the line $y + 2z = 3$, the weighted projective space for the equation is $\mathbb{C}P^2(1,2,3)$. See [4] for the analysis of the Airy equation by means of the weighted projective space. The last column of Table 3 gives the weight of each Riccati equation. It is interesting to note that these weights are obtained by deleting the first or second numbers from the weights of the corresponding Painlevé equations.

	weight	parameter	Riccati	weight
$\mathrm{P_{II}}$	$(2,1,2,3)$	$\alpha = 1/2$	Airy	$(1,2,3)$
$\mathrm{P_{IV}}$	$(1,1,1,2)$	$\alpha = 0$	Hermite	$(1,1,2)$
$\mathrm{P}_{\mathrm{III}(D_6)}$	$(0,1,2,1)$	$\alpha = 0$	Bessel	$(1,2,1)$
$\mathrm{P_V}$	$(1,0,1,1)$	$\alpha_1 = 0$	CHG	$(1,1,1)$
$\mathrm{P_{VI}}$	$(1,0,0,1)$	$\alpha_1 = 0$	HG	$(1,0,1)$

Table 3: The type of Riccati equations and their weights. CHG and HG denote the confluent hypergeometric and hypergeometric equations, respectively.

3.4 Boutroux's coordinates

For the first and second Painlevé equations, the third local chart $(X_3, Y_3, \varepsilon_3)$ of $\mathbb{C}P^3(p,q,r,s)$ defined by Eq.(2.2) is equivalent to Boutroux's coordinates introduced

in [1] to investigate the irregular singular point $z = \infty$. For the other Painlevé equations, the chart $(X_3, Y_3, \varepsilon_3)$ plays the same role as Boutroux's coordinates; as $\varepsilon_3 \to 0$, Eq.(3.5) is reduced to the autonomous Hamiltonian system

$$\begin{cases} \dot{X}_3 = 2X_3Y_3 - 2X_3^2Y_3, \\ \dot{Y}_3 = 1 - Y_3^2 + 2X_3Y_3^2, \end{cases} \tag{3.24}$$

which is often called the autonomous limit. Since a generic integral curve given by $\mathcal{H}_{III(D_6)} = X_3^2Y_3^2 - X_3Y_3^2 + X_3 = \text{const.}$ is an elliptic curve, a general solution can be expressed by Weierstrass's elliptic functions. Then, the system (3.5) with small ε_3 can be studied by a perturbation method.

Let us calculate the action of the extended affine Weyl group $\widetilde{W}((2A_1)^{(1)})$ restricted on the set $\{\varepsilon_3 = 0\}$, which leaves the autonomous limit (3.24) invariant.

Proposition 3.4. The birational transformation group $\widetilde{W}((2A_1)^{(1)})$ is extended to the birational transformation group acting on $\mathbb{C}P^3(0, 1, 2, 1)$. The transformation group which leaves the autonomous limit (3.24) invariant is given by $\mathbb{Z}_2 \ltimes \mathrm{Aut}((2A_1)^{(1)}) = \mathbb{Z}_2 \ltimes \langle \pi_1, \pi_2, \sigma_1 \rangle$.

Proof. The first part of Proposition is verified by a straightforward calculation. To show the second part, we should write down the actions in the $(X_3, Y_3, \varepsilon_3)$-chart. For example, the action of s_1 in the $(X_3, Y_3, \varepsilon_3)$-chart is given by

$$(X_3, Y_3, \varepsilon_3, \alpha, \beta) \mapsto (X_3, Y_3 + \alpha\frac{\varepsilon_3}{X_3}, \varepsilon_3, -\alpha, \beta).$$

On the set $\{\varepsilon_3 = 0\}$, it is reduced to

$$(X_3, Y_3, \alpha, \beta) \mapsto (X_3, Y_3, -\alpha, \beta).$$

Since the autonomous limit (3.24) is independent of the parameters α, β, the action of s_1 to (3.24) is trivial. Similarly, the actions of s_0, s_0' and s_1' to (3.24) are reduced to the trivial one. On the other hand, it is easy to confirm that the restriction of the actions of π_1, π_2, σ_1 are not trivial, which are explicitly given by

$$\begin{aligned} &\pi_1 : (X_3, Y_3) \mapsto (X_3Y_3^2 - Y_3^2 + 1,\ -1/Y_3), \\ &\pi_2 : (X_3, Y_3) \mapsto (-X_3Y_3^2,\ 1/Y_3), \\ &\sigma_1 : (X_3, Y_3) \mapsto (1 - X_3,\ \sqrt{-1}Y_3). \end{aligned}$$

Furthermore, (3.24) is invariant under the $\mathbb{Z}_2$ action $Y_3 \mapsto -Y_3$ due to the orbifold structure of $\mathbb{C}P^3(0, 1, 2, 1)$. Since $r = \deg(z) = 2$, $(X_3, Y_3, \varepsilon_3)$ are coordinates on the lift $\mathbb{C}^3$ of the quotient $\mathbb{C}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ action is defined by $(X_3, Y_3, \varepsilon_3) \mapsto (X_3, -Y_3, -\varepsilon_3)$. This is reduced to the action $Y_3 \mapsto -Y_3$ on the set $\{\varepsilon_3 = 0\}$. □

A similar result also holds for the other Painlevé equations except for ($\mathrm{P_{VI}}$) and summarized in Table 4. If $r = \deg(z) = 1$, the symmetry group of the autonomous limit is given by the Dynkin automorphism group $G = \mathrm{Aut}(R^{(1)})$ because the action of $W(R^{(1)})$ on the set $\{\varepsilon_3 = 0\}$ is reduced to the trivial action as above. If $r > 1$, the

	weight	$\mathcal{H}_J$	symmetry
$\mathrm{P_I}$	$(3,2,4,5)$	X^2-4Y^3-2Y	$\mathbb{Z}_4$
$\mathrm{P_{II}}$	$(2,1,2,3)$	$X^2-Y^4-Y^2$	$\mathbb{Z}_2 \ltimes \mathrm{Aut}(A_1^{(1)})$
$\mathrm{P_{IV}}$	$(1,1,1,2)$	X^2Y-XY^2-2XY	$\mathrm{Aut}(A_2^{(1)})$
$\mathrm{P_{III(D_8)}}$	$(-1,2,4,1)$	$X^2Y^2-\frac{1}{2}Y-\frac{1}{2Y}$	$\mathbb{Z}_4 \ltimes \mathbb{Z}_2$
$\mathrm{P_{III(D_7)}}$	$(-1,2,3,1)$	X^2Y^2+X+Y	$\mathbb{Z}_3 \ltimes \mathrm{Aut}(A_1^{(1)})$
$\mathrm{P_{III(D_6)}}$	$(0,1,2,1)$	$X^2Y^2-XY^2+X$	$\mathbb{Z}_2 \ltimes \mathrm{Aut}((2A_1)^{(1)})$
$\mathrm{P_V}$	$(1,0,1,1)$	$X^2Y^2-X^2Y+XY^2-XY$	$\mathrm{Aut}(A_3^{(1)})$

Table 4: Hamiltonian functions of the autonomous limit defined on the set $\{\varepsilon_3=0\}$. (X_3,Y_3) is denoted by (X,Y) for simplicity.

autonomous limit is further invariant under the $\mathbb{Z}_r$ action arising from the orbifold structure of $\mathbb{C}P^3(p,q,r,s)$. The autonomous limit on the set $\{\varepsilon_3=0\}$ is not defined for $(\mathrm{P_{VI}})$ in this way because $r=0$.

In the rest of this paper, the other Painlevé equations $(\mathrm{P_{III(D_7)}})$, $(\mathrm{P_{III(D_8)}})$, $(\mathrm{P_V})$ and $(\mathrm{P_{VI}})$ are studied with the aid of the weighted projective spaces and dynamical systems theory (see [2] for $(\mathrm{P_I})$, $(\mathrm{P_{II}})$ and $(\mathrm{P_{IV}})$). Since the strategy is completely the same as that for $(\mathrm{P_{III(D_6)}})$, we only show important steps and formulae.

4 The third Painlevé equation of type D_7

The orbifold $\mathbb{C}P^3(-1,2,3,1)$ for $(\mathrm{P_{III(D_7)}})$ is covered by four inhomogeneous coordinates $(Y_1,Z_1,\varepsilon_1),(X_2,Z_2,\varepsilon_2),(X_3,Y_3,\varepsilon_3)$ and (x,y,z) related as

$$\left\{\begin{array}{llll} x= & \varepsilon_1= & X_2\varepsilon_2= & X_3\varepsilon_3 \\ y= & Y_1\varepsilon_1^{-2}= & \varepsilon_2^{-2}= & Y_3\varepsilon_3^{-2} \\ z= & Z_1\varepsilon_1^{-3}= & Z_2\varepsilon_2^{-3}= & \varepsilon_3^{-3}. \end{array}\right. \tag{4.1}$$

We give the third Painlevé equation of type D_7 on the local chart (x,y,z). On the other local charts, $(\mathrm{P_{III(D_7)}})$ is expressed as rational differential equations. By rewriting them as 3-dim autonomous vector fields, we obtain

$$\left\{\begin{array}{l} \dot{Y}_1=2Y_1+2Y_1^2-Z_1+\alpha\varepsilon_1Y_1, \\ \dot{Z}_1=3Z_1+6Y_1Z_1-\varepsilon_1Z_1+3\alpha\varepsilon_1Z_1, \\ \dot{\varepsilon}_1=\varepsilon_1(1+2Y_1+\alpha\varepsilon_1), \end{array}\right. \tag{4.2}$$

$$\left\{\begin{array}{l} \dot{X}_2=2+2X_2^2-X_2Z_2+\alpha\varepsilon_2X_2, \\ \dot{Z}_2=6X_2Z_2+3Z_2^2+3\alpha\varepsilon_2Z_2-2\varepsilon_2Z_2, \\ \dot{\varepsilon}_2=\varepsilon_2(2X_2+Z_2+\alpha\varepsilon_2), \end{array}\right. \tag{4.3}$$

and

$$\begin{cases} \dot{X}_3 = -1 - 2X_3^2 Y_3 + \varepsilon_3(\frac{1}{3}X_3 - \alpha X_3), \\ \dot{Y}_3 = 1 + 2X_3 Y_3^2 + \varepsilon_3(-\frac{2}{3}Y_3 + \alpha Y_3), \\ \dot{\varepsilon}_3 = -\frac{1}{3}\varepsilon_3^2. \end{cases} \tag{4.4}$$

We have the decomposition

$$\mathbb{C}P^3(-1,2,3,1) \;=\; \mathbb{C}P^2(-1,2,3) \cup \mathbb{C}^3 \quad \text{(disjoint)}.$$

where $\mathbb{C}^3 = \{(x,y,z)\}$ and $\mathbb{C}P^2(-1,2,3) = \{\varepsilon_1 = 0\} \cup \{\varepsilon_2 = 0\} \cup \{\varepsilon_3 = 0\}$ in coordinates. This implies that the set $\mathbb{C}P^2(-1,2,3)$ is attached at infinity of the phase space $\mathbb{C}^3$. Thus, the asymptotic behavior of solutions can be studied by the limit $\varepsilon_1 \to 0$ or $\varepsilon_2 \to 0$ or $\varepsilon_3 \to 0$.

The autonomous limit is a Hamiltonian system obtained by putting $\varepsilon_3 = 0$ for Eq.(4.4). On the set $\varepsilon_3 = 0$, the action of the extended affine Weyl group $\widetilde{W}(A_1^{(1)}) \simeq \mathrm{Aut}(A_1^{(1)}) \ltimes W(A_1^{(1)})$ is reduced to the action of $\mathrm{Aut}(A_1^{(1)}) \simeq \mathbb{Z}_2$ defined by $(X_3, Y_3) \mapsto (Y_3, X_3)$. Further, the autonomous limit is invariant under the $\mathbb{Z}_3$ action ($r = 3$) induced by the orbifold structure, see Table 4.

The vector field (4.2) has fixed points on the infinity set $\mathbb{C}P^2(-1,2,3)$ given by $(Y_1, Z_1, \varepsilon_1) = (-1,0,0)$ and $(-1/2,-1/2,0)$. The Jacobi matrices of (4.2) at the fixed points are

$$-\begin{pmatrix} 2 & 1 & \alpha \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad -\begin{pmatrix} 0 & 1 & \alpha/2 \\ 3 & 0 & (3\alpha-1)/2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.5}$$

respectively. The latter fixed point having a zero eigenvalue corresponds to the irregular singular point, and the former one corresponds to a movable singularity associated with the regular Laurent series solution (i) given in Sec.2.2; The Laurent series (i) converges to the point $(Y_1, Z_1, \varepsilon_1) = (-1,0,0)$ as $z \to z_0$. The exceptional Laurent series solution (ii) does not converge to some point on $\mathbb{C}P^3(-1,2,3,1)$ as $z \to z_0$.

Remark 4.1. These fixed points are also included in the chart $(X_2, Z_2, \varepsilon_2)$. However, it is better to use the first chart $(Y_1, Z_1, \varepsilon_1)$ because the second chart has to be divided by the $\mathbb{Z}_2$ action due to the orbifold structure.

To treat the Laurent series (ii), we use the Bäcklund transformation σ defined by

$$(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\alpha}) = \sigma(x,y,z,\alpha) = \left(-\frac{y}{z},\, zx,\, -z,\, 1-\alpha\right). \tag{4.6}$$

We consider another space $\mathbb{C}P^3(-1,2,3,1)$ with inhomogeneous coordinates denoted by $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ etc. We glue two copies of $\mathbb{C}P^3(-1,2,3,1)$ by the transformation σ.

Then, it is easy to show that the exceptional Laurent series (ii) is converted to the regular series (i) in the $(\widetilde{x},\widetilde{y},\widetilde{z})$-chart, and it approaches to the fixed point $(\widetilde{Y}_1,\widetilde{Z}_1,\widetilde{\varepsilon}_1)=(-1,0,0)$ as $z\to z_0$.

To construct the space of initial conditions of ($\mathrm{P}_{\mathrm{III}(D_7)}$), we perform the weighted blow-ups at the points $(Y_1,Z_1,\varepsilon_1)=(-1,0,0)$ and $(\widetilde{Y}_1,\widetilde{Z}_1,\widetilde{\varepsilon}_1)=(-1,0,0)$.

(i) blow-up at $(Y_1,Z_1,\varepsilon_1)=(-1,0,0)$.

For the vector field (4.2), put $\hat{Y}_1=Y_1+1$ and

$$u=\hat{Y}_1-Z_1+\alpha\varepsilon_1,\quad v=Z_1,\quad w=\varepsilon_2.$$

Then, the linear part is diagonalized around $(-1,0,0)$ and we obtain

$$\begin{cases}\dot{u}=2u-2u^2+2uv+4v^2-vw+\alpha(uw-2vw),\\ \dot{v}=3v+v(w-6u-6v+3\alpha w),\\ \dot{w}=w-w(2u+2v-\alpha w).\end{cases}$$

The origin $(u,v,w)=(0,0,0)$ is a singularity of the foliation of integral curves. To resolve it, we introduce the weighted blow-up defined by

$$\begin{cases} u &= u_1^2 &= v_2^2u_2 &= w_3^2u_3,\\ v &= u_1^3v_1 &= v_2^3 &= w_3^3v_3,\\ w &= u_1w_1 &= v_2w_2 &= w_3.\end{cases}\tag{4.7}$$

The weight $(2,3,1)$ is taken from the eigenvalues of the Jacobi matrix at the singularity. The relation between the original coordinates (x,y,z) and the new coordinates (u_3,v_3,w_3) is given by

$$\begin{cases}x=w_3,\\ y=u_3+v_3w_3-\alpha/w_3-1/w_3^2,\\ z=v_3,\end{cases}\qquad\begin{cases}u_3=y-xz+\alpha/x+1/x^2,\\ w_3=x,\\ v_3=z.\end{cases}\tag{4.8}$$

In the new coordinates, ($\mathrm{P}_{\mathrm{III}(D_7)}$) is transformed into the Hamiltonian system

$$\frac{dw_3}{dz}=-\frac{\partial H_1}{\partial u_3},\quad \frac{du_3}{dz}=\frac{\partial H_1}{\partial w_3},$$

with the polynomial Hamiltonian function

$$zH_1=-u+u^2w^2-\frac{1}{2}w^2z+2uw^3z+w^4z^2-\alpha(uw+w^2z),\tag{4.9}$$

(here, the subscript is omitted for simplicity). Furthermore, we can verify the symplectic relation (3.15).

(ii) blow-up at $(\widetilde{Y}_1,\widetilde{Z}_1,\widetilde{\varepsilon}_1)=(-1,0,0)$.

This singularity is resolved by the same way as above by the weighted blow-up with the weight $(2,3,1)$. The result is easily obtained by the Bäcklund transformation σ as follows. Define the new coordinates (u_6, v_6, w_6) by

$$\begin{cases} u_6 = \widetilde{y} - \widetilde{x}\widetilde{z} + \widetilde{\alpha}/\widetilde{x} + 1/\widetilde{x}^2, \\ w_6 = \widetilde{x}, \\ v_6 = \widetilde{z}. \end{cases} \tag{4.10}$$

Substituting (4.6) yields

$$\begin{cases} u_6 = xz + xy - (1-\alpha)z/y + z^2/y^2, \\ w_6 = -y/z, \\ v_6 = -z. \end{cases} \tag{4.11}$$

By this coordinates change, $(\mathrm{P}_{\mathrm{III}(D_7)})$ is transformed into the Hamiltonian system of (u_6, w_6), whose Hamiltonian function has the same form as (4.9), for which α is replaced by $\widetilde{\alpha} = 1 - \alpha$.

In this manner, all singularities are resolved and the space of initial conditions of $(\mathrm{P}_{\mathrm{III}(D_7)})$ is given by $\mathbb{C}^2_{(x,y)} \cup \mathbb{C}^2_{(u_3,w_3)} \cup \mathbb{C}^2_{(u_6,w_6)}$ glued by the symplectic transformations (4.8), (4.11).

5 The third Painlevé equation of type D_8

The orbifold $\mathbb{C}P^3(-1,2,4,1)$ for $(\mathrm{P}_{\mathrm{III}(D_8)})$ is covered by four inhomogeneous coordinates related as

$$\begin{cases} x = & \varepsilon_1 = & X_2\varepsilon_2 = & X_3\varepsilon_3 \\ y = & Y_1\varepsilon_1^{-2} = & \varepsilon_2^{-2} = & Y_3\varepsilon_3^{-2} \\ z = & Z_1\varepsilon_1^{-4} = & Z_2\varepsilon_2^{-4} = & \varepsilon_3^{-4}. \end{cases} \tag{5.1}$$

We give the third Painlevé equation of type D_8 on the local chart (x, y, z). On the other local charts, $(\mathrm{P}_{\mathrm{III}(D_8)})$ is expressed as rational differential equations. By rewriting them as 3-dim autonomous vector fields, we obtain

$$\begin{cases} \dot{Y}_1 = Y_1^3 - 2Y_1^4 - Y_1Z_1, \\ \dot{Z}_1 = 2Y_1^2Z_1 - 2Z_1^2 - 8Y_1^3Z_1 + \varepsilon_1 Y_1^2 Z_1, \\ \dot{\varepsilon}_1 = \varepsilon_1(\dfrac{1}{2}Y_1^2 - 2Y_1^3 - \dfrac{1}{2}Z_1), \end{cases} \tag{5.2}$$

$$\begin{cases} \dot{X}_3 = -2X_3^2Y_3 + \dfrac{1}{2} - \dfrac{1}{2Y_3^2} + \dfrac{1}{4}\varepsilon_3 X_3, \\ \dot{Y}_3 = 2X_3Y_3^2 - \dfrac{1}{2}\varepsilon_3 Y_3, \\ \dot{\varepsilon}_3 = -\dfrac{1}{4}\varepsilon_3^2. \end{cases} \tag{5.3}$$

We will not use the vector field written in $(X_2, Z_2, \varepsilon_2)$-chart.

The autonomous limit is a Hamiltonian system obtained by putting $\varepsilon_3 = 0$ for Eq.(5.3). It is known that $(\mathrm{P}_{\mathrm{III}(D_8)})$ is invariant under the transformation π defined by

$$(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \pi(x, y, z) = \left(-\frac{xy^2}{z} + \frac{y}{2z},\ \frac{z}{y},\ z \right). \tag{5.4}$$

On the set $\varepsilon_3 = 0$, this action is reduced to the $\mathbb{Z}_2$ action given by $(X_3, Y_3) \mapsto (-X_3 Y_3^2,\ 1/Y_3)$. Further, the autonomous limit is invariant under the $\mathbb{Z}_4$ action $(r = 4)$ induced by the orbifold structure, see Table 4.

The vector field (5.2) has the fixed point $(Y_1, Z_1, \varepsilon_1) = (1/2, 0, 0)$. The Jacobi matrix of (5.2) at the fixed point is

$$-\frac{1}{8} \begin{pmatrix} 2 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.5}$$

The regular Laurent series solution (i) given in Sec.2.2 converges to this point as $z \to z_0$, while the exceptional Laurent series solution (ii) does not converge to some point on $\mathbb{C}P^3(-1, 2, 4, 1)$. To treat the Laurent series (ii), consider another space $\mathbb{C}P^3(-1, 2, 4, 1)$ with inhomogeneous coordinates denoted by $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ etc. We glue two copies of $\mathbb{C}P^3(-1, 2, 4, 1)$ by the transformation (5.4). Then, it is easy to show that the exceptional Laurent series (ii) is converted to the regular series (i) in the $(\widetilde{x}, \widetilde{y}, \widetilde{z})$-chart, and it approaches to the fixed point $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (1/2, 0, 0)$ as $z \to z_0$.

To construct the space of initial conditions of $(\mathrm{P}_{\mathrm{III}(D_8)})$, we perform the weighted blow-ups at the points $(Y_1, Z_1, \varepsilon_1) = (1/2, 0, 0)$ and $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (1/2, 0, 0)$.

(i) blow-up at $(Y_1, Z_1, \varepsilon_1) = (1/2, 0, 0)$.

For the vector field (5.2), put $\hat{Y}_1 = Y_1 - 1/2$ and

$$u = \hat{Y}_1 - 2Z_1, \quad v = Z_1, \quad w = \varepsilon_2.$$

Then, the linear part is diagonalized around $(-1, 0, 0)$. To resolve the singularity $(u, v, w) = (0, 0, 0)$ of the resultant equation, we introduce the weighted blow-up defined by

$$\begin{cases} u &= u_1^2 &= v_2^2 u_2 &= w_3^2 u_3, \\ v &= u_1^4 v_1 &= v_2^4 &= w_3^4 v_3, \\ w &= u_1 w_1 &= v_2 w_2 &= w_3. \end{cases} \tag{5.6}$$

The relation between the original coordinates (x, y, z) and the new coordinates (u_3, v_3, w_3) is given by

$$\begin{cases} x = w_3, \\ y = u_3 + 2v_3 w_3^2 + \dfrac{1}{2w_3^2}, \\ z = v_3, \end{cases} \qquad \begin{cases} u_3 = y - 2x^2 z - \dfrac{1}{2x^2}, \\ w_3 = x, \\ v_3 = z. \end{cases} \tag{5.7}$$

In the new coordinates, ($\mathrm{P}_{\mathrm{III}(D_8)}$) is transformed into the Hamiltonian system with the Hamiltonian function

$$zH_1 = \frac{u}{2} + u^2w^2 + w^2z - \frac{2}{3}w^3z + 4uw^4z + 4w^6z^2 - \frac{w^2z}{1+2uw^2+4w^4z}, \tag{5.8}$$

(here, the subscript is omitted for simplicity). Furthermore, we can verify the symplectic relation (3.15).

The blow-up at $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (-1, 0, 0)$ is done in the same way and the Hamiltonian function is easily obtained by applying the transformation (5.4) to the above H_1. In this manner, we can obtain the space of initial conditions.

6 The fifth Painlevé equation

The orbifold $\mathbb{C}P^3(1,0,1,1)$ for (P_{V}) is covered by three inhomogeneous coordinates $(Y_1, Z_1, \varepsilon_1), (X_3, Y_3, \varepsilon_3)$ and (x, y, z) related as

$$\begin{cases} x = & \varepsilon_1^{-1} = & X_3\varepsilon_3^{-1} \\ y = & Y_1 = & Y_3 \\ z = & Z_1\varepsilon_1^{-1} = & \varepsilon_3^{-1}. \end{cases} \tag{6.1}$$

The second chart does not appear because $q = 0$. We give the fifth Painlevé equation on the local chart (x, y, z). On the other local charts, (P_{V}) is expressed as rational differential equations. By rewriting them as 3-dim autonomous vector fields, we obtain

$$\begin{cases} \dot{Y}_1 = -2Y_1 + 2Y_1^2 - Y_1Z_1 + Y_1^2Z_1 + \alpha_1\varepsilon_1 - (\alpha_1+\alpha_3)Y_1\varepsilon_1, \\ \dot{Z}_1 = -Z_1 + 2Y_1Z_1 - Z_1^2 + \varepsilon_1Z_1 + 2Y_1Z_1^2 - (\alpha_1+\alpha_3)Z_1\varepsilon_1 + \alpha_2\varepsilon_1Z_1^2, \\ \dot{\varepsilon}_1 = \varepsilon_1(-1 + 2Y_1 - Z_1 + 2Y_1Z_1 - (\alpha_1+\alpha_3)\varepsilon_1 + \alpha_2Z_1\varepsilon_1), \end{cases} \tag{6.2}$$

$$\begin{cases} \dot{X}_3 = X_3 + X_3^2 - 2X_3Y_3 - 2X_3^2Y_3 - \alpha_2\varepsilon_3 + (\alpha_1+\alpha_3-1)\varepsilon_3X_3, \\ \dot{Y}_3 = -Y_3 + Y_3^2 - 2X_3Y_3 + 2X_3Y_3^2 + \alpha_1\varepsilon_3 - (\alpha_1+\alpha_3)\varepsilon_3Y_3, \\ \dot{\varepsilon}_3 = -\varepsilon_3^2. \end{cases} \tag{6.3}$$

The autonomous limit is a Hamiltonian system obtained by putting $\varepsilon_3 = 0$ for Eq.(6.3). On the set $\{\varepsilon_3 = 0\}$, the action of the extended affine Weyl group $\widetilde{W}(A_3^{(1)}) \simeq \mathrm{Aut}(A_3^{(1)}) \ltimes W(A_3^{(1)})$ is reduced to the action of $\mathrm{Aut}(A_3^{(1)})$ generated by $(X_3, Y_3) \mapsto (Y_3 - 1, -X_3)$ and $(X_3, Y_3) \mapsto (X_3, 1 - Y_3)$.

The vector field (6.2) has fixed points on the infinity set $\mathbb{C}P^2(1,0,1)$ given by

$$(Y_1, Z_1, \varepsilon_1) = (0,0,0),\ (1,0,0),\ (0,-1,0),\ (1/2,-2,0),\ (1,-1,0). \tag{6.4}$$

The Jacobi matrices of (6.2) at these fixed points are

$$-\begin{pmatrix} 2 & 0 & -\alpha_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -\alpha_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & \alpha_1 \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1/4 & * \\ 4 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\alpha_3 \\ 0 & -1 & * \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

respectively, where $*$ denotes certain long numerical expressions. The latter three fixed points having zero eigenvalues correspond to the irregular singular point. The center-(un)stable manifolds at these points can be exactly calculated for certain specific values of parameters, which give the Riccati equations of confluent hypergeometric type (Table 3). The former two points correspond to movable singularities associated with the regular Laurent series solutions (ii) and (i) given in Sec.2.2. The Laurent series (ii) and (i) converge to the points $(Y_1, Z_1, \varepsilon_1) = (0, 0, 0)$ and $(1, 0, 0)$ as $z \to z_0$, respectively. The exceptional Laurent series solutions (iii) and (iv) do not converge to some point on $\mathbb{C}P^3(1, 0, 1, 1)$ as $z \to z_0$.

To treat the Laurent series (iii) and (iv), we use the Bäcklund transformation π defined by

$$\begin{aligned}(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3) &= \pi(x, y, z, \alpha_1, \alpha_2, \alpha_3) \\ &= \left(z(y-1),\, -\frac{x}{z},\, z,\, \alpha_2,\, \alpha_3,\, 1-\alpha_1-\alpha_2-\alpha_3\right). \end{aligned} \tag{6.6}$$

We consider another space $\mathbb{C}P^3(1, 0, 1, 1)$ with inhomogeneous coordinates denoted by $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ etc. We glue two copies of $\mathbb{C}P^3(1, 0, 1, 1)$ by the transformation π. Then, the exceptional Laurent series (iii) and (iv) are converted to the regular series (i) and (ii), respectively, in the $(\widetilde{x}, \widetilde{y}, \widetilde{z})$-chart. They converge to the fixed points $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (1, 0, 0)$ and $(0, 0, 0)$ as $z \to z_0$, respectively.

To construct the space of initial conditions of ($\mathrm{P_V}$), we perform the weighted blow-ups at these four points.

(i) blow-up at $(Y_1, Z_1, \varepsilon_1) = (0, 0, 0)$.

For the vector field (6.2), put $u = Y_1 - \alpha_1\varepsilon_1$, $v = Z_1$, $w = \varepsilon_1$. Then, the linear part is diagonalized. For the system of (u, v, w), we introduce the weighted blow-up defined by

$$\left\{\begin{array}{llll} u &= u_1^2 & = v_2^2 u_2 & = w_3^2 u_3, \\ v &= u_1 v_1 & = v_2 & = w_3 v_3, \\ w &= u_1 w_1 & = v_2 w_2 & = w_3. \end{array}\right. \tag{6.7}$$

The weight $(2, 1, 1)$ is taken from the eigenvalues of the Jacobi matrix at the singularity. The relation between the original coordinates (x, y, z) and the new coordinates (u_3, v_3, w_3) is given by

$$\left\{\begin{array}{l} x = 1/w_3, \\ y = u_3 w_3^2 + \alpha_1 w_3, \\ z = v_3, \end{array}\right. \qquad \left\{\begin{array}{l} u_3 = x^2 y - \alpha_1 x, \\ w_3 = 1/x, \\ v_3 = z. \end{array}\right. \tag{6.8}$$

In the new coordinates, ($\mathrm{P_V}$) is transformed into the Hamiltonian system, whose Hamiltonian function is

$$zH_1 = u - u^2w^2 - u^2w^3z + (z - \alpha_1 + \alpha_3)uw - (2\alpha_1 + \alpha_2)uw^2z - \alpha_1(\alpha_1 + \alpha_2)wz, \tag{6.9}$$

(here, the subscript is omitted for simplicity). Furthermore, we can verify the symplectic relation (3.15).

(ii) blow-up at $(Y_1, Z_1, \varepsilon_1) = (1, 0, 0)$.

For the vector field (6.2), put $\hat{Y}_1 = Y_1 - 1$ and $u = \hat{Y}_1 - \alpha_3\varepsilon_1$, $v = Z_1$, $w = \varepsilon_1$. Then, the linear part is diagonalized around $(1, 0, 0)$. For the system of (u, v, w), we introduce the weighted blow-up defined by

$$\begin{cases} u &= u_4^2 &= v_5^2 u_5 &= w_6^2 u_6, \\ v &= u_4 v_4 &= v_5 &= w_6 v_6, \\ w &= u_4 w_4 &= v_5 w_5 &= w_6. \end{cases} \tag{6.10}$$

The relation between the original coordinates (x, y, z) and the coordinates (u_6, v_6, w_6) is given by

$$\begin{cases} x = 1/w_6, \\ y = u_6 w_6^2 + \alpha_3 w_6 + 1, \\ z = v_6, \end{cases} \qquad \begin{cases} u_6 = x^2 y - x^2 - \alpha_3 x, \\ w_6 = 1/x, \\ v_6 = z. \end{cases} \tag{6.11}$$

In the new coordinates, $(\mathrm{P_V})$ is transformed into the Hamiltonian system, whose Hamiltonian function is

$$zH_1 = -u - u^2 w^2 - u^2 w^3 z + (-z + \alpha_1 - \alpha_3)uw - (2\alpha_3 + \alpha_2)uw^2 z - \alpha_3(\alpha_2 + \alpha_3)wz, \tag{6.12}$$

(here, the subscript is omitted for simplicity). Again, we can verify the symplectic relation (3.15).

The singularities $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (1, 0, 0)$ and $(0, 0, 0)$ are resolved by the same way as above by the weighted blow-up with the weight $(2, 1, 1)$. The result is easily obtained by the Bäcklund transformation π as in the previous sections. In this manner, all singularities are resolved and it turns out that the space of initial conditions of $(\mathrm{P_V})$ is given by five copies of $\mathbb{C}^2$ glued by the symplectic transformations.

7 The sixth Painlevé equation

The orbifold $\mathbb{C}P^3(1, 0, 0, 1)$ for $(\mathrm{P_{VI}})$ is covered by two inhomogeneous coordinates $(Y_1, Z_1, \varepsilon_1)$ and (x, y, z) related as

$$\begin{cases} x = & \varepsilon_1^{-1} \\ y = & Y_1 \\ z = & Z_1. \end{cases} \tag{7.1}$$

We give the sixth Painlevé equation on the local chart (x, y, z). On the other local chart $(Y_1, Z_1, \varepsilon_1)$, $(\mathrm{P_{VI}})$ is expressed as a rational differential equation. As before,

we rewrite it as a 3-dim autonomous polynomial vector field, whose expression is too long and omitted here. This vector field has fixed points

$$(Y_1, Z_1, \varepsilon_1) = (0, c, 0),\ (1, c, 0),\ (c, c, 0), \tag{7.2}$$

where $c \in \mathbb{C}$ is an arbitrary constant. The Jacobi matrices at these fixed points are

$$c\begin{pmatrix} 2 & 0 & -\alpha_4 \\ 0 & 0 & c-1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1-c)\begin{pmatrix} 2 & 0 & -\alpha_3 \\ 0 & 0 & -c \\ 0 & 0 & 1 \end{pmatrix}, \quad c(c-1)\begin{pmatrix} 2 & -2 & 1-\alpha_0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.3}$$

respectively. These fixed points correspond to movable singularities associated with the regular Laurent series solutions (iii), (ii) and (i) given in Sec.2.2. The Laurent series (iii), (ii) and (i) converge to these points as $z \to z_0$, respectively. The exceptional Laurent series solutions (iv) and (v) do not converge to some point on $\mathbb{C}P^3(1,0,0,1)$ as $z \to z_0$.

To treat the Laurent series (iv) and (v), we use the Bäcklund transformation π_1 defined by

$$\begin{aligned} (\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\alpha}_0, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\alpha}_4) &= \pi_1(x, y, z, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \left(-\frac{y}{z}(xy + \alpha_2),\ \frac{z}{y},\ z,\ \alpha_3,\ \alpha_4,\ \alpha_2, \alpha_0, \alpha_1\right). \end{aligned} \tag{7.4}$$

We consider another space $\mathbb{C}P^3(1,0,0,1)$ with inhomogeneous coordinates denoted by $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ etc. We glue two copies of $\mathbb{C}P^3(1,0,0,1)$ by the transformation π_1. Then, the exceptional Laurent series (iv) becomes the regular series (iii) in the $(\widetilde{x}, \widetilde{y}, \widetilde{z})$-chart. It converges to the fixed point $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (0, c, 0)$ as $z \to z_0$. On the other hand, the exceptional Laurent series (v) is expressed as $\widetilde{x}(z) \sim O(1)$, $\widetilde{y}(z) \sim O(T)$ in the $(\widetilde{x}, \widetilde{y}, \widetilde{z})$-chart. Since the solution is holomorphic at $z = z_0$, we need not resolve the singularity.

To construct the space of initial conditions of ($\mathrm{P_{VI}}$), we perform the weighted blow-ups at four points.

(i) blow-up at $(Y_1, Z_1, \varepsilon_1) = (0, c, 0)$.

For the vector field written in $(Y_1, Z_1, \varepsilon_1)$-chart, put $u = Y_1 - \alpha_4\varepsilon_1$, $v = Z_1$, $w = \varepsilon_1$. Then, we introduce the weighted blow-up with the weight $(2, 0, 1)$ defined by

$$\begin{cases} u &= u_1^2 &= w_2^2 u_2, \\ v &= v_1 &= v_2, \\ w &= u_1 w_1 &= w_2. \end{cases} \tag{7.5}$$

This is a blow-up along the line $\{v = \text{const}\}$. The relation between the original coordinates (x, y, z) and the new coordinates (u_2, v_2, w_2) is given by

$$\begin{cases} x = 1/w_2, \\ y = u_2 w_2^2 + \alpha_2 w_2, \\ z = v_2, \end{cases} \qquad \begin{cases} u_2 = x^2 y - \alpha_2 x, \\ w_2 = 1/x, \\ v_2 = z. \end{cases} \tag{7.6}$$

In the new coordinates, ($\mathrm{P_{VI}}$) is transformed into the Hamiltonian system, whose Hamiltonian function is

$$\begin{aligned} z(z-1)H_1 &= u^3w^4+uz-u^2w^2(1+z) \\ &\quad +(1-\alpha_0-\alpha_3+2\alpha_4)u^2w^3+(\alpha_0-\alpha_4+z\alpha_3-z\alpha_4-1)uw \\ &\quad +\alpha_4\left(\alpha_1\alpha_2+\alpha_2^2+(1-\alpha_0-\alpha_3)\alpha_4\right)w \\ &\quad +\left(\alpha_1\alpha_2+\alpha_2^2+2\alpha_4-2\alpha_0\alpha_4-2\alpha_3\alpha_4+\alpha_4^2\right)uw^2. \end{aligned}$$

(here, the subscript is omitted for simplicity). Furthermore, we can verify the symplectic relation (3.15).

The other singularities $(Y_1, Z_1, \varepsilon_1) = (1, c, 0), (c, c, 0)$ and $(\widetilde{Y}_1, \widetilde{Z}_1, \widetilde{\varepsilon}_1) = (0, c, 0)$ are resolved by the same way as above by the weighted blow-up with the weight $(2, 0, 1)$. In this manner, all singularities are resolved and the space of initial conditions of ($\mathrm{P_{VI}}$) is obtained by six copies of $\mathbb{C}^2$ glued by the symplectic transformations; i.e. $(x, y), (\widetilde{x}, \widetilde{y})$ and four charts arising from blow-ups at four points. We need $(\widetilde{x}, \widetilde{y})$-chart for the exceptional Laurent series (v).

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