Multi-Poisson approach to the Painlevé equations:
from the isospectral deformation to the isomonodromic deformation

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Abstract

A multi-Poisson structure on a Lie algebra $\mathfrak{g}$ provides a systematic way to construct completely integrable Hamiltonian systems on $\mathfrak{g}$ expressed in Lax form $\partial X_\lambda/\partial t = [X_\lambda, A_\lambda]$ in the sense of the isospectral deformation, where $X_\lambda, A_\lambda \in \mathfrak{g}$ depend rationally on the indeterminate λ called the spectral parameter. In this paper, a method for modifying the isospectral deformation equation to the Lax equation $\partial X_\lambda/\partial t = [X_\lambda, A_\lambda] + \partial A_\lambda/\partial \lambda$ in the sense of the isomonodromic deformation, which exhibits the Painlevé property, is proposed. This method gives a few new Painlevé systems of dimension four.

Keywords: Painlevé equations; Lax equations; multi-Poisson structure

1 Introduction

A differential equation defined on a complex region is said to have the Painlevé property if any movable singularity of any solution is a pole. Painlevé and his group classified second order ODEs having the Painlevé property and found new six differential equations called the Painlevé equations. Nowadays, it is known that they are written in Hamiltonian forms

$$(\mathrm{P_J}) : \frac{dq}{dt} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_J}{\partial q}, \quad J = \mathrm{I}, \cdots, \mathrm{VI}. \tag{1.1}$$

Among six Painlevé equations, the Hamiltonian functions of the first, second and fourth Painlevé equations are polynomials in both of the independent variable t and the dependent variables (q, p). They are given by

$$H_\mathrm{I} = \frac{1}{2}p^2 - 2q^3 - tq, \tag{1.2}$$

$$H_\mathrm{II} = \frac{1}{2}p^2 - \frac{1}{2}q^4 - \frac{1}{2}tq^2 - \alpha q, \tag{1.3}$$

$$H_\mathrm{IV} = -pq^2 + p^2q - 2pqt - \alpha p + \beta q, \tag{1.4}$$

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respectively, where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters. Another important property of the Painlevé equations is that they are expressed as Lax equations. Let L_λ and A_λ be square matrices which depend rationally on the indeterminate λ called the spectral parameter. The Painlevé equations are written in Lax form as

$$\frac{\partial L_\lambda}{\partial t} = [L_\lambda, A_\lambda] + \frac{\partial A_\lambda}{\partial \lambda}, \tag{1.5}$$

for some choice of L_λ and A_λ. This equation arises from the compatibility condition of the two differential systems

$$\frac{\partial \Psi}{\partial \lambda} = L_\lambda \Psi, \quad \frac{\partial \Psi}{\partial t} = A_\lambda \Psi. \tag{1.6}$$

Since the monodromy of the former system $\partial \Psi / \partial \lambda = L_\lambda \Psi$ is independent of t if the equation (1.5) is satisfied, (1.5) is called the isomonodromic deformation equation.

Another type of the Lax equation is of the form

$$\frac{\partial X_\lambda}{\partial t} = [X_\lambda, A_\lambda], \tag{1.7}$$

which is called the isospectral deformation equation because the eigenvalues of the matrix X_λ is independent of t. There are several systematic ways to construct isospectral deformation equations [1]. In particular, a Lie algebraic method have been often employed. Let $\mathfrak{g}$ be a Lie algebra. On the dual space $\mathfrak{g}^*$, there exists a canonical Poisson structure called the Lie-Poisson structure. If $\mathfrak{g}$ is equipped with a nondegenerate bilinear symmetric form, the Lie-Poisson structure is also defined on $\mathfrak{g}$. Let $P : T^*\mathfrak{g} \to T\mathfrak{g}$ be the Poisson tensor and $F : \mathfrak{g} \to \mathbb{C}$ a smooth function. Then, the vector field PdF on $\mathfrak{g}$ can be expressed as the Lax equation (1.7) with some $X_\lambda, A_\lambda \in \mathfrak{g}$ [1].

It is notable that the isospectral deformation equation (1.7) is completely integrable for most examples, although the isomonodromic deformation equation (1.5) is not in general; it is believed that solutions of an isomonodromic deformation equation define new functions called the Painlevé transcendents.

In Nakamura [16], a way to obtain the isospectral deformation equation (1.7) from the isomonodromic deformation equation (1.5) by a certain scaling of the time t is proposed, which is called the autonomous limit. She proved that the autonomous limits of 6-types of two dimensional Painlevé equations and 40-types of four dimensional Painlevé equations are completely integrable.

The purpose in the present paper is opposite; a way to construct the isomonodromic deformation equation (1.5) from the isospectral deformation equation (1.7) will be proposed. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Consider the set of $\mathfrak{g}$-valued polynomials of degree n

$$\mathfrak{g}_n := \{X_\lambda := X_0\lambda^n + X_1\lambda^{n-1} + \cdots + X_n \,|\, X_i \in \mathfrak{g}\},$$

with the indeterminate λ. This set $\mathfrak{g}_n$ is equipped with a structure of a Lie algebra by a certain Lie bracket. At first, the isospectral deformation equation (1.7) on

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$\mathfrak{g}_n$ is constructed with the aid of the bi-Poisson theory of Magri et. al [12, 13, 14, 15]. Isospectral deformation equations obtained in this method are shown to be completely integrable (Thm.2.4). Next, we restrict the equations onto a symplectic leaf. Let $\varphi_1, \cdots, \varphi_N$ be Casimir functions of an underlying Poisson structure on $\mathfrak{g}_n$. A symplectic leaf S is defined by the level surface of them as

$$S := \{\varphi_i = \alpha_i \,(\text{constant}) \,|\, i = 1, \cdots, N\}.$$

Restricted on the leaf S, the isospectral deformation equation (1.7) becomes an integrable Hamiltonian system. Since the matrix $X_\lambda \in \mathfrak{g}_n$ depends on the parameters $\alpha := (\alpha_1, \cdots, \alpha_N)$, it is denoted as $X_\lambda = X_\lambda(t, \alpha)$.

Now suppose that there exists a parameter, say α_j, such that the following condition holds

$$\frac{\partial X_\lambda}{\partial \alpha_j}(t, \alpha) = \frac{\partial A_\lambda}{\partial \lambda}. \tag{1.8}$$

Eq.(1.7) is put together with Eq.(1.8) to yield

$$\frac{\partial X_\lambda}{\partial t}(t, \alpha) + \frac{\partial X_\lambda}{\partial \alpha_j}(t, \alpha) = [X_\lambda, A_\lambda] + \frac{\partial A_\lambda}{\partial \lambda}.$$

Define the Lax matrix L_λ by

$$L_\lambda := X_\lambda(t, \alpha)|_{\alpha_j = t},$$

where the parameter α_j satisfying the condition (1.8) is replaced by t. Then, the above equation is rewritten as the isomonodromic deformation equation (1.5).

Remark that the isomonodromic deformation equation (1.5) is equivalent to the zero curvature condition of the connection 1 form $L_\lambda d\lambda + A_\lambda dt$ on a vector bundle over the (t, λ)-space, while the condition (1.8) is the exactness condition of the connection 1 form $X_\lambda d\lambda + A_\lambda d\alpha_j$.

This method is demonstrated for the following three cases (I) $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $n = 2$, (II) $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $n = 3$ and (III) $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$, $n = 1$. For the case (I), the first, second and fourth Painlevé equations (1.2), (1.3), (1.4) will be obtained in Section 3.

More generally, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with general n, one can obtain several Painlevé hierarchies of dimension $2n - 2$, including the first Painlevé hierarchy $(P_I)_m$[10, 11, 17], the second-first Painlevé hierarchy $(P_{II-1})_m$[5, 6, 10, 11], the second-second Painlevé hierarchy $(P_{II-2})_m$ and the fourth Painlevé hierarchy $(P_{IV})_m$[7, 10]. They are $2m$-dimensional Hamiltonian PDEs of the form $(m = n - 1)$

$$\begin{cases} \dfrac{\partial q_j}{\partial t_i} = \dfrac{\partial H_i}{\partial p_j}, \ \dfrac{\partial p_j}{\partial t_i} = -\dfrac{\partial H_i}{\partial q_j}, \quad j = 1, \cdots, m; \ i = 1, \cdots, m \\ H_i = H_i(q_1, \cdots, q_m, p_1, \cdots, p_m, t_1, \cdots, t_m) \end{cases} \tag{1.9}$$

consisting of m Hamiltonians $H_1, \cdots, H_m$ with m independent variables $t_1, \cdots, t_m$. When $m = 1$ (the case (I)), $(P_I)_1$ and $(P_{IV})_1$ are reduced to the first and fourth

Painlevé equations, respectively. Both of $(P_{II\text{-}1})_1$ and $(P_{II\text{-}2})_1$ coincide with the second Painlevé equation, while they are different systems for $m \geq 2$. When $m = 2$ (the case (II)), Hamiltonians of $(P_I)_2$, $(P_{II\text{-}1})_2$, $(P_{II\text{-}2})_2$ and $(P_{IV})_2$ are given by

$$(P_I)_2 \begin{cases} H_1 = 2p_2 p_1 + 3p_2^2 q_1 + q_1^4 - q_1^2 q_2 - q_2^2 - t_1 q_1 + t_2(q_1^2 - q_2), \\ H_2 = p_1^2 + 2p_2 p_1 q_1 - q_1^5 + p_2^2 q_2 + 3q_1^3 q_2 - 2q_1 q_2^2 \\ \qquad + t_1(q_1^2 - q_2) + t_2(t_2 q_1 + q_1 q_2 - p_2^2), \end{cases} \tag{1.10}$$

$$(P_{II\text{-}1})_2 \begin{cases} H_1 = 2p_1 p_2 - p_2^3 - p_1 q_1^2 + q_2^2 - t_1 p_2 + t_2 p_1 + 2\alpha q_1, \\ H_2 = -p_1^2 + p_1 p_2^2 + p_1 p_2 q_1^2 + 2p_1 q_1 q_2 \\ \qquad + t_1 p_1 + t_2(t_2 p_1 - p_1 q_1^2 + p_1 p_2) - \alpha(2p_2 q_1 + 2q_2 + 2t_2 q_1), \end{cases} \tag{1.11}$$

$$(P_{II\text{-}2})_2 \begin{cases} H_1 = p_1 p_2 - p_1 q_1^2 - 2p_1 q_2 + p_2 q_1 q_2 + q_1 q_2^2 + q_2 t_1 + t_2(q_1 q_2 - p_1) + \alpha q_1, \\ H_2 = p_1^2 - p_1 p_2 q_1 + p_2^2 q_2 - 2p_1 q_1 q_2 - p_2 q_2^2 + q_1^2 q_2^2 \\ \qquad + t_1(q_1 q_2 - p_1) - t_2(p_1 q_1 + q_2^2 + q_2 t_2) + \alpha p_2, \end{cases} \tag{1.12}$$

$$(P_{IV})_2 \begin{cases} H_1 = p_1^2 + p_1 p_2 - p_1 q_1^2 + p_2 q_1 q_2 - p_2 q_2^2 - t_1 p_1 + t_2 p_2 q_2 + \alpha q_2 + \beta q_1, \\ H_2 = p_1 p_2 q_1 - 2p_1 p_2 q_2 - p_2^2 q_2 + p_2 q_1 q_2^2 \\ \qquad + p_2 q_2 t_1 + t_2(p_1 p_2 - p_2 q_2^2 + p_2 q_2 t_2) + (p_1 - q_1 q_2 + q_2 t_2)\alpha - \beta p_2, \end{cases} \tag{1.13}$$

respectively, with arbitrary parameters $\alpha, \beta \in \mathbb{C}$. These systems will be obtained from the case (II) $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $n = 3$ in Section 4. In our method, such Hamiltonian PDEs are obtained if there are several Hamiltonian systems written in Lax form (1.7), and if there are several parameters satisfying (1.8); such parameters will be replaced by distinct times $t_1, t_2, \cdots$.

We will find other 4-dimensional Painlevé systems with Hamiltonian functions

$$\begin{aligned} H_{(1,1,2,0)} &= -p_1^2 q_1 - 2p_1 q_1^2 + 2p_1 q_2 - 2p_1 p_2 q_2 - 2p_2 q_1 q_2 \\ &\quad + (2p_1 q_1 + 2p_2 q_2)t + (2\alpha_2 + 2\beta_2)q_1 + 2\beta_2 p_1 + 2\beta_3 p_2, \end{aligned} \tag{1.14}$$

$$H_{(-1,4,1,2)} = p_1 - p_2^2 - 2p_1 q_1 q_2 - p_2 q_2^2 + 2\beta_3 q_2 + 2\beta_5 q_1 + p_2 t, \tag{1.15}$$

$$\begin{aligned} H_{\text{Cosgrove}} &= -4p_1 p_2 - 2p_2^2 q_1 - \frac{73}{128}q_1^4 + \frac{11}{8}q_1^2 q_2 - \frac{1}{2}q_2^2 \\ &\quad - q_1 t - \frac{\alpha_2}{48}\left(q_1 + \frac{\alpha_2}{6}\right)q_1^2, \end{aligned} \tag{1.16}$$

where $\alpha_i, \beta_i \in \mathbb{C}$ are arbitrary parameters (the subscripts for parameters are related to the weighted degrees so that the Hamiltonian functions become quasihomogeneous, see below). The first two systems will be also obtained from the case (II). As far as the author knows, these systems have not appeared in the literature. The last one H_{Cosgrove} will be obtained from the case (III) $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$, $n = 1$ in Section 5. If we rewrite the system as a fourth order single equation of $q_1 = y$, we obtain

$$y'''' = 18yy'' + 9(y')^2 - 24y^3 + 16t + \alpha y(y + \frac{1}{9}\alpha). \tag{1.17}$$

This equation was given in Cosgrove [8], denoted by F-VI, without a proof that it has the Painlevé property. Since this system is obtained as the isomonodromic

deformation equation in this paper, this equation actually enjoys the Painlevé property. In his paper [8], it is conjectured that this equation defines a new Painlevé transcendent (i.e. it is not reduced to known equations). Another expression of the Hamiltonian function of the same system is

$$\widetilde{H}_{\text{Cosgrove}} = 2p_1p_2 - \frac{18}{13}p_2^2q_1 - \frac{2}{169}q_1^4 - \frac{180}{13}q_1^2q_2 + 6q_2^2 - 8q_1t + \frac{8}{9}\alpha_2q_1^3 + \frac{8}{27}\alpha_2^2q_1^2. \quad (1.18)$$

The corresponding Hamiltonian system is also reduced to (1.17).

Note that all of the Hamiltonian functions above are polynomials in both of the independent variables and the dependent variables. Furthermore, they are semi-quasihomogeneous functions. In general, a polynomial $H(x_1, \cdots, x_n)$ is called a quasihomogeneous polynomial if there are integers $a_1, \cdots, a_n$ and h such that

$$H(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_n) = \lambda^h H(x_1, \cdots, x_n) \quad (1.19)$$

for any $\lambda \in \mathbb{C}$. A polynomial H is called a semi-quasihomogeneous if H is decomposed into two polynomials as $H = H^P + H^N$, where H^P satisfies (1.19) and H^N satisfies

$$H^N(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_n) \sim o(\lambda^h), \quad |\lambda| \to \infty.$$

The integer $\text{wdeg}(H) := h$ is called the weighted degree of H with respect to the weight $\text{wdeg}(x_1, \cdots, x_n) := (a_1, \cdots, a_n)$. For example, if we define degrees of variables by $\text{wdeg}(q, p, t) = (2, 3, 4)$ for H_{I}, $\text{wdeg}(q, p, t) = (1, 2, 2)$ for H_{II} and $\text{wdeg}(q, p, t) = (1, 1, 1)$ for H_{IV}, then Hamiltonian functions have the weighted degrees $6, 4$ and 3, respectively (Table 1). The weights for four dimensional systems above are shown in Table 2. In this paper, these weights are naturally obtained from a suitable definition of weights of entries of a matrix $X_\lambda \in \mathfrak{g}_n$ and the spectral parameter λ. In particular, the weights of the Hamiltonian functions are closely related to the exponents of simple Lie algebras because the Hamiltonian functions are essentially Ad-invariant polynomials of simple Lie algebras. See Chiba [2, 3, 4] for the detailed study of the weights of the Painlevé equations.

	$\text{wdeg}(q, p, t)$	$\text{wdeg}(H)$
P_{I}	$(2, 3, 4)$	6
P_{II}	$(1, 2, 2)$	4
P_{IV}	$(1, 1, 1)$	3

Table 1: Weights for two dimensional Painlevé equations.

2 Settings

2.1 Lie-Poisson structure on $\mathfrak{g}_n$

We define a multi-Poisson structure on a certain Lie algebra following Magri et. al [12, 13, 14, 15]. Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be a simple Lie algebra over $\mathbb{C}$. Consider the set

	wdeg(q_1, p_1, q_2, p_2)	wdeg(t_1, t_2)	wdeg(H_1, H_2)
$(P_I)_2$	$(2, 5, 4, 3)$	$6, 4$	$8, 10$
$(P_{II\text{-}1})_2$	$(1, 4, 3, 2)$	$4, 2$	$6, 8$
$(P_{II\text{-}2})_2$	$(1, 3, 2, 2)$	$3, 2$	$5, 6$
$(P_{IV})_2$	$(1, 2, 1, 2)$	$2, 1$	$4, 5$
$H_{(1,1,2,0)}$	$(1, 1, 2, 0)$	1	3
$H_{(-1,4,1,2)}$	$(-1, 4, 1, 2)$	2	4
H_{Cosgrove}	$(2, 5, 4, 3)$	6	8

Table 2: Weights for four dimensional Painlevé equations.

of $\mathfrak{g}$-valued polynomials of degree n

$$\mathfrak{g}_n := \{X_\lambda := X_0 \lambda^n + X_1 \lambda^{n-1} + \cdots + X_n \mid X_i \in \mathfrak{g}\}, \tag{2.1}$$

with the indeterminate λ. The bracket defined by

$$
\begin{aligned}
[X_\lambda, Y_\lambda]_n \quad := \quad & [X_n, Y_n] + \lambda([X_n, Y_{n-1}] + [X_{n-1}, Y_n]) + \cdots \\
& + \lambda^n \left([X_0, Y_n] + [X_1, Y_{n-1}] + \cdots + [X_n, Y_0]\right)
\end{aligned}
$$

introduces the structure of a Lie algebra on $\mathfrak{g}_n$. Note that $[X_\lambda, Y_\lambda]_n$ coincides with $[X_\lambda, Y_\lambda]$ expanded in λ and truncated at degree n.

It is known that the dual space $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$ is equipped with a canonical Poisson structure called the Lie-Poisson structure. If a nondegenerate symmetric bilinear form $\eta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is defined on $\mathfrak{g}$, it induces a Lie-Poisson structure on $\mathfrak{g}$. For functions $F, G : \mathfrak{g} \to \mathbb{C}$, the Poisson bracket on $\mathfrak{g}$ is defined by $\{F, G\}(X) = \eta(X, [\nabla F(X), \nabla G(X)])$, where $\nabla F(X) \in \mathfrak{g}$ is defined through $(dF)_X(Y) = \eta(\nabla F(X), Y)$. To give the Lie-Poisson structure on $\mathfrak{g}_n$, we define a nondegenerate symmetric bilinear form η on $\mathfrak{g}_n$ by

$$\eta(X_\lambda, Y_\lambda) := \sum_{i=0}^{n} \text{Tr}(X_i Y_{n-i}),$$

by which $\mathfrak{g}_n$ is identified with its dual. For a smooth function $F : \mathfrak{g}_n \to \mathbb{C}$, define the gradient $\nabla F \in \mathfrak{g}_n$ through $(dF)(Y_\lambda) = \eta(\nabla F, Y_\lambda)$, and define $\nabla_i F \in \mathfrak{g}$ by

$$\nabla F = (\nabla_n F)\lambda^n + (\nabla_{n-1} F)\lambda^{n-1} + \cdots + \nabla_0 F.$$

Using them, the Lie-Poisson bracket on $\mathfrak{g}_n$ is given by

$$
\begin{aligned}
\{F, G\}_0 \quad := \quad & \eta(X_\lambda, [\nabla F, \nabla G]_n) \\
= \quad & \text{Tr}(X_0 \cdot [\nabla_0 F, \nabla_0 G]) + \text{Tr}\left(X_1 \cdot ([\nabla_0 F, \nabla_1 G] + [\nabla_1 F, \nabla_0 G])\right) + \cdots \\
& + \text{Tr}(X_n \cdot ([\nabla_0 F, \nabla_n G] + \cdots + [\nabla_n F, \nabla_0 G])) \\
= \quad & -\text{Tr}(\nabla_0 F \cdot ([X_0, \nabla_0 G] + [X_1, \nabla_1 G] + \cdots + [X_n, \nabla_n G])) - \cdots \\
& - \text{Tr}(\nabla_{n-1} F \cdot ([X_{n-1}, \nabla_0 G] + [X_n, \nabla_1 G])) - \text{Tr}(\nabla_n F \cdot [X_n, \nabla_0 G]).
\end{aligned}
$$

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The Poisson tensor (bivector) $P_0 : T^*\mathfrak{g}_n \to T\mathfrak{g}_n$ is defined so that

$$\{F, G\}_0 = dF(P_0 dG) = \eta(\nabla F, P_0 dG) = \sum_{i=0}^{n} \mathrm{Tr}(\nabla_i F \cdot (P_0 dG)_i).$$

This implies

$$-(P_0 dG)_0 = [X_0, \nabla_0 G] + [X_1, \nabla_1 G] + \cdots + [X_n, \nabla_n G]$$
$$\vdots$$
$$-(P_0 dG)_{n-1} = [X_{n-1}, \nabla_0 G] + [X_n, \nabla_1 G]$$
$$-(P_0 dG)_n = [X_n, \nabla_0 G].$$

The following expression is useful

$$
P_0 : dG \ \mapsto \ -\begin{pmatrix} [X_0, \cdot] & [X_1, \cdot] & \cdots & [X_n, \cdot] \\ \vdots & & \cdot{\cdot}{\cdot} & \\ [X_{n-1}, \cdot] & [X_n, \cdot] & & \\ [X_n, \cdot] & & & \end{pmatrix} \begin{pmatrix} \nabla_0 G \\ \vdots \\ \nabla_{n-1} G \\ \nabla_n G \end{pmatrix}
$$

$$
= \begin{pmatrix} [\nabla_0 G, X_0] + [\nabla_1 G, X_1] + \cdots + [\nabla_n G, X_n] \\ \vdots \\ [\nabla_0 G, X_{n-1}] + [\nabla_1 G, X_n] \\ [\nabla_0 G, X_n] \end{pmatrix}. \tag{2.2}
$$

It is also represented as a matrix as follows. Let $A = A(X)$ be a representation matrix of the mapping

$$T^*\mathfrak{g} \, (\simeq \mathfrak{g}) \to \mathfrak{g}, \quad dG \mapsto [X, \nabla G], \quad G : \mathfrak{g} \to \mathbb{C}, \, X \in \mathfrak{g}$$

with respect to some coordinates on $\mathfrak{g}$ (here ∇G is the gradient on $\mathfrak{g}$). By the definition, $-A$ is a Poisson tensor of the Lie-Poisson structure on $\mathfrak{g}$. Since $A(X)$ is linear in X, $A(X_\lambda)$ is expanded as $A(X_\lambda) = \lambda^n A(X_0) + \lambda^{n-1} A(X_1) + \cdots + A(X_n)$. Putting $A(X_j) = A_j$, P_0 is represented as an $(n+1)\mathrm{dim}(\mathfrak{g}) \times (n+1)\mathrm{dim}(\mathfrak{g})$ matrix

$$
P_0 = -\begin{pmatrix} A_0 & \cdots & A_{n-1} & A_n \\ A_1 & \cdots & A_n & \\ \vdots & \cdot{\cdot}{\cdot} & & \\ A_n & & & \end{pmatrix}. \tag{2.3}
$$

In what follows, suppose $\mathrm{dim}(\mathfrak{g}) = d$, $\mathrm{rank}(\mathfrak{g}) = h$ and let $m_1, \cdots, m_h$ be exponents of $\mathfrak{g}$. Let $(y_1, \cdots, y_d)$ be coordinates on $\mathfrak{g}$. It is known that the Casimir functions of the Lie-Poisson structure on $\mathfrak{g}$ (i.e. a function φ satisfying $\{F, \varphi\} = 0$ for any $F : \mathfrak{g} \to \mathbb{C}$) are the Ad-invariant polynomials denoted by $\varphi_i(y_1, \cdots, y_d)$, $i = 1, \cdots, h$, and they satisfy $\mathrm{deg}(\varphi_i) = m_i + 1$.

Let $x_j := (x_{j,1}, \cdots, x_{j,d})$ be coordinates on the j-th copy of $\mathfrak{g}$ (coordinate expression for X_j) and $(x_0, \cdots, x_n)$ coordinates on $\mathfrak{g}_n$. We define the weighted degrees of variables to be

$$\mathrm{wdeg}(x_j) = \mathrm{wdeg}(x_{j,\alpha}) = j, \quad \mathrm{wdeg}(\lambda) = 1. \tag{2.4}$$

Then, X_λ is quasihomogeneous (homogeneous in the weighted sense) of $\mathrm{wdeg}(X_\lambda) = n$. Substituting $y_\alpha = x_{0,\alpha}\lambda^n + x_{1,\alpha}\lambda^{n-1} + \cdots + x_{n,\alpha}$ into $\varphi_i(y_1, \cdots, y_d)$ and expanding it in λ provide

$$\begin{aligned}
\varphi_i(y_1, \cdots, y_d) &= \varphi_{i,0}(x_0, \cdots, x_n)\lambda^{(m_i+1)n} + \varphi_{i,1}(x_0, \cdots, x_n)\lambda^{(m_i+1)n-1} + \\
&\quad \cdots + \varphi_{i,(m_i+1)n}(x_0, \cdots, x_n), \quad i = 1, \cdots, h,
\end{aligned}$$

which defines polynomials $\varphi_{i,j}$ on $\mathfrak{g}_n$ satisfying

$$\deg(\varphi_{i,j}) = m_i + 1, \quad \mathrm{wdeg}(\varphi_{i,j}) = j. \tag{2.5}$$

Proposition 2.1.
(i) $\varphi_{i,j}$ depends only on $(x_0, \cdots, x_j)$ for $0 \le j \le n-1$.
(ii) $\varphi_{i,j}(x_0, x_1, \cdots, x_n) = \varphi_{i,(m_i+1)n-j}(x_n, \cdots, x_1, x_0)$.
(iii) For each i, j, α, the derivative $\partial\varphi_{i,j+k}/\partial x_{k,\alpha}$ is independent of $k = 0, \cdots, n$.
(iv) For each i, j, the gradient $\nabla_k \varphi_{i,j+k}$ is independent of $k = 0, \cdots, n$.
(v) For each i, j, k, the equality

$$\sum_{l=0}^{n} A_l \frac{\partial\varphi_{i,j+k-l}}{\partial x_k} = \sum_{l=0}^{n} A_l \frac{\partial\varphi_{i,j-l}}{\partial x_0} = 0 \tag{2.6}$$

holds.
(vi) The Casimir functions of the Lie-Poisson structure P_0 on $\mathfrak{g}_n$ are

$$\varphi_{i,(m_i+1)n-j}, \quad i = 1, \cdots, h; \; j = 0, \cdots, n.$$

Proof. (i) and (ii) follow from the definition of $\varphi_{i,j}$.
(iii) For $y_\alpha = \sum_{k=0}^{n} \lambda^{n-k} x_{k,\alpha}$, we have

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial y_\alpha} &= \frac{\partial x_{k,\alpha}}{\partial y_\alpha} \frac{\partial}{\partial x_{k,\alpha}} \sum_{j=0}^{(m_i+1)n} \lambda^{(m_i+1)n-j} \varphi_{i,j} \\
&= \sum_{j=0}^{(m_i+1)n} \lambda^{m_i n - j + k} \frac{\partial\varphi_{i,j}}{\partial x_{k,\alpha}} \\
&= \sum_{j=k}^{m_i n + k} \lambda^{m_i n - j + k} \frac{\partial\varphi_{i,j}}{\partial x_{k,\alpha}}.
\end{aligned}$$

For the last equality, we used Part (i) combined with Part (ii). Thus we obtain

$$\frac{\partial\varphi_i}{\partial y_\alpha} = \sum_{j=0}^{m_i n} \lambda^{m_i n - j} \frac{\partial\varphi_{i,j+k}}{\partial x_{k,\alpha}}.$$

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Since the left hand side is independent of k, so is each coefficient of $\lambda^{m_i n - j}$ in the right hand side. Part (iv) immediately follows from (iii).

(v) The first equality is a consequence of Part (iii). Since $\varphi_i(y)$ is a Casimir function of the Lie-Poisson structure on $\mathfrak{g}$, $Ad\varphi_i = 0$, where A is a matrix defined before. Substituting $y = \sum_{k=0}^{n} \lambda^{n-k} x_k$ yields

$$
\begin{aligned}
0 &= A\frac{\partial \varphi_i}{\partial y} = (\lambda^n A_0 + \lambda^{n-1} A_1 + \cdots + A_n) \sum_{j=0}^{m_i n} \lambda^{m_i n - j} \frac{\partial \varphi_{i,j+k}}{\partial x_k} \\
&= \sum_{j,l} \lambda^{m_i n - j + n - l} A_l \frac{\partial \varphi_{i,j+k}}{\partial x_k} \\
&= \sum_{j=l}^{m_i n + l} \lambda^{m_i n + n - j} \sum_{l=0}^{n} A_l \frac{\partial \varphi_{i,j+k-l}}{\partial x_k}.
\end{aligned}
$$

This proves the second equality of (v).

To prove (vi), it is sufficient to show

$$
\begin{pmatrix}
A_0 & \cdots & A_{n-1} & A_n \\
A_1 & \cdots & A_n & \\
\vdots & \ddots & & \\
A_n & & &
\end{pmatrix}
\begin{pmatrix}
\partial \varphi_{i,(m_i+1)n-j} / \partial x_0 \\
\partial \varphi_{i,(m_i+1)n-j} / \partial x_1 \\
\vdots \\
\partial \varphi_{i,(m_i+1)n-j} / \partial x_n
\end{pmatrix}
= 0
$$

for $j = 0, \cdots, n$. This is verified with the aid of Part (v). $\square$

Example 2.2. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, we have $d = 3, h = 1$ and $m_i = m_1 = 1$. Denote a general element $X_\lambda \in \mathfrak{g}_n$ as

$$
\begin{aligned}
X_\lambda &= \lambda^n X_0 + \lambda^{n-1} X_1 + \cdots + X_n \\
&= \lambda^n \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} + \lambda^{n-1} \begin{pmatrix} u_1 & v_1 \\ w_1 & -u_1 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & v_n \\ w_n & -u_n \end{pmatrix}.
\end{aligned}
$$

Let (u_j, v_j, w_j) be coordinates on the j-th copy of $\mathfrak{g}$ and $(u_0, v_0, w_0, \cdots, u_n, v_n, w_n)$ coordinates on $\mathfrak{g}_n$. Then,

$$
\nabla_j F = \begin{pmatrix} \dfrac{1}{2}\dfrac{\partial F}{\partial u_j} & \dfrac{\partial F}{\partial w_j} \\[2mm] \dfrac{\partial F}{\partial v_j} & -\dfrac{1}{2}\dfrac{\partial F}{\partial u_j} \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & v_j & -w_j \\ -v_j & 0 & 2u_j \\ w_j & -2u_j & 0 \end{pmatrix}.
$$

The Casimir function on $\mathfrak{g}$ is given by $\varphi_i = \varphi = u^2 + vw$. Then, the functions $\varphi_{i,j} = \varphi_j$ are defined by expanding

$$
(\lambda^n u_0 + \cdots + u_n)^2 + (\lambda^n v_0 + \cdots + v_n)(\lambda^n w_0 + \cdots + w_n)
$$

in λ. This gives

$$
\varphi_j = \sum_{k+l=j} (u_k u_l + v_k w_l), \quad j = 0, \cdots, 2n.
$$

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Note that they are coefficients of $-\det X_\lambda$. The Casimir functions of $\mathfrak{g}_n$ are given by φ_j for $j = n, \cdots, 2n$.

2.2 Multi-Poisson structure on $\mathfrak{g}_n^0$

In general, a manifold M is called a bi-Poisson manifold if
(i) there are two Poisson brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_1$, and
(ii) the linear combination $\{\ ,\ \}_0 + t\{\ ,\ \}_1$ is also a Poisson bracket for any $t \in \mathbb{C}$.

See [12, 13, 14, 15] for applications of bi-Poisson manifolds to integrable systems. Here, we introduce a bi-Poisson structure on $\mathfrak{g}_n$ following [13]. The shift operator $X_\lambda \mapsto X_{\lambda+t}$ defines an automorphism of $\mathfrak{g}_n$ with a parameter $t \in \mathbb{C}$. It induces a deformation, denoted by $\{\ ,\ \}_t$, of the Lie-Poisson bracket $\{\ ,\ \}_0$. Let

$$\{\ ,\ \}_t = \{\ ,\ \}_0 + t\{\ ,\ \}_1 + \cdots + t^{n+1}\{\ ,\ \}_{n+1} + \cdots$$

be its expansion. Magnano and Magri [13] proved that each $\{\ ,\ \}_i$ $(i = 0, \cdots, n+1)$ and their any linear combination satisfy the axiom of a Poisson bracket. Hence, $\mathfrak{g}_n$ has $n+2$ compatible Poisson brackets and it becomes a multi-Poisson manifold. Their Poisson tensors are

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & -A_1 & -A_2 & \cdots & -A_n \\ \vdots & \vdots & \vdots & \iddots & \\ 0 & -A_{n-1} & -A_n & & \\ 0 & -A_n & & & \end{pmatrix},$$

$$P_{k+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & & & A_0 & & & \\ \vdots & & \iddots & \vdots & & & \\ 0 & A_0 & \cdots & A_{k-1} & & & \\ \hline 0 & & & & -A_{k+1} & \cdots & -A_n \\ \vdots & & & & \vdots & \iddots & \\ 0 & & & & -A_n & & \end{pmatrix}, \quad k = 1, \cdots, n-1,$$

$$P_{n+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & A_0 \\ 0 & & & A_0 & A_1 \\ \vdots & & \iddots & \vdots & \vdots \\ 0 & A_0 & \cdots & A_{n-2} & A_{n-1} \end{pmatrix}.$$

(P_0 is the same as before). Let $\mathfrak{g}_n^0$ be a submanifold of $\mathfrak{g}_n$ defined by $x_0 = \text{constant}$;

$$\mathfrak{g}_n^0 := \{X_\lambda = X_0\lambda^n + X_1\lambda^{n-1} + \cdots + X_n \,|\, X_0 = \text{constant}\} \subset \mathfrak{g}_n. \qquad (2.7)$$

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Since the first row and column of $P_1, \cdots, P_{n+1}$ are zero (i.e. $x_0 = (x_{0,1}, \cdots, x_{0,d})$ are Casimir functions of them), the restrictions of them on $\mathfrak{g}_n^0$ define a multi-Poisson structure on $\mathfrak{g}_n^0$, whose brackets and tensors are again denoted by $(\{\ ,\ \}_i, P_i)$. The tensors are given by

$$
P_1 = \begin{pmatrix}
-A_1 & -A_2 & \cdots & -A_n \\
\vdots & \vdots & \iddots & \\
-A_{n-1} & -A_n & & \\
-A_n & & &
\end{pmatrix},
$$

$$
P_{k+1} = \left(
\begin{array}{ccc|ccc}
& & A_0 & & & \\
& \iddots & \vdots & & & \\
A_0 & \cdots & A_{k-1} & & & \\
\hline
& & & -A_{k+1} & \cdots & -A_n \\
& & & \vdots & \iddots & \\
& & & -A_n & &
\end{array}
\right), \quad k = 1, \cdots, n-1,
$$

$$
P_{n+1} = \begin{pmatrix}
& & & A_0 \\
& & A_0 & A_1 \\
& \iddots & \vdots & \vdots \\
A_0 & \cdots & A_{n-2} & A_{n-1}
\end{pmatrix}.
$$

For $i = 1, \cdots, h$ and $j = 1, \cdots, (m_i + 1)n$, define functions $\psi_{i,j}$ on $\mathfrak{g}_n^0$ by

$$
\psi_{i,j}(x_1, \cdots, x_n) := \varphi_{i,j}|_{\mathfrak{g}_n^0} = \varphi_{i,j}|_{x_0 = \text{constant}}.
$$

(we do not define $\psi_{i,0}$ because $\varphi_{i,0}$ is constant on $\mathfrak{g}_n^0$).

Proposition 2.3.
(i) Casimir functions of P_{k+1} are $\psi_{i,j}$ ($i = 1, \cdots, h$) for $j = 1, 2, \cdots, k$ and for $j = m_i n + k + 1, m_i n + k + 2, \cdots, (m_i + 1)n$.
(ii) Casimir functions of the combination $\lambda P_{k+1} - P_k$ are $\psi_{i,j}$ ($i = 1, \cdots, h$) for $j = 1, 2, \cdots, k-1$ and for $j = m_i n + k + 1, m_i n + k + 2, \cdots, (m_i + 1)n$, and

$$
\lambda^{m_i n} \psi_{i,k} + \lambda^{m_i n - 1} \psi_{i,k+1} + \cdots + \psi_{i,m_i n + k}, \quad (i = 1, \cdots, h).
$$

(iii) Let $F : \mathfrak{g}_n^0 \to \mathbb{C}$ be a smooth function. The differential equation for the vector field $(\lambda P_{k+1} - P_k)dF$ is expressed in Lax form as

$$
\frac{d}{dt} X_\lambda = [X_\lambda, \nabla_k F], \quad X_\lambda = \lambda^n X_0 + \lambda^{n-1} X_1 + \cdots + X_n.
$$

(iv) Define the function $G_{i,k,j}$ to be

$$
G_{i,k,j} = - \left(\lambda^{j-1} \psi_{i,k} + \lambda^{j-2} \psi_{i,k+1} + \cdots + \psi_{i,k+j-1} \right).
$$

Then, the equality

$$P_{k+1}d\psi_{i,k+j} = P_k d\psi_{i,k+j-1} = (\lambda P_{k+1} - P_k)dG_{i,k,j}$$

holds for $i = 1, \cdots, h$, $j = 1, \cdots, m_i n$ and $k = 1, \cdots, n$. In particular, the vector field $P_{k+1}d\psi_{i,k+j}$ is independent of k and the equation for it is expressed in Lax form as

$$\frac{d}{dt}X_\lambda = [X_\lambda, \nabla_k G_{i,k,j}].$$

(v) The vector fields $P_{k+1}d\psi_{i,k+j}$ for $i = 1, \cdots, h$ and $j = 1, \cdots, m_i n$ commute with each other (note that it is zero when $j \notin \{1, \cdots, m_i n\}$).

Proof. (i) and (ii) can be verified by a straightforward calculation with the aid of Prop.2.1 (v). To prove (iii), note that the vector field $P_{k+1}dF$ is written as

$$P_{k+1}dF = \left(\begin{array}{ccc|ccc} & & [X_0, \cdot] & & & \\ & \iddots & \vdots & & & \\ [X_0, \cdot] & \cdots & [X_{k-1}, \cdot] & & & \\ \hline & & & -[X_{k+1}, \cdot] & \cdots & -[X_n, \cdot] \\ & & & \vdots & \iddots & \\ & & & -[X_n, \cdot] & & \end{array} \right) \left(\begin{array}{c} \nabla_1 F \\ \vdots \\ \nabla_k F \\ \nabla_{k+1} F \\ \vdots \\ \nabla_n F \end{array} \right),$$

and similarly for $P_k dF$. Using them, write down the equation of X_j for the vector field $(\lambda P_{k+1} - P_k)dF$. For example, the equation for X_1 is $dX_1/dt = \lambda[X_0, \nabla_k F] - [X_0, \nabla_{k-1}F]$. Summing up the equations of $\lambda^{n-j}X_j$ proves the desired result.

(iv) Since $\lambda^{m_i n}\psi_{i,k} + \cdots + \psi_{i,m_i n+k}$ is the Casimir of $\lambda P_{k+1} - P_k$, we have

$$(\lambda P_{k+1} - P_k)d(\lambda^{m_i n}\psi_{i,k} + \lambda^{m_i n-1}\psi_{i,k+1} + \cdots + \psi_{i,m_i n+k}) = 0.$$

Expanding this yields the first equality. The second equality is confirmed by a straightforward calculation.

(v) Due to Part (iv), we can assume that $k = n$. Because of the property $[P_{n+1}dF, P_{n+1}dG] = P_{n+1}d\{G, F\}$ of a Poisson bracket (the left hand side is the Lie bracket for vector fields), it is sufficient to show the equality $\{\psi_{i',j'}, \psi_{i,j}\}_{n+1} = 0$ for $i, i' = 1, \cdots, h$ and $j, j' = 1, \cdots, (m_i + 1)n$. When $j = 1, \cdots, n$, it is trivial because $\psi_{i,j}$ is the Casimir of P_{n+1}. Next, we have

$$\begin{aligned}
&\{\lambda^{m_i n}\psi_{i',k} + \lambda^{m_i n-1}\psi_{i',k+1} + \cdots + \psi_{i',m_i n+k}, \ \psi_{i,n+j}\}_{n+1} \\
=\ & \langle d(\lambda^{m_i n}\psi_{i',k} + \lambda^{m_i n-1}\psi_{i',k+1} + \cdots + \psi_{i',m_i n+k}), \ P_{n+1}d\psi_{i,n+j}\rangle \\
=\ & \langle d(\lambda^{m_i n}\psi_{i',k} + \lambda^{m_i n-1}\psi_{i',k+1} + \cdots + \psi_{i',m_i n+k}), \ P_{k+1}d\psi_{i,k+j}\rangle \\
=\ & \langle d(\lambda^{m_i n}\psi_{i',k} + \lambda^{m_i n-1}\psi_{i',k+1} + \cdots + \psi_{i',m_i n+k}), \ (\lambda P_{k+1} - P_k)dG_{i,k,j}\rangle \\
=\ & -\langle dG_{i,k,j}, \ (\lambda P_{k+1} - P_k)d(\lambda^{m_i n}\psi_{i',k} + \lambda^{m_i n-1}\psi_{i',k+1} + \cdots + \psi_{i',m_i n+k})\rangle = 0.
\end{aligned}$$

This provides

$$\{\psi_{i',k}, \psi_{i,n+j}\}_{n+1} = \cdots = \{\psi_{i',m_i n+k}, \psi_{i,n+j}\}_{n+1} = 0,$$

for any $k = 1, \cdots, n$ and any $j = 1, \cdots, m_i n$, which completes the proof. □

Theorem 2.4. Suppose that the constant x_0 for the definition of $\mathfrak{g}_n^0$ is chosen so that the functions $\{\psi_{i,j}\}_{i,j}$ are functionally independent. Then, the vector field $P_{k+1} d\psi_{i,k+j}$, which is independent of k, is completely integrable in the Liouville sense for any i and j.

Proof. Recall $\dim(\mathfrak{g}) = d$ and $\operatorname{rank}(\mathfrak{g}) = h$. Thus, $\dim(\mathfrak{g}_n^0) = nd$. Since P_{k+1} has nh Casimir functions, the dimension of a symplectic leaf S of P_{k+1} is $n(d - h)$. On the leaf S, the vector fields $\{P_{k+1} d\psi_{i,k+j}\}_{i,j}$ define $n(d-h)$-dim Hamiltonian systems, among which nonzero vector fields are for $i = 1, \cdots, h$ and $j = 1, \cdots, m_i n$. Further, these nonzero vector fields commute with each other and they are linearly independent due to the assumption. The number of the nonzero vector fields is

$$\sum_{i=1}^{h} m_i n = \frac{1}{2}(\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}))n = \frac{1}{2}n(d - h) = \frac{1}{2}\dim(S).$$

Hence, the Liouville theorem shows that the vector fields are integrable. □

In what follows, we suppose the above assumption; the constant x_0 for the definition of $\mathfrak{g}_n^0$ is chosen so that the functions $\{\psi_{i,j}\}_{i,j}$ are functionally independent. That is, the differentials $\{d\psi_{i,j}\}_{i,j}$ are linearly independent except for finite points.

2.3 Symplectic reduction

The next purpose is to perform a symplectic reduction [12, 13, 14, 15].

Lemma 2.5. The h dimensional distribution D defined by

$$D = \operatorname{span}\{P_k d\psi_{i,k} \,|\, i = 1, \cdots, h\}$$

is integrable in the Frobenius sense. The vector fields $P_k d\psi_{i,k}$ are linear for $i = 1, \cdots, h$.

Proof. The first statement follows from Prop.2.3(v). Since $P_k d\psi_{i,k}$ is independent of k, we obtain $P_k d\psi_{i,k} = P_1 d\psi_{i,1}$. Since $\operatorname{wdeg}(\psi_{i,1}) = 1$, $d\psi_{1,i}$ is a constant, while P_1 is linear in $(x_1, \cdots, x_n)$. □

The differential equation for $P_k d\psi_{i,k} = P_1 d\psi_{i,1}$ is given by

$$\frac{d}{dt} X_\lambda = [X_\lambda, \nabla_1 G_{i,1,1}] = [\nabla_1 \psi_{i,1}, X_\lambda].$$

Since $\nabla_1 \psi_{i,1}$ is independent of λ, this is decomposed as

$$\frac{d}{dt} X_k = [\nabla_1 \psi_{i,1}, X_k], \quad k = 1, \cdots, n.$$

In coordinates, it is expressed as

$$\frac{dx_k}{dt} = -A_k \frac{\partial \psi_{i,1}}{\partial x_1}(x_1), \quad k = 1, \cdots, n. \tag{2.8}$$

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Let us consider the orbit space $\pi : \mathfrak{g}_n^0 \to \mathfrak{g}_n^0/D$, which is a smooth manifold if points on $\mathfrak{g}_n^0$ at which $\dim(D) < h$ is removed if necessary. The Marsden-Ratiu reduction theorem states that the orbit space $\mathfrak{g}_n^0/D$ is again a multi-Poisson manifold with compatible Poisson tensors denoted by $\widetilde{P}_1, \cdots, \widetilde{P}_{n+1}$. They are defined by $\widetilde{P}_k = \pi_* P_k \pi^*$. Let $\{ \ , \ \}_k$ and $\{ \ , \ \}_k'$ be Poisson brackets associated with P_k and $\widetilde{P}_k$, respectively. For a function F on $\mathfrak{g}_n^0$ which is constant along each integral manifold of D, a function $\widetilde{F}$ on $\mathfrak{g}_n^0/D$ is well-defined through $\widetilde{F} \circ \pi = \pi^* \widetilde{F} = F$. Conversely, for a function $\widetilde{F}$ on $\mathfrak{g}_n^0/D$, we can find a function F on $\mathfrak{g}_n^0$, which is constant along D, such that $\widetilde{F} \circ \pi = F$. Then, $\{ \ , \ \}_k'$ is given by $\{\widetilde{F}, \widetilde{G}\}_k' \circ \pi = \{F, G\}_k$.

Because of Prop.2.3(v), $\psi_{i,j}$ is constant along each integral manifold of D and the projection $\widetilde{\psi}_{i,j}$ is well-defined. The projected vector field is given by $\widetilde{P}_k d\widetilde{\psi}_{i,j} = \pi_*(P_k d\psi_{i,j})$.

It is convenient to realize $\mathfrak{g}_n^0/D$ as a submanifold of $\mathfrak{g}_n^0$. Let $\sigma : \mathfrak{g}_n^0/D \to \mathfrak{g}_n^0$ be a smooth section. The image $\sigma(\mathfrak{g}_n^0/D)$ is a submanifold of $\mathfrak{g}_n^0$ which is diffeomorphic to $\mathfrak{g}_n^0/D$. In Prop.2.6(iii) below, $\mathfrak{g}_n^0/D$ is identified with a submanifold in this manner.

Proposition 2.6.
(i) Casimir functions of $\widetilde{P}_{k+1}$ are $\widetilde{\psi}_{i,j}$ $(i = 1, \cdots, h)$ for $j = 1, 2, \cdots, k + 1$ and for $j = m_i n + k + 1, m_i n + k + 2, \cdots, (m_i + 1)n$.
(ii) Casimir functions of the combination $\lambda \widetilde{P}_{k+1} - \widetilde{P}_k$ are $\widetilde{\psi}_{i,j}$ $(i = 1, \cdots, h)$ for $j = 1, 2, \cdots, k$ and for $j = m_i n + k + 1, m_i n + k + 2, \cdots, (m_i + 1)n$, and

$$\lambda^{m_i n - 1}\widetilde{\psi}_{i,k+1} + \lambda^{m_i n - 2}\widetilde{\psi}_{i,k+2} + \cdots + \widetilde{\psi}_{i,m_i n + k}, \quad (i = 1, \cdots, h).$$

(iii) For a smooth function $\widetilde{F} : \mathfrak{g}_n^0/D \to \mathbb{C}$, there exist scalar-valued functions $\beta_1, \cdots, \beta_h : \mathfrak{g}_n^0/D \to \mathbb{C}$ such that the equation for the vector field $(\lambda \widetilde{P}_{k+1} - \widetilde{P}_k)d\widetilde{F}$ is expressed in Lax form as

$$\frac{d}{dt}\widetilde{X}_\lambda = [\widetilde{X}_\lambda, \nabla_k \widetilde{F}] - \sum_{i=1}^h \beta_i[\widetilde{X}_\lambda, \nabla_1 \widetilde{\psi}_{i,1}] = [\widetilde{X}_\lambda, \nabla_k \widetilde{F} - \sum_{i=1}^h \beta_i \nabla_1 \widetilde{\psi}_{i,1}],$$

where $\widetilde{X}_\lambda = X_\lambda|_{\mathfrak{g}_n^0/D}$ and $\nabla_k \widetilde{F} = (\nabla_k F)|_{\mathfrak{g}_n^0/D}$.
(iv) Define the function $\widetilde{G}_{i,k,j}$ to be

$$\widetilde{G}_{i,k,j} = -\left(\lambda^{j-1}\widetilde{\psi}_{i,k} + \lambda^{j-2}\widetilde{\psi}_{i,k+1} + \cdots + \widetilde{\psi}_{i,k+j-1}\right).$$

Then, the equality

$$\widetilde{P}_{k+1}d\widetilde{\psi}_{i,k+j} = \widetilde{P}_k d\widetilde{\psi}_{i,k+j-1} = (\lambda \widetilde{P}_{k+1} - \widetilde{P}_k)d\widetilde{G}_{i,k,j}$$

holds for $i = 1, \cdots, h$, $j = 2, \cdots, m_i n$ and $k = 1, \cdots, n$.
(v) The vector fields $\widetilde{P}_{k+1}d\widetilde{\psi}_{i,k+j}$ for $i = 1, \cdots, h$ and $j = 2, \cdots, m_i n$ commute with each other (note that it is zero when $j \notin \{2, \cdots, m_i n\}$).

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Proof. (i) $\widetilde{\psi}_{i,j}$ for $j = 1, \cdots, k$ and $j = m_i n + k + 1, \cdots, (m_i + 1)n$ are Casimir of $\widetilde{P}_{k+1}$ because they are Casimir of P_{k+1}. For $\widetilde{\psi}_{i,k+1}$, we have

$$\{\widetilde{F}, \widetilde{\psi}_{i,k+1}\}'_{k+1} = \{F, \psi_{i,k+1}\}_{k+1} = \langle dF, P_{k+1} d\psi_{i,k+1} \rangle = (P_k d\psi_{i,k})(F).$$

The right hand side becomes zero because F is constant along D.

(ii) The first statement (on $\widetilde{\psi}_{i,j}$) is trivial because they are common Casimir of $\widetilde{P}_{k+1}$ and $\widetilde{P}_k$. The last function $\lambda^{m_i n - 1}\widetilde{\psi}_{i,k+1} + \cdots$ is a projection of the function given in Prop.2.3(ii).

The results of (iv) and (v) are projections of those of Prop.2.3(iv) and (v).

To prove (iii), $\mathfrak{g}_n^0/D$ is identified with a submanifold of $\mathfrak{g}_n^0$ as above. Put $F = \pi^*\widetilde{F}$. We have to calculate the projection of the vector field $[X_\lambda, \nabla_k F]$ onto $\mathfrak{g}_n^0/D$ (see Prop.2.3(iii)). At first, we restrict the domain to $\mathfrak{g}_n^0/D$ as

$$[X_\lambda, \nabla_k F]|_{\mathfrak{g}_n^0/D} = [X_\lambda|_{\mathfrak{g}_n^0/D}, (\nabla_k F)|_{\mathfrak{g}_n^0/D}] = [\widetilde{X}_\lambda, \nabla_k \widetilde{F}].$$

Since this is not tangent to $T(\mathfrak{g}_n^0/D)$, we calculate the projection of it according to the decomposition $T\mathfrak{g}_n^0 = T(\mathfrak{g}_n^0/D) \oplus D$. Then, (iii) follows from the fact that the distribution D is spanned by the vector fields of the form $[X_\lambda, \nabla_1 \psi_{i,1}]$. $\square$

2.4 Isospectral deformation to isomonodromic deformation

Now we have $(m_i n - 1)h$ distinct vector fields on $\mathfrak{g}_n^0/D$

$$j = 2: \quad \widetilde{P}_1 d\widetilde{\psi}_{i,2} = \cdots = \widetilde{P}_{k+1} d\widetilde{\psi}_{i,k+2} = \cdots = \widetilde{P}_{n+1} d\widetilde{\psi}_{i,n+2}$$

$$\vdots$$

$$j = j: \quad \widetilde{P}_1 d\widetilde{\psi}_{i,j} = \cdots = \widetilde{P}_{k+1} d\widetilde{\psi}_{i,k+j} = \cdots = \widetilde{P}_{n+1} d\widetilde{\psi}_{i,n+j}$$

$$\vdots$$

$$j = m_i n: \quad \widetilde{P}_1 d\widetilde{\psi}_{i,m_i n} = \cdots = \widetilde{P}_{k+1} d\widetilde{\psi}_{i,k+m_i n} = \cdots = \widetilde{P}_{n+1} d\widetilde{\psi}_{i,(m_i+1)n}$$

They are $(nd - h)$-dimensional integrable systems. For fixed k, a symplectic leaf of the Poisson structure $\widetilde{P}_k$ is given by a level surface of Casimir functions as

$$S_k = \{\widetilde{\psi}_{i,j} = \alpha_{i,j} \text{ (constant)} \,| $$
$$i = 1, \cdots, h; \, j = 1, \cdots, k; \, j = m_i n + k, \cdots, (m_i + 1)n\}.$$

Restricted on the symplectic leaf, the vector fields become $(nd - nh - 2h)$-dimensional completely integrable Hamiltonian systems of the form

$$\widetilde{P}_{k+1} d\widetilde{\psi}_{i,k+j}: \quad \frac{d}{dt}\widetilde{X}_\lambda = [\widetilde{X}_\lambda, A_\lambda], \qquad \text{on } S_k, \qquad (2.9)$$

$$A_\lambda := \nabla_k \widetilde{G}_{i,k,j} - \sum_{i=1}^{h} \beta_i \nabla_1 \widetilde{\psi}_{i,1}.$$

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Both of $\widetilde{X}_\lambda$ and A_λ depend on parameters $\{\alpha_{i,j}\}_{i,j}$ which define the symplectic leaf. Thus, we write $\widetilde{X}_\lambda$ as $\widetilde{X}_\lambda(t, \alpha)$, where α denotes the collection of parameters $\alpha_{i,j}$.

Now suppose that there exists a parameter $\alpha_{i',j'}$ such that the following condition holds

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha_{i',j'}}(t, \alpha) = \lambda^l \frac{\partial A_\lambda}{\partial \lambda} \tag{2.10}$$

for some integer l. Eq.(2.9) is put together with Eq.(2.10) to yield

$$\frac{\partial \widetilde{X}_\lambda}{\partial t}(t, \alpha) + \frac{\partial \widetilde{X}_\lambda}{\partial \alpha_{i',j'}}(t, \alpha) = [\widetilde{X}_\lambda, A_\lambda] + \lambda^l \frac{\partial A_\lambda}{\partial \lambda}.$$

Define the Lax matrix L_λ by

$$L_\lambda := \frac{1}{\lambda^l} \widetilde{X}_\lambda(t, \alpha)|_{\alpha_{i',j'}=t},$$

where the parameter $\alpha_{i',j'}$ satisfying the condition (2.10) is replaced by t. Then, the above equation is rewritten as

$$\frac{dL_\lambda}{dt} = [L_\lambda, A_\lambda] + \frac{\partial A_\lambda}{\partial \lambda}, \tag{2.11}$$

which is known as the isomonodromic deformation equation. It is known that a system written as the isomonodromic deformation equation enjoys the Painlevé property. The function $\widetilde{\psi}_{i,k+j}$ restricted on S_k will be a Hamiltonian function of the Painlevé equation after replacing $\alpha_{i',j'} \mapsto t$ and changing to Darboux's coordinates if necessary.

For ABCD-type simple Lie algebras, the dimensions of Painlevé systems obtained in this manner are summarized in Table 3. In particular, the dimension is 2 when $\mathfrak{g} = \mathfrak{sl}_2$, $n = 2$, and it is 4 when $\mathfrak{g} = \mathfrak{sl}_2$, $n = 3$ or $\mathfrak{g} = \mathfrak{so}_5$, $n = 1$. From the next sections, we will demonstrate our method for these cases. In particular, the Hamiltonian functions shown in Sec.1 will be obtained.

		dimension	$h = 1$	$h = 2$	$h = 3$
$A_h (h \geq 1)$	$\mathfrak{sl}_{h+1}$	$nh^2 + nh - 2h$	$2n - 2$	$6n - 4$	$12n - 6$
$B_h (h \geq 2)$	$\mathfrak{so}_{2h+1}$	$2nh^2 - 2h$	$-$	$8n - 4$	$18n - 6$
$C_h (h \geq 3)$	$\mathfrak{sp}_{2h}$	$2nh^2 - 2h$	$-$	$-$	$18n - 6$
$D_h (h \geq 4)$	$\mathfrak{so}_{2h}$	$2nh^2 - 2nh - 2h$	$-$	$-$	$-$

Table 3: The dimensions of Painlevé systems.

3 2-dim Painlevé equations: $\mathfrak{g} = \mathfrak{sl}_2$, $n = 2$

In this case, a general element of $\mathfrak{g}_n$ is written by

$$X_\lambda = \lambda^2 \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} + \lambda \begin{pmatrix} u_1 & v_1 \\ w_1 & -u_1 \end{pmatrix} + \begin{pmatrix} u_2 & v_2 \\ w_2 & -u_2 \end{pmatrix}.$$

The Painlevé equation obtained by our method depends on a choice of $x_0 = (u_0, v_0, w_0)$. We consider the following two cases.

$$\text{(I)} \quad \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{(II)} \quad \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

From the former case, we will obtain the second and fourth Painlevé equations P_{II}, P_{IV}, and from the latter one, we will obtain the first and second Painlevé equations P_I, P_{II}.

3.1 Case (I)

In this case, the functions $\psi_{i,j} = \psi_j$ (since $h = \text{rank}(\mathfrak{g}) = 1$, we omit the subscript i) are given by

$$\begin{cases} \psi_1 = 2u_1 \\ \psi_2 = 2u_2 + u_1^2 + v_1 w_1 \\ \psi_3 = 2u_1 u_2 + v_2 w_1 + v_1 w_2 \\ \psi_4 = u_2^2 + v_2 w_2. \end{cases}$$

(see Example 2.2). The differential equation (2.8) defining the distribution D is $u'_j = 0$, $v'_j = 2v_j$, $w'_j = -2w_j$ for $j = 1, 2$. This is solved as a function of w_1 as

$$u_1 = U_1, \ u_2 = U_2, \ v_1 = V_2/w_1, \ v_2 = V_3/w_1, \ w_2 = W_1 w_1,$$

where U_1, U_2, V_2, V_3, W_1 are integral constants (initial values at $w_1 = 1$), for which the subscripts are given so that they are consistent with the weighted degrees (for example, since $\text{wdeg}(v_2 w_1) = 2 + 1 = 3$, the weighted degree of V_3 is three). This relation defines a coordinate transformation

$$(u_1, v_1, w_1, u_2, v_2, w_2) \mapsto (U_1, V_2, w_1, U_2, V_3, W_1).$$

In the new coordinates, integral manifolds of the distribution D are straight lines along w_1-axis. In particular, the subset $\{w_1 = 1\} \subset \mathfrak{g}_n^0$ gives the realization of the orbit space $\mathfrak{g}_n^0/D$ as a submanifold and $(U_1, V_2, U_2, V_3, W_1)$ provides a global coordinate system of $\mathfrak{g}_n^0/D$.

At this stage, we have on $\mathfrak{g}_n^0/D$

$$\begin{cases} \widetilde{\psi}_1 = 2U_1 \\ \widetilde{\psi}_2 = 2U_2 + U_1^2 + V_2 \\ \widetilde{\psi}_3 = 2U_1 U_2 + V_3 + V_2 W_1 \\ \widetilde{\psi}_4 = U_2^2 + V_3 W_1, \end{cases}$$

and three Poisson structures $\widetilde{P}_1$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_3, \widetilde{\psi}_4$), $\widetilde{P}_2$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_4$), $\widetilde{P}_3$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3$), and vector fields $\widetilde{P}_3 d\widetilde{\psi}_4 = \widetilde{P}_2 d\widetilde{\psi}_3 = \widetilde{P}_1 d\widetilde{\psi}_2$ which are

expressed as the Lax equation $d\widetilde{X}_\lambda/dt = [A_\lambda, \widetilde{X}_\lambda]$, where

$$
\begin{cases}
\widetilde{X}_\lambda = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} + \begin{pmatrix} U_2 & V_3 \\ W_1 & -U_2 \end{pmatrix}, \\
A_\lambda = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} - \begin{pmatrix} W_1 & 0 \\ 0 & -W_1 \end{pmatrix}.
\end{cases}
$$

The next purpose is to restrict the vector fields on a symplectic leaf. We will consider $\widetilde{P}_3 d\widetilde{\psi}_4$ and $\widetilde{P}_2 d\widetilde{\psi}_3$ separately ($\widetilde{P}_1 d\widetilde{\psi}_2$ will not be considered because there are no parameters satisfying (2.10)).

(i) Consider the vector field $\widetilde{P}_3 d\widetilde{\psi}_4$. For the Poisson tensor $\widetilde{P}_3$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \; j = 1, 2, 3\}$. In order for the condition

$$
\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \frac{\partial A_\lambda}{\partial \lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}
$$

to be satisfied, we find that U_2 in $\widetilde{X}_\lambda$ has to include a parameter α which will be replaced by t later. For this purpose, we take the symplectic leaf

$$
S = \{2U_1 = 0, \; 2U_2 + U_1^2 + V_2 = 2\alpha_2, \; 2U_1 U_2 + V_3 + V_2 W_1 = \alpha_3\}.
$$

Hence, we put $U_1 = 0$, $U_2 = \alpha_2 - V_2/2$, $V_3 = \alpha_3 - V_2 W_1$, and (V_2, W_1) gives a global coordinate system for the symplectic leaf. Then, it turns out that $\widetilde{X}_\lambda$ satisfies the condition (3.1) with $\alpha = \alpha_2$ on the symplectic leaf. Finally, by replacing α_2 by t, we obtain the isomonodromic deformation equation (2.11) with

$$
\begin{cases}
L_\lambda = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & V_2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} t - V_2/2 & \alpha_3 - V_2 W_1 \\ W_1 & -(t - V_2/2) \end{pmatrix}, \\
A_\lambda = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -W_1 & V_2 \\ 1 & W_1 \end{pmatrix}.
\end{cases}
$$

The Poisson tensor $\widetilde{P}_3$ on the symplectic leaf with coordinates (V_2, W_1) is given by

$$
\widetilde{P}_3 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.
$$

To change to Darboux's coordinates, we put $V_2 = -2p_2$, $W_1 = q_1$. Then, $\widetilde{P}_3$ becomes the canonical symplectic matrix. In the coordinates (q_1, p_2), the isomonodromic deformation equation is a Hamiltonian system. The Hamiltonian function $\widetilde{\psi}_4$ for the vector field $\widetilde{P}_3 d\widetilde{\psi}_4$ is written as

$$
\begin{aligned}
\widetilde{\psi}_4 &= U_2^2 + V_3 W_1 \\
&= (t - V_2/2)^2 + (\alpha_3 - V_2 W_1) W_1 \\
&= p_2^2 + 2q_1^2 p_2 + 2t p_2 + \alpha_3 q_1 + t^2.
\end{aligned}
$$

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This is reduced to the Hamiltonian function (1.3) of the second Painlevé equation by a certain coordinate change and the isomonodromic deformation equation (2.11) is equivalent to the second Painlevé equation.

(ii) Consider the vector field $\widetilde{P}_2 d\widetilde{\psi}_3$. For the Poisson tensor $\widetilde{P}_2$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 4\}$. In order for the condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \lambda \frac{\partial A_\lambda}{\partial \lambda} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

to be satisfied, we find that U_1 in $\widetilde{X}_\lambda$ has to include a parameter α which will be replaced by t later, and the other components of $\widetilde{X}_\lambda$ cannot include α. For this purpose, we take the symplectic leaf

$$S = \{2U_1 = 2\alpha_1, \ 2U_2 + U_1^2 + V_2 = \alpha_2 + \alpha_1^2, \ U_2^2 + V_3 W_1 = \alpha_4\}.$$

This relation is rewritten as

$$U_1 = \alpha_1, \quad V_2 = \alpha_2 - 2U_2, \quad V_3 = (\alpha_4 - U_2^2)/W_1.$$

By substituting them, $\widetilde{X}_\lambda$ satisfies the condition (3.2) with $\alpha = \alpha_1$. Finally, by replacing α_1 by t, we obtain the isomonodromic deformation equation (2.11).

The Poisson tensor $\widetilde{P}_2$ on the symplectic leaf with coordinates (U_2, W_1) is given by

$$\widetilde{P}_2 = \begin{pmatrix} 0 & W_1 \\ -W_1 & 0 \end{pmatrix}.$$

For Darboux's coordinates, we put $U_2 = p_1 q_1 - \beta_2$ and $W_1 = q_1$, where β_2 is an arbitrary constant. Then, $\widetilde{P}_2$ is transformed to the canonical symplectic matrix. In the coordinates (q_1, p_1), the isomonodromic deformation equation is a Hamiltonian system. The Hamiltonian function $\widetilde{\psi}_3$ for the vector field $\widetilde{P}_2 d\widetilde{\psi}_3$ is written as

$$
\begin{aligned}
\widetilde{\psi}_3 &= 2U_1 U_2 + V_3 + V_2 W_1 \\
&= 2tU_2 + (\alpha_4 - U_2^2)/W_1 + (\alpha_2 - 2U_2)W_1 \\
&= -p_1^2 q_1 - 2p_1 q_1^2 + 2tp_1 q_1 + 2\beta_2 p_1 + (\alpha_2 + 2\beta_2)q_1 - 2\beta_2 t + \frac{\alpha_4 - \beta_2^2}{q_1}.
\end{aligned}
$$

We choose the free parameter β_2 to be $\alpha_4 = \beta_2^2$ so that the Hamiltonian becomes a polynomial. This is the Hamiltonian function (1.4) of the fourth Painlevé equation up to some scaling. The isomonodromic deformation equation (2.11) is equivalent to the fourth Painlevé equation, where

$$
\begin{cases}
L_\lambda = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} t & \alpha_2 + 2\beta_2 - 2p_1 q_1 \\ 1 & -t \end{pmatrix} + \dfrac{1}{\lambda} \begin{pmatrix} p_1 q_1 - \beta_2 & -p_1^2 q_1 + 2\beta_2 p_1 \\ q_1 & -(p_1 q_1 - \beta_2) \end{pmatrix}, \\
A_\lambda = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} t - q_1 & \alpha_2 + 2\beta_2 - 2p_1 q_1 \\ 1 & -(t - q_1) \end{pmatrix}.
\end{cases}
$$

3.2   Case (II)

In this case, the functions $\psi_{i,j} = \psi_j$ are given by

$$\begin{cases} \psi_1 = v_1 \\ \psi_2 = u_1^2 + v_2 + v_1 w_1 \\ \psi_3 = 2u_1 u_2 + v_2 w_1 + v_1 w_2 \\ \psi_4 = u_2^2 + v_2 w_2. \end{cases}$$

The differential equation (2.8) defining the distribution D is $u_j' = -v_j$, $v_j' = 0$, $w_j' = 2u_j$ for $j = 1, 2$. We can assume without loss of generality that $v_1 = 1$ by a suitable scaling of variables (indeed, v_1 is a common Casimir of P_1, P_2, P_3). Thus, the equations are solved as a function of u_1 as

$$v_1 = 1, \ v_2 = V_2, \ u_2 = V_2 u_1 + U_3, \ w_1 = -u_1^2 + W_2, \ w_2 = -V_2 u_1^2 - 2U_3 u_1 + W_4,$$

where V_2, U_3, W_2, W_4 are integral constants (initial values at $u_1 = 0$). This relation defines a coordinate transformation

$$(u_1, w_1, u_2, v_2, w_2) \mapsto (u_1, W_2, U_3, V_2, W_4).$$

In the new coordinates, integral manifolds of the distribution D are straight lines along u_1-axis. In particular, the subset $\{u_1 = 0\} \subset \mathfrak{g}_n^0$ gives the realization of the orbit space $\mathfrak{g}_n^0/D$ and (W_2, U_3, V_2, W_4) provides a global coordinate system of $\mathfrak{g}_n^0/D$ restricted to $v_1 = 1$.

On $\mathfrak{g}_n^0/D$, we have functions

$$\begin{cases} \widetilde{\psi}_1 = 1 \\ \widetilde{\psi}_2 = V_2 + W_2 \\ \widetilde{\psi}_3 = V_2 W_2 + W_4 \\ \widetilde{\psi}_4 = U_3^2 + V_2 W_4, \end{cases}$$

and three Poisson structures $\widetilde{P}_1$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_3, \widetilde{\psi}_4$), $\widetilde{P}_2$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_4$), $\widetilde{P}_3$ (Casimirs are $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3$), and vector fields $\widetilde{P}_3 d\widetilde{\psi}_4 = \widetilde{P}_2 d\widetilde{\psi}_3 = \widetilde{P}_1 d\widetilde{\psi}_2$ which are expressed as the Lax equation $d\widetilde{X}_\lambda/dt = [A_\lambda, \widetilde{X}_\lambda]$, where

$$\begin{cases} \widetilde{X}_\lambda = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} + \begin{pmatrix} U_3 & V_2 \\ W_4 & -U_3 \end{pmatrix}, \\ A_\lambda = \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ V_2 & 0 \end{pmatrix}. \end{cases}$$

The next purpose is to restrict the vector fields on a symplectic leaf.

(i) Consider the vector field $\widetilde{P}_3 d\widetilde{\psi}_4$. For the Poisson tensor $\widetilde{P}_3$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 3\}$. For the condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \frac{\partial A_\lambda}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

we find that W_4 in $\widetilde{X}_\lambda$ has to include a parameter α. To this end, we take the symplectic leaf as

$$S = \{\widetilde{\psi}_1 = 1, \ \widetilde{\psi}_2 = V_2 + W_2 = 0, \ \widetilde{\psi}_3 = V_2 W_2 + W_4 = \alpha_4\}.$$

Hence, we put $V_2 = -W_2$, $W_4 = \alpha_4 + W_2^2$, so that (W_2, U_3) gives a global coordinate system on the leaf. Then, $\widetilde{X}_\lambda$ satisfies the condition (3.3) with $\alpha = \alpha_4$. Finally, by replacing α_4 by t, we obtain the isomonodromic deformation equation (2.11) with

$$\begin{cases} L_\lambda = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} + \begin{pmatrix} U_3 & -W_2 \\ t + W_2^2 & -U_3 \end{pmatrix}, \\ A_\lambda = \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2W_2 & 0 \end{pmatrix}. \end{cases}$$

The Poisson tensor $\widetilde{P}_3$ on the symplectic leaf is given by

$$\widetilde{P}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is already in canonical form. On the symplectic leaf, the function $\widetilde{\psi}_4$ is written as

$$\widetilde{\psi}_4 \ = \ U_3^2 + V_2 W_4 = U_3^2 - W_2^3 - t W_2.$$

This is the Hamiltonian function (1.2) of the first Painlevé equation (up to some scaling) and the isomonodromic deformation equation (2.11) coincides with the first Painlevé equation.

(ii) For the vector field $\widetilde{P}_2 d\widetilde{\psi}_3$, we again obtain the second Painlevé equation and the detailed calculation is omitted.

4   4-dim Painlevé equations: $\mathfrak{g} = \mathfrak{sl}_2$, $n = 3$

In this case, a general element of $\mathfrak{g}_n$ is written as

$$X_\lambda = \lambda^3 \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} + \lambda^2 \begin{pmatrix} u_1 & v_1 \\ w_1 & -u_1 \end{pmatrix} + \lambda \begin{pmatrix} u_2 & v_2 \\ w_2 & -u_2 \end{pmatrix} + \begin{pmatrix} u_3 & v_3 \\ w_3 & -u_3 \end{pmatrix}.$$

For the definition of $\mathfrak{g}_n^0$, we again consider the following two cases.

$$\text{(I)} \ \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{(II)} \ \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

From the former case, we will obtain Hamiltonian functions (1.12),(1.13),(1.14), and from the latter one, we will obtain (1.10),(1.11),(1.15).

4.1 Case (I)

In this case, the functions $\psi_{i,j} = \psi_j$ (since $h = \mathrm{rank}(\mathfrak{g}) = 1$, we omit the subscript i) are given by

$$
\begin{cases}
\psi_1 = 2u_1 \\
\psi_2 = 2u_2 + u_1^2 + v_1 w_1 \\
\psi_3 = 2u_1 u_2 + 2u_3 + v_2 w_1 + v_1 w_2 \\
\psi_4 = u_2^2 + 2u_1 u_3 + v_3 w_1 + v_2 w_2 + v_1 w_3 \\
\psi_5 = 2u_2 u_3 + v_3 w_2 + v_2 w_3 \\
\psi_6 = u_3^2 + v_3 w_3.
\end{cases}
$$

The differential equation (2.8) defining the distribution D is $u_j' = 0$, $v_j' = 2v_j$, $w_j' = -2w_j$ for $j = 1, 2, 3$. This is solved as a function of w_1 as

$$
u_1 = U_1, \ \ u_2 = U_2, \ \ u_3 = U_3,
$$
$$
v_1 = V_2/w_1, \ \ v_2 = V_3/w_1, \ \ v_3 = V_4/w_1, \ \ w_2 = W_1 w_1, \ \ w_3 = W_2 w_1,
$$

where $U_1, U_2, U_3, V_2, V_3, V_4, W_1, W_2$ are integral constants (initial values at $w_1 = 1$). This relation defines a coordinate transformation

$$
(u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3) \mapsto (U_1, V_2, w_1, U_2, V_3, W_1, U_3, V_4, W_2).
$$

In the new coordinates, integral manifolds of the distribution D are straight lines along w_1-axis. In particular, the subset $\{w_1 = 1\} \subset \mathfrak{g}_n^0$ gives the realization of the orbit space $\mathfrak{g}_n^0/D$ as a submanifold and $(U_1, V_2, U_2, V_3, W_1, U_3, V_4, W_2)$ provides a global coordinate system of $\mathfrak{g}_n^0/D$.

At this stage, we have on $\mathfrak{g}_n^0/D$

$$
\begin{cases}
\widetilde{\psi}_1 = 2U_1 \\
\widetilde{\psi}_2 = 2U_2 + U_1^2 + V_2 \\
\widetilde{\psi}_3 = 2U_1 U_2 + 2U_3 + V_3 + V_2 W_1 \\
\widetilde{\psi}_4 = 2U_1 U_3 + U_2^2 + V_4 + V_2 W_2 + V_3 W_1, \\
\widetilde{\psi}_5 = 2U_2 U_3 + V_4 W_1 + V_3 W_2 \\
\widetilde{\psi}_6 = U_3^2 + V_4 W_2,
\end{cases}
$$

and two vector fields

$$
\widetilde{P}_2 d\widetilde{\psi}_3 = \widetilde{P}_3 d\widetilde{\psi}_4 = \widetilde{P}_4 d\widetilde{\psi}_5, \tag{4.1}
$$
$$
\widetilde{P}_2 d\widetilde{\psi}_4 = \widetilde{P}_3 d\widetilde{\psi}_5 = \widetilde{P}_4 d\widetilde{\psi}_6. \tag{4.2}
$$

The differential equations of these vector fields are expressed in Lax form as

$$
\frac{\partial \widetilde{X}_\lambda}{\partial t_1} = [A_1, \widetilde{X}_\lambda], \quad \frac{\partial \widetilde{X}_\lambda}{\partial t_2} = [A_2, \widetilde{X}_\lambda], \tag{4.3}
$$

respectively, where

$$\begin{cases} \widetilde{X}_\lambda = \lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda^2 \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} + \lambda \begin{pmatrix} U_2 & V_3 \\ W_1 & -U_2 \end{pmatrix} + \begin{pmatrix} U_3 & V_4 \\ W_2 & -U_3 \end{pmatrix}, \\[2mm] A_1 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} - \begin{pmatrix} W_1 & 0 \\ 0 & -W_1 \end{pmatrix}, \\[2mm] A_2 = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} + \begin{pmatrix} U_2 & V_3 \\ W_1 & -U_2 \end{pmatrix} - \begin{pmatrix} W_2 & 0 \\ 0 & -W_2 \end{pmatrix}. \end{cases}$$

The next purpose is to restrict the vector fields on a symplectic leaf.

(i) Consider the pair of vector fields $\widetilde{P}_4 d\widetilde{\psi}_5$ and $\widetilde{P}_4 d\widetilde{\psi}_6$. For the Poisson tensor $\widetilde{P}_4$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 3, 4\}$. In order for the two conditions

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \frac{\partial A_1}{\partial \lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \frac{\partial A_2}{\partial \lambda} = 2\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix} \quad (4.4)$$

to be satisfied, we find that U_3 in $\widetilde{X}_\lambda$ has to include a parameter α, which will be replaced by t_1 later, and U_2 and W_2 have to include a parameter α', which will be replaced by t_2 later. For this purpose, we take the symplectic leaf

$$S = \{\widetilde{\psi}_1 = 0, \ \widetilde{\psi}_2 = 4\alpha_2, \ \widetilde{\psi}_3 = 2\alpha_3, \ \widetilde{\psi}_4 = \alpha_4 + 4\alpha_2^2\}.$$

Further, we change the coordinate as $W_2 = \widetilde{W}_2 + \alpha_2$ because W_2 should include a parameter. Then, the above relation for S is rearranged as

$$U_1 = 0, \ U_2 = 2\alpha_2 - \frac{1}{2}V_2, \ U_3 = \alpha_3 - \frac{1}{2}V_3 - \frac{1}{2}V_2 W_1,$$

$$V_4 = \alpha_4 + \alpha_2 V_2 - \frac{1}{4}V_2^2 - V_2 \widetilde{W}_2 - V_3 W_1.$$

Substituting them into $\widetilde{X}_\lambda$, A_1 and A_2, we can verify the condition (4.4) with $\alpha = \alpha_3$ and $\alpha' = \alpha_2$. By replacing α_3, α_2 by t_1, t_2, respectively, we obtain the isomonodromic deformation equations

$$\frac{\partial L_\lambda}{\partial t_1} = [A_1, L_\lambda] + \frac{\partial A_1}{\partial \lambda}, \quad \frac{\partial L_\lambda}{\partial t_2} = [A_2, L_\lambda] + \frac{\partial A_2}{\partial \lambda}, \quad (4.5)$$

which are equations of $(V_2, V_3, W_1, \widetilde{W}_2)$ with two independent variables t_1, t_2, where

$$\begin{cases} L_\lambda = \lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & V_2 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 2t_2 - \frac{1}{2}V_2 & V_3 \\ W_1 & -(2t_2 - \frac{1}{2}V_2) \end{pmatrix} \\[2mm] \qquad + \begin{pmatrix} t_1 - \frac{1}{2}V_3 - \frac{1}{2}V_2 W_1 & \alpha_4 + t_2 V_2 - \frac{1}{4}V_2^2 - V_2\widetilde{W}_2 - V_3 W_1 \\ \widetilde{W}_2 + t_2 & -(t_1 - \frac{1}{2}V_3 - \frac{1}{2}V_2 W_1) \end{pmatrix}, \\[3mm] A_1 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -W_1 & V_2 \\ 1 & W_1 \end{pmatrix}, \\[2mm] A_2 = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & V_2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} t_2 - \frac{1}{2}V_2 - \widetilde{W}_2 & V_3 \\ W_1 & -(t_2 - \frac{1}{2}V_2 - \widetilde{W}_2) \end{pmatrix}. \end{cases}$$

The Poisson tensor $\widetilde{P}_4$ on the symplectic leaf written in the coordinates $(V_2, V_3, W_1, \widetilde{W}_2)$ is given by

$$\widetilde{P}_4 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}.$$

The above two isomonodromic deformation equations can be written as Hamiltonian systems. The Hamiltonian functions of these equations are obtained by deleting U_1, U_2, U_3, V_4 from $\widetilde{\psi}_5$ and $\widetilde{\psi}_6$ by using the above relations, and changing to Darboux's coordinates by a scaling so that the above $\widetilde{P}_4$ is transformed to the canonical symplectic matrix. In this manner, we obtain Hamiltonian functions (1.12) of $(\mathrm{P_{II\text{-}2}})_2$ given in Sec.1.

(ii) Consider the pair of vector fields $\widetilde{P}_3 d\widetilde{\psi}_4$ and $\widetilde{P}_3 d\widetilde{\psi}_5$. For the Poisson tensor $\widetilde{P}_3$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 3, 6\}$. For the two conditions

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \lambda \frac{\partial A_1}{\partial \lambda} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \lambda \frac{\partial A_2}{\partial \lambda} = 2\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} U_1 & V_2 \\ 1 & -U_1 \end{pmatrix}, \quad (4.6)$$

we find that U_2 in $\widetilde{X}_\lambda$ has to include a parameter α, and U_1 and W_1 have to include a parameter α'. For this purpose, we take the symplectic leaf

$$S = \{\widetilde{\psi}_1 = 4\alpha_1, \ \widetilde{\psi}_2 = 2\alpha_2 + 6\alpha_1^2, \ \widetilde{\psi}_3 = \alpha_3 + 4\alpha_1(\alpha_2 + \alpha_1^2), \ \widetilde{\psi}_6 = \alpha_6\}.$$

Further, we change the coordinate as $W_1 = \widetilde{W}_1 + \alpha_1$. Then, the above relations for S yield

$$U_1 = 2\alpha_1, \ U_2 = \alpha_2 + \alpha_1^2 - \frac{1}{2}V_2, \ V_3 = \alpha_3 - 2U_3 - V_2\widetilde{W}_1 + \alpha_1 V_2,$$

$$V_4 = (\alpha_6 - U_3^2)/W_2.$$

Substituting them into $\widetilde{X}_\lambda$, A_1 and A_2, we can verify the condition (4.6) with $\alpha = \alpha_2$ and $\alpha' = \alpha_1$. By replacing α_2, α_1 by t_1, t_2, respectively, we obtain the isomonodromic deformation equations (4.5), which are equations of $(V_2, \widetilde{W}_1, U_3, W_2)$ with two independent variables t_1, t_2. The Poisson tensor $\widetilde{P}_3$ on the symplectic leaf expressed in the coordinates $(V_2, \widetilde{W}_1, U_3, W_2)$ is given by

$$\widetilde{P}_3 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_2 \\ 0 & 0 & -W_2 & 0 \end{pmatrix}.$$

To change to Darboux's coordinates, put

$$(V_2, \widetilde{W}_1, U_3, W_2) = (2p_1, q_1, q_2 p_2 - \beta_3, p_2),$$

where β_3 is an arbitrary parameter. In the new coordinates, we obtain

$$\widetilde{P_3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.7}$$

Therefore, the two isomonodromic deformation equations (4.5) are Hamiltonian systems in this coordinate system. The Hamiltonian functions are obtained by deleting (U_1, U_2, V_3, V_4) from $\widetilde{\psi}_4$ and $\widetilde{\psi}_5$ and by changing to the coordinates (p_1, q_1, p_2, q_2). It is easy to verify that if we set $\beta_3^2 = \alpha_6$, then two functions become polynomials, which give Hamiltonian functions (1.13) of $(\mathrm{P_{IV}})_2$ given in Sec.1.

(iii) Consider the pair of vector fields $\widetilde{P}_2 d\widetilde{\psi}_3$ and $\widetilde{P}_2 d\widetilde{\psi}_4$. For the Poisson tensor $\widetilde{P}_2$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 5, 6\}$. In this case, we cannot find an integer l and a parameter α' such that the condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \lambda^l \frac{\partial A_2}{\partial \lambda}$$

holds. Hence, we impose only one condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \lambda^2 \frac{\partial A_1}{\partial \lambda} = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.8}$$

For it, U_1 in $\widetilde{X}_\lambda$ has to include a parameter α. To this end, take the symplectic leaf

$$S = \{\widetilde{\psi}_1 = 2\alpha_1, \ \widetilde{\psi}_2 = 2\alpha_2 + \alpha_1^2, \ \widetilde{\psi}_5 = \alpha_5, \ \widetilde{\psi}_6 = \alpha_6\}.$$

This is rearranged as

$$U_1 = \alpha_1, \ V_2 = 2\alpha_2 - 2U_2, \ V_3 = (\alpha_5 - 2U_2 U_3 - V_4 W_1)/W_2,$$
$$V_4 = (\alpha_6 - U_3^2)/W_2.$$

Substituting them into $\widetilde{X}_\lambda$ and A_1, we can verify the condition (4.8) with $\alpha = \alpha_1$. By replacing α_1 by t, we obtain the isomonodromic deformation equation

$$\frac{\partial L_\lambda}{\partial t} = [A_1, L_\lambda] + \frac{\partial A_1}{\partial \lambda}.$$

The Poisson tensor $\widetilde{P}_2$ on the symplectic leaf with coordinates (U_2, W_1, U_3, W_2) is given by

$$\widetilde{P}_2 = \begin{pmatrix} 0 & W_1 & 0 & W_2 \\ -W_1 & 0 & -W_2 & 0 \\ 0 & W_2 & 0 & 0 \\ -W_2 & 0 & 0 & 0 \end{pmatrix}.$$

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To change to Darboux's coordinates, put

$$(U_2, W_1, U_3, W_2) = (p_1q_1 + p_2q_2 - \beta_2,\ q_1,\ p_1q_2 - \beta_3,\ q_2),$$

where β_2 and β_3 are arbitrary parameters. In the new coordinates (q_1, p_1, q_2, p_2), $\widetilde{P}_2$ is reduced to the same form as (4.7). The isomonodromic deformation equation is a Hamiltonian system whose Hamiltonian function is $\widetilde{\psi}_3$ written in this coordinate system. It is easy to verify that if we set $\alpha_6 = \beta_3^2$ and $\alpha_5 = 2\beta_2\beta_3$, then $\widetilde{\psi}_3$ written in the coordinates (q_1, p_1, q_2, p_2) becomes a polynomial. In this manner, the Hamiltonian function (1.14) given in Sec.1 is obtained.

4.2 Case (II)

In this case, the functions $\psi_{i,j} = \psi_j$ are given by

$$\begin{cases}
\psi_1 = v_1 \\
\psi_2 = u_1^2 + v_2 + v_1 w_1 \\
\psi_3 = 2u_1 u_2 + v_3 + v_2 w_1 + v_1 w_2 \\
\psi_4 = u_2^2 + 2u_1 u_3 + v_3 w_1 + v_2 w_2 + v_1 w_3 \\
\psi_5 = 2u_2 u_3 + v_3 w_2 + v_2 w_3 \\
\psi_6 = u_3^2 + v_3 w_3.
\end{cases}$$

The differential equation (2.8) defining the distribution D is $u_j' = -v_j$, $v_j' = 0$, $w_j' = 2u_j$ for $j = 1, 2, 3$. We can assume without loss of generality that $v_1 = 1$ by a suitable scaling of variables. These equations are solved with respect to u_1 as

$$v_1 = 1,\ w_1 = -u_1^2 + W_2,\ u_2 = V_2 u_1 + U_3,\ v_2 = V_2, w_2 = -V_2 u_1^2 - 2U_3 u_1 + W_4,$$
$$u_3 = V_4 u_1 + U_5,\ v_3 = V_4,\ w_3 = -V_4 u_1^2 - 2U_5 u_1 + W_6,$$

where $W_2, U_3, V_2, W_4, U_5, V_4, W_6$ are integral constants (initial values at $u_1 = 0$). This relation defines a coordinate transformation

$$(u_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3) \mapsto (u_1, W_2, U_3, V_2, W_4, U_5, V_4, W_6).$$

In the new coordinates, integral manifolds of the distribution D are straight lines along u_1-axis. In particular, the subset $\{u_1 = 0\} \subset \mathfrak{g}_n^0$ gives the realization of the orbit space $\mathfrak{g}_n^0/D$ as a submanifold and $(W_2, U_3, V_2, W_4, U_5, V_4, W_6)$ provides a global coordinate system of $\mathfrak{g}_n^0/D$ restricted to $v_1 = 1$.

On $\mathfrak{g}_n^0/D$, we have functions

$$\begin{cases}
\widetilde{\psi}_1 = 1 \\
\widetilde{\psi}_2 = V_2 + W_2 \\
\widetilde{\psi}_3 = V_4 + W_4 + V_2 W_2 \\
\widetilde{\psi}_4 = U_3^2 + W_6 + V_2 W_4 + V_4 W_2 \\
\widetilde{\psi}_5 = 2U_3 U_5 + V_2 W_6 + V_4 W_4 \\
\widetilde{\psi}_6 = U_5^2 + V_4 W_6,
\end{cases}$$

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and two vector fields (4.1), (4.2) expressed in Lax form (4.3) with

$$
\begin{cases}
\widetilde{X}_\lambda = \lambda^3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} + \lambda \begin{pmatrix} U_3 & V_2 \\ W_4 & -U_3 \end{pmatrix} + \begin{pmatrix} U_5 & V_4 \\ W_6 & -U_5 \end{pmatrix}, \\
A_1 = \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ V_2 & 0 \end{pmatrix}, \\
A_2 = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} + \begin{pmatrix} U_3 & V_2 \\ W_4 & -U_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ V_4 & 0 \end{pmatrix}.
\end{cases}
$$

The next purpose is to restrict the vector fields on a symplectic leaf.

(i) Consider the pair of vector fields $\widetilde{P}_4 d\widetilde{\psi}_5$ and $\widetilde{P}_4 d\widetilde{\psi}_6$. For the Poisson tensor $\widetilde{P}_4$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 3, 4\}$. In order for the two conditions

$$
\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \frac{\partial A_1}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \frac{\partial A_2}{\partial \lambda} = 2\lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix} \tag{4.9}
$$

to be satisfied, we find that W_6 in $\widetilde{X}_\lambda$ has to include a parameter α, which will be replaced by t_1 later, and W_4 and V_4 have to include a parameter α', which will be replaced by t_2 later. For this purpose, we take the symplectic leaf

$$
S = \{\widetilde{\psi}_1 = 1, \ \widetilde{\psi}_2 = 0, \ \widetilde{\psi}_3 = 3\alpha_4, \ \widetilde{\psi}_4 = \alpha_6\}.
$$

Further, we change the coordinate as $W_4 = \widetilde{W}_4 + 2\alpha_4$. Then, the above relation for S is rearranged as

$$
V_2 = -W_2, \ V_4 = \alpha_4 - \widetilde{W}_4 + W_2^2, \ W_6 = \alpha_6 - U_3^2 + 2W_2\widetilde{W}_4 - W_2^3 + \alpha_4 W_2.
$$

Substituting them into $\widetilde{X}_\lambda$, A_1 and A_2, we can verify the condition (4.9) with $\alpha = \alpha_6$ and $\alpha' = \alpha_4$. By replacing α_6, α_4 by t_1, t_2, respectively, we obtain the isomonodromic deformation equations (4.5), which are equations of $(W_2, U_3, \widetilde{W}_4, U_5)$ with two independent variables t_1, t_2. The Poisson tensor $\widetilde{P}_4$ on the symplectic leaf written in this coordinate system is given by

$$
\widetilde{P}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & W_2 \\ -1 & 0 & -W_2 & 0 \end{pmatrix}.
$$

To change to Darboux's coordinates, put

$$
(W_2, U_3, \widetilde{W}_4, U_5) = (q_1, p_2, q_2, p_1 + p_2 q_1).
$$

Then, $\widetilde{P}_4$ is transformed to the canonical symplectic matrix. In the coordinates (q_1, p_1, q_2, p_2), the above two isomonodromic deformation equations are Hamiltonian

systems. The Hamiltonian functions are obtained by deleting V_2, V_4, W_6 from $\widetilde{\psi}_5$ and $\widetilde{\psi}_6$ by using the above relations, and changing to Darboux's coordinates. In this manner, we obtain Hamiltonian functions (1.10) of $(\mathrm{P_I})_2$ given in Sec.1.

(ii) Consider the pair of vector fields $\widetilde{P}_3 d\widetilde{\psi}_4$ and $\widetilde{P}_3 d\widetilde{\psi}_5$. For the Poisson tensor $\widetilde{P}_3$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 3, 6\}$. For the two conditions

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \lambda \frac{\partial A_1}{\partial \lambda} = \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \lambda \frac{\partial A_2}{\partial \lambda} = 2\lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ W_2 & 0 \end{pmatrix}, \quad (4.10)$$

we find that W_4 in $\widetilde{X}_\lambda$ has to include a parameter α, and W_2 and V_2 have to include a parameter α'. For this purpose, we take the symplectic leaf

$$S = \{\widetilde{\psi}_1 = 1, \ \widetilde{\psi}_2 = 3\alpha_2, \ \widetilde{\psi}_3 = \alpha_4 + 3\alpha_2^2, \ \widetilde{\psi}_6 = \alpha_{10}\}.$$

Further, we change the coordinate as $V_2 = \widetilde{V}_2 + \alpha_2$. Then, the above relation for S is rearranged as

$$W_2 = 2\alpha_2 - \widetilde{V}_2, \ W_4 = \alpha_4 + \alpha_2^2 - \alpha_2 \widetilde{V}_2 - V_4 + V_2^2, \ W_6 = (\alpha_{10} - U_5^2)/V_4.$$

Substituting them into $\widetilde{X}_\lambda$, A_1 and A_2, we can verify the condition (4.10) with $\alpha = \alpha_4$ and $\alpha' = \alpha_2$. By replacing α_4, α_2 by t_1, t_2, respectively, we obtain the isomonodromic deformation equations (4.5). The Poisson tensor $\widetilde{P}_3$ on the symplectic leaf written in the coordinates $(\widetilde{V}_2, U_3, V_4, U_5)$ is given by

$$\widetilde{P}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -V_4 \\ 0 & 0 & V_4 & 0 \end{pmatrix}.$$

To change to Darboux's coordinates, put

$$(\widetilde{V}_2, U_3, V_4, U_5) = (p_2, q_2, p_1, q_1 p_1 - \alpha_5),$$

where α_5 is an arbitrary parameter. Then, $\widetilde{P}_3$ is transformed to the canonical symplectic matrix. In the coordinates (q_1, p_1, q_2, p_2), the above two isomonodromic deformation equations are Hamiltonian systems. The Hamiltonian functions are obtained by deleting W_2, W_4, W_6 from $\widetilde{\psi}_4$ and $\widetilde{\psi}_5$ by using the above relations, and by changing to Darboux's coordinates. It is easy to verify that if we set $\alpha_{10} = \alpha_5^2$, then two functions become polynomials. In this manner, we obtain Hamiltonian functions (1.11) of $(\mathrm{P_{II\text{-}1}})_2$ given in Sec.1.

(iii) Consider the vector fields $\widetilde{P}_2 d\widetilde{\psi}_3$ and $\widetilde{P}_2 d\widetilde{\psi}_4$. For the Poisson tensor $\widetilde{P}_2$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_j = \text{constant}, \ j = 1, 2, 5, 6\}$. In this case, we cannot find an integer l and a parameter α' such that the condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha'} = \lambda^l \frac{\partial A_2}{\partial \lambda}$$

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holds. Hence, we impose only one condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \lambda^2 \frac{\partial A_1}{\partial \lambda} = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.11}$$

For it, W_2 in $\widetilde{X}_\lambda$ has to include a parameter α. To this end, take the symplectic leaf

$$S = \{\widetilde{\psi}_1 = 1, \ \widetilde{\psi}_2 = \alpha_2, \ \widetilde{\psi}_5 = \alpha_8, \ \widetilde{\psi}_6 = \alpha_{10}\}.$$

This is rearranged as

$$W_2 = \alpha_2 - V_2, \ W_4 = (\alpha_8 - 2U_3 U_5 - V_2 W_6)/V_4, \ W_6 = (\alpha_{10} - U_5^2)/V_4.$$

Substituting them into $\widetilde{X}_\lambda$ and A_1, we can verify the condition (4.11) with $\alpha = \alpha_2$. By replacing α_2 by t, we obtain the isomonodromic deformation equation

$$\frac{\partial L_\lambda}{\partial t} = [A_1, L_\lambda] + \frac{\partial A_1}{\partial \lambda}.$$

The Poisson tensor $\widetilde{P}_2$ on the symplectic leaf with coordinates (U_3, V_2, U_5, V_4) is given by

$$\widetilde{P}_2 = \begin{pmatrix} 0 & -V_2 & 0 & -V_4 \\ V_2 & 0 & V_4 & 0 \\ 0 & -V_4 & 0 & 0 \\ V_4 & 0 & 0 & 0 \end{pmatrix}.$$

To change to Darboux's coordinates, put

$$(U_3, V_2, U_5, V_4) = (p_1 q_1 + p_2 q_2 - \beta_3, \ p_2, \ p_1 q_2 - \beta_5, \ p_1),$$

where β_3 and β_5 are arbitrary parameters. Then, $\widetilde{P}_2$ is transformed to the canonical symplectic matrix. In the coordinates (q_1, p_1, q_2, p_2), the above isomonodromic deformation equation is a Hamiltonian system. The Hamiltonian function is obtained by deleting W_2, W_4, W_6 from $\widetilde{\psi}_3$ by using the above relations, and by changing to Darboux's coordinates. It is easy to verify that if we set $\alpha_{10} = \beta_5^2$ and $\alpha_8 = 2\beta_3\beta_5$, then $\widetilde{\psi}_3$ written in the coordinates (q_1, p_1, q_2, p_2) becomes a polynomial. This procedure yields the Hamiltonian function (1.15) given in Sec.1.

5   4-dim Painlevé equations: $\mathfrak{g} = \mathfrak{so}_5, \ n = 1$

According to [9], we use the following representation for the Lie algebra $\mathfrak{g} \simeq \mathfrak{so}_5$ of type B_2

$$X_i = \begin{pmatrix} p_i & q_i & r_i & s_i & 0 \\ t_i & u_i & v_i & 0 & s_i \\ w_i & x_i & 0 & v_i & -r_i \\ y_i & 0 & x_i & -u_i & q_i \\ 0 & y_i & -w_i & t_i & -p_i \end{pmatrix}.$$

Consider the Lie algebra $\mathfrak{g}_1 = \{X_\lambda = \lambda X_0 + X_1 \mid X_i \in \mathfrak{g} \simeq \mathfrak{so}_5\}$. For the definition of $\mathfrak{g}_1^0$, we only consider the following case

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

(i.e. $x_0 = t_0 = 1$ and zeros otherwise). The purpose in this section is to derive the Hamiltonian (1.16) for Cosgrove's equation. The other choice of X_0 may yield different Painlevé systems. Note that $n = 1, \operatorname{rank}(\mathfrak{g}) = h = 2, m_1 = 1$ and $m_2 = 3$. We have the following functions

$$\begin{cases} \psi_{1,1} = -2q_1 - 2v_1 \\ \psi_{1,2} = -2q_1t_1 - u_1^2 - 2v_1x_1 - 2s_1y_1 \\ \psi_{2,1} = -2s_1 \\ \psi_{2,2} = q_1^2 - 2s_1t_1 + 2q_1v_1 - 4s_1x_1 \\ \psi_{2,3} = 2q_1^2t_1 + 2q_1t_1v_1 + 2s_1u_1w_1 - 4s_1t_1x_1 + 2q_1v_1x_1 - 2s_1x_1^2 - 2q_1s_1y_1 + 2s_1v_1y_1 \\ \psi_{2,4} = q_1^2t_1^2 - 2q_1u_1v_1w_1 + 2q_1s_1w_1^2 + 2q_1t_1v_1x_1 + 2s_1u_1w_1x_1 \\ \qquad\quad -2s_1t_1x_1^2 - 2q_1s_1t_1y_1 - 2q_1v_1^2y_1 + 2s_1v_1x_1y_1 + s_1^2y_1^2, \end{cases}$$

which are coefficients of the characteristic polynomial $\det(\mu - X_\lambda)$.

We solve the differential equations for the two dimensional distribution D as functions of (p_1, r_1) with the initial condition $(q_1, s_1, t_1, u_1, v_1, w_1, x_1, y_1) = (Q, S, T, U, V, W, X, Y)$ at $(p_1, r_1) = (0, 0)$. The expressions of solutions are too long and omitted here. These solutions define a coordinate transformation

$$(p_1, r_1, q_1, s_1, t_1, u_1, v_1, w_1, x_1, y_1) \mapsto (p_1, r_1, Q, S, T, U, V, W, X, Y).$$

In the new coordinates, integral manifolds of the distribution D are plains which are parallel to the (p_1, r_1)-plain. In particular, the subset $\{p_1 = r_1 = 0\} \subset \mathfrak{g}_1^0$ gives the realization of the orbit space $\mathfrak{g}_1^0/D$ as a submanifold and (Q, S, T, U, V, W, X, Y) provides a global coordinate system of $\mathfrak{g}_1^0/D$.

At this stage, we have on $\mathfrak{g}_1^0/D$

$$\begin{cases} \widetilde{\psi}_{1,1} = -2Q - 2V \\ \widetilde{\psi}_{1,2} = -2QT - U^2 - 2VX - 2SY \\ \widetilde{\psi}_{2,1} = -2S \\ \widetilde{\psi}_{2,2} = Q^2 - 2ST + 2QV - 4SX \\ \widetilde{\psi}_{2,3} = 2Q^2T + 2QTV + 2SUW - 4STX + 2QVX - 2SX^2 - 2QSY + 2SVY, \end{cases}$$

and the vector field $\widetilde{P}_2 d\widetilde{\psi}_{2,3}$, whose Casimir functions are $\widetilde{\psi}_{1,1}, \widetilde{\psi}_{1,2}, \widetilde{\psi}_{2,1}$ and $\widetilde{\psi}_{2,2}$. The corresponding differential equation is expressed in Lax form as $d\widetilde{X}_\lambda/dt = [A_\lambda, \widetilde{X}_\lambda]$,

where

$$X_\lambda = \lambda \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & Q & 0 & S & 0 \\ T & U & V & 0 & S \\ W & X & 0 & V & 0 \\ Y & 0 & X & -U & Q \\ 0 & Y & -W & T & 0 \end{pmatrix},$$

$$A_\lambda = \lambda \nabla_1 \widetilde{\psi}_{2,1} + \nabla_1 \widetilde{\psi}_{2,2} + (V - Q)\nabla_1 \widetilde{\psi}_{1,1} + \frac{2}{S}(Q^2 - SX)\nabla_1 \widetilde{\psi}_{2,1},$$

$$\nabla_1 \widetilde{\psi}_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}, \quad \nabla_1 \widetilde{\psi}_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix},$$

$$\nabla_1 \widetilde{\psi}_{2,2} = \begin{pmatrix} 0 & -2S & 0 & 0 & 0 \\ 2(Q+V) & 0 & -4S & 0 & 0 \\ -2U & 2Q & 0 & -4S & 0 \\ -2(T+2X) & 0 & 2Q & 0 & -2S \\ 0 & -2(T+2X) & 2U & 2(Q+V) & 0 \end{pmatrix}.$$

The next purpose is to restrict the vector field on a symplectic leaf. For the Poisson tensor $\widetilde{P}_2$, a symplectic leaf is defined by the level surface $\{\widetilde{\psi}_{i,j} = \text{constant}, \ i,j = 1,2\}$. In order for the condition

$$\frac{\partial \widetilde{X}_\lambda}{\partial \alpha} = \frac{\partial A_\lambda}{\partial \lambda} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

to be satisfied, we find that Y in $\widetilde{X}_\lambda$ has to include a parameter α which will be replaced by t later. For this purpose, we take the symplectic leaf

$$S = \{\widetilde{\psi}_{2,1} = -2, \ \widetilde{\psi}_{1,1} = -2\alpha_2, \ \widetilde{\psi}_{2,2} = -2\alpha_4 + \alpha_2^2, \ \widetilde{\psi}_{1,2} = 4\alpha_6 - 2\alpha_2\alpha_4\}.$$

This is rewritten as

$$S = 1, \ Q = \alpha_2 - V, \ T = \alpha_4 - 2X - \frac{1}{2}V^2,$$

$$Y = -2\alpha_6 - \frac{1}{2}U^2 - \frac{1}{2}V^3 - 3VX + \frac{\alpha_2}{2}V^2 + 2\alpha_2 X + \alpha_4 V.$$

Substituting them into $\widetilde{X}_\lambda$ and A_λ, it turns out that the condition (5.1) is satisfied with $\alpha = \alpha_6$. Finally, by replacing α_6 by t, we obtain the isomonodromic deformation equation (2.11). The Poisson tensor $\widetilde{P}_2$ written with respect to the

coordinates (U, V, W, X) is already in the canonical symplectic matrix. Thus, the isomonodromic deformation equation is a Hamiltonian system with the Hamiltonian function $\widetilde{\psi}_{2,3}$ written in the coordinate system (U, V, W, X). Since this expression is too complicated, we further introduce the symplectic transformation

$$
\begin{aligned}
&(U, V, W, X) \\
= \ &(p_2, \ q_1 + \frac{13}{18}\alpha_2, \ p_1 + \frac{4}{13}p_2(q_1 + \frac{13}{18}\alpha_2), \ q_2 + \frac{1}{3}\alpha_4 + \frac{7}{108}\alpha_2^2 - \frac{2}{13}(q_1 + \frac{13}{18}\alpha_2)^2).
\end{aligned}
$$

Then, the Hamiltonian function (1.18), which is equivalent to (1.16), is obtained.

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