A Hopf bifurcation in the Kuramoto-Daido model

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Abstract

A Hopf bifurcation in the Kuramoto-Daido model is investigated based on the generalized spectral theory and the center manifold reduction for a certain class of frequency densities. The dynamical system of the order parameter on a four-dimensional center manifold is derived. It is shown that the dynamical system undergoes a Hopf bifurcation as the coupling strength increases, which proves the existence of a periodic two-cluster state of oscillators.

1 Introduction

Collective synchronization phenomena are observed in a variety of areas such as chemical reactions, engineering circuits and biological populations [13]. In order to investigate such phenomena, a system of globally coupled phase oscillators called the Kuramoto-Daido model [7]

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i),\ \ i = 1, \cdots, N, \tag{1.1}$$

is often used, where $\theta_i = \theta_i(t) \in [0, 2\pi)$ is a dependent variable which denotes the phase of an i-th oscillator on a circle, $\omega_i \in \mathbb{R}$ denotes its natural frequency drawn from some density function $g(\omega)$, $K > 0$ is a coupling strength, and where $f(\theta)$ is a 2π-periodic function. The complex order parameter defined by

$$re^{i\psi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \quad i = \sqrt{-1} \tag{1.2}$$

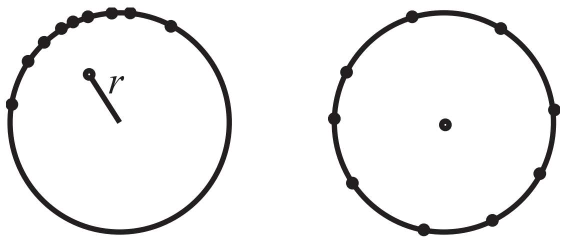

Figure 1: Collective behavior of oscillators. The points represent the quantities $e^{i\theta_j}$ in the complex plane. r is the magnitude of the average of them.

is used to measure the amount of coherence in the system; if r is nearly equal to zero, oscillators are uniformly distributed (called the incoherent state), while if $r > 0$, the synchronization occurs, see Fig. 1.

In this paper, the continuous limit (thermodynamics limit) of the following model

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\big(\sin(\theta_j - \theta_i) + h\cdot\sin 2(\theta_j - \theta_i)\big), \tag{1.3}$$

will be considered, where h is a parameter which controls the strength of the second harmonic. For the continuous limit of the system, a Hopf bifurcation from the incoherent state to the two-cluster periodic state will be investigated based on the generalized spectral theory.

It is known that when the frequency density $g(\omega)$ is an even and unimodal function, the transition from the incoherent state to the partially synchronized state occurs at the critical coupling strength $K = K_c = 2/(\pi g(0))$. In Chiba [2, 3], this result is proved based on the generalized spectral theory [4] under the assumption that $g(\omega)$ has an analytic continuation near the real axis. With the aid of the generalized spectral theory, it is proved that the order parameter is locally governed by the dynamical system on a center manifold as

$$\frac{dr}{dt} = \text{const.}\left(K - K_c + \frac{\pi g''(0)K_c^4}{16}r^2\right) r + O(r^4),$$

for $h = 0$, and

$$\frac{dr}{dt} = \text{const.}\left(K - K_c - \frac{K_c^2 Ch}{1-h}r\right) r + O(r^3),$$

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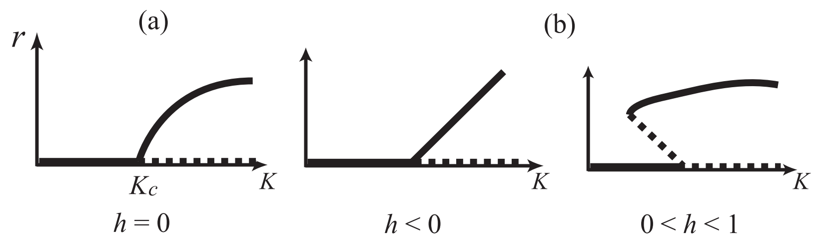

Figure 2: Bifurcation diagrams of the order parameter for (a) $f(\theta) = \sin\theta$ and (b) $f(\theta) = \sin\theta + h\sin 2\theta$. The solid lines denote stable solutions, and the dotted lines denote unstable solutions.

for $h \neq 0$, where C is a certain negative constant. As a result, a bifurcation diagram of r is given as Fig.2. When $h = 0$, the synchronous state emerges through a pitchfork bifurcation, though when $h \neq 0$, it is a transcritical bifurcation. Our method based on the generalized spectral theory is applicable to any coupling function $f(\theta)$ (cf. Eq.(1.1)), however, a bifurcation scenario strongly depends on a balance of its Fourier coefficients. Thus, we consider only the coupling function of the form (1.3) for simplicity.

The purpose in this paper is to investigate a Hopf bifurcation of the system (1.3) under certain assumptions for the density function $g(\omega)$. In particular, the dynamics of the order parameter on a center manifold will be derived. For this purpose, we need five assumptions (A1) to (A5) given after Section 3. Here, we give a rough explanation of these assumptions.

(A1) We assume that $h < 1$ so that $\sin\theta$ is a dominant term in the coupling function.
(A2) We assume that the density $g(\omega)$ of natural frequencies is an analytic function near the real axis. This is the essential assumption to apply the generalized spectral theory.
(A3) We assume that at a bifurcation value $K = K_c$, a pair of generalized eigenvalues of a certain linear operator obtained by the linearization of the system are at the points $\pm iy_c$ on the imaginary axis. Furthermore, such a pair is unique and they are simple eigenvalues.
(A4) We assume that as K increases, the pair of generalized eigenvalues transversally crosses the imaginary axis at the point $\pm iy_c$ from the left to the right.
(A5) Assume that $g(\omega)$ is an even function.

It seems that (A3) and (A4) are satisfied for a wide class of even and *bimodal* densities $g(\omega)$ as long as the distance of two peaks are sufficiently far apart, though we do not assume explicitly that $g(\omega)$ is bimodal. The main results in the present paper are stated as follows;

Theorem 1.1 (Instability of the incoherent state).
Suppose (A1) and $g(\omega)$ is continuous. There exists a number $\varepsilon > 0$ such that when $K_c < K < K_c + \varepsilon$, the incoherent state is linearly unstable, where

$$K_c = \frac{2}{\pi g(y_c)}$$

and y_c is a certain real number, see (A3) above.

Theorem 1.2 (Local stability of the incoherent state).
Suppose (A1) and (A2). When $0 < K < K_c$, the incoherent state is linearly asymptotically stable in the weak sense (see Section 4 for the weak stability).

Theorem 1.3 (Bifurcation).
Suppose (A1) to (A5). There exists a positive constant ε_0 such that if $K_c - \varepsilon_0 < K < K_c + \varepsilon_0$ and if an initial condition is close to the incoherent state, the dynamics of the order parameter is locally governed by a certain four dimensional dynamical system on the center manifold given in Section 6. At $K = K_c$ the system undergoes a Hopf bifurcation and when $K_c < K < K_c + \varepsilon_0$, $h \leq 0$ and $\mathrm{Re}(p_2) < 0$ (see below), the system has a family of asymptotically stable periodic orbits.

(i) Suppose $h = 0$. On the family of stable periodic orbits, the complex order parameter η_1 defined in Section 2 is given by

$$\eta_1(t) = 2\sqrt{\frac{-\mathrm{Re}(p_1)}{\mathrm{Re}(p_2)}}\sqrt{K - K_c}\, e^{i\beta} \cos(y_c t + O(K - K_c)) + O(K - K_c),$$

where p_1 and p_2 are certain complex constants explicitly given in Section 6, and $\beta \in \mathbb{R}$ is an arbitrary constant specified by an initial condition. (The assumption (A4) implies $\mathrm{Re}(p_1) > 0$; i.e. $-\mathrm{Re}(p_1)/\mathrm{Re}(p_2) > 0$.)

(ii) Suppose $h < 0$. On the family of stable periodic orbits, the complex order parameter η_1 is given by

$$\eta_1(t) = -2\frac{1-h}{hK_c}\mathrm{Re}(p_1)(K - K_c)\, e^{i\beta} \cos(y_c t + O(K - K_c)) + O((K - K_c)^2),$$

where p_1 is the same constant as (i).

The constants p_1 and p_2 are determined only by the frequency density $g(\omega)$. The condition $\mathrm{Re}(p_2) < 0$ seems to be satisfied for most even and bimodal densities. If $\mathrm{Re}(p_2) > 0$, a family of unstable periodic orbits exists when $K_c - \varepsilon_0 < K < K_c$; that is, a bifurcation occurs in the subcritical regime, while the expression for η_1 is the same as above. Similarly, if $0 <$

$h < 1$, a bifurcation is subcritical and a family of unstable periodic orbits exists for $K_c - \varepsilon_0 < K < K_c$, see the right figure of Fig.2. This case looks important because a hysteresis seems to occur.

In Martens et al.[9], the following bimodal frequency density defined as the sum of two Lorentzian density

$$g(\omega) = \frac{1}{2\pi}\left(\frac{1}{(\omega-\omega_0)^2+1} + \frac{1}{(\omega+\omega_0)^2+1}\right), \tag{1.4}$$

is considered. They revealed the dynamics of the order parameter in detail by using the Ott-Antonsen ansatz [12], though the ansatz is applicable only when $h = 0$. The present paper extends their result for the case $h \neq 0$. For this bimodal density, we can verify that $K_c = 4$, $y_c = \sqrt{\omega_0^2 - 1}$, $\mathrm{Re}(p_1) = 1/4$ and $\mathrm{Re}(p_2) = -4$. Hence, there exists a family of stable periodic solutions for both of $h = 0$ and $h < 0$, and unstable one for $h > 0$. See also Example 3.5 and Example 5.3.

Our physical motivation for considering a bimodal $g(\omega)$ is as follows. Recently, a study of multiagent systems that are formed by several groups consisting of a large number of interacting elements becomes more and more important. Such systems often show collective phenomena such as synchronization and clustering. Typical examples of multiagent systems include social sciences and biological systems. For instance, a system of interacting two populations of fireflies, where each population inhabits a separate tree, consists of two subgroups of oscillators. It has been shown experimentally that synchronization arises between different neighboring visual cortex columns.

In [1], [11] and [10], they considered the Kuramoto-type model with a bimodal (or multimodal) frequency density for the model of such systems. In particular, Mikhailov et al.[10] experimentally realized such a system by using the electrochemical oscillators.

Let us consider two populations of coupled oscillators governed by the Kuramoto model. Let $\hat{g}(\omega)$ be an even and unimodal density function. We suppose that a natural frequency of one population is drawn from the function $\hat{g}(\omega - \Omega_1)$, and a natural frequency of the other population is drawn from $\hat{g}(\omega - \Omega_2)$, where Ω_1 and Ω_2 are constants. For such a system, it is natural to consider the Kuramoto model with the density function of the form

$$g(\omega) := \frac{1}{2}(\hat{g}(\omega - \Omega_1) + \hat{g}(\omega - \Omega_2)),$$

whose average is given by $(\Omega_1 + \Omega_2)/2$. This is a bimodal function if $|\Omega_1 - \Omega_2|$ is sufficiently large. Because of the rotation symmetry, a change of variables $\theta_i \mapsto \theta_i + (\Omega_1 + \Omega_2)t/2$ gives the same model, whose frequency density is an even and bimodal function. For such a system, our main result implies that under suitable assumptions shown in Theorem 1.3, there exists a stable periodic two-cluster state. The velocities of them are approximately given by $y_c + (\Omega_1 + \Omega_2)/2$ and $-y_c + (\Omega_1 + \Omega_2)/2$, respectively.

2 The continuous model

For the finite dimensional Kuramoto-Daido model (1.1), the l-th order parameter is defined by

$$\hat{\eta}_l(t) := \frac{1}{N}\sum_{j=1}^{N} e^{il\theta_j(t)}. \tag{2.1}$$

By using it, Eq.(1.1) is rewritten as

$$\frac{d\theta_j}{dt} = \omega_j + K\sum_{l=-\infty}^{\infty} f_l \hat{\eta}_l(t) e^{-il\theta_j}, \quad f(\theta) := \sum_{l=-\infty}^{\infty} f_l e^{il\theta}.$$

The continuous limit of this model is an evolution equation of a density

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} + \dfrac{\partial}{\partial \theta}(\rho_t v) = 0, \quad \rho_t = \rho_t(\theta, \omega), \\ v := \omega + K \displaystyle\sum_{l=-\infty}^{\infty} f_l \eta_l(t) e^{-il\theta}, \\ \eta_l(t) := \displaystyle\int_{\mathbb{R}}\int_0^{2\pi} e^{il\theta} \rho_t(\theta, \omega) g(\omega) d\theta d\omega. \end{cases} \tag{2.2}$$

Here, $g(\omega)$ is a given probability density function for natural frequencies, and the unknown function $\rho_t = \rho_t(\theta, \omega)$ is a probability measure on $[0, 2\pi)$ parameterized by $t, \omega \in \mathbb{R}$. $\eta_l(t)$ is a continuous analog of $\hat{\eta}_l(t)$ in (2.1). In particular, $\eta_1(t)$ is a continuous version of Kuramoto's order parameter (1.2). The trivial solution $\rho_t = 1/(2\pi)$ of the system is a uniform distribution on the circle, which is called the incoherent state (de-synchronous state). Our purpose is to investigate the stability and bifurcation of the incoherent state and the order parameter η_1.

Define the Fourier coefficients

$$Z_j(t, \omega) := \int_0^{2\pi} e^{ij\theta} \rho_t(\theta, \omega) d\theta.$$

Then, the continuous model is rewritten as a system of evolution equations of Z_j

$$\frac{dZ_j}{dt} = ij\omega Z_j + ijK f_j \eta_j + ijK \sum_{l \neq j} f_l \eta_l Z_{j-l}. \tag{2.3}$$

The trivial solution $Z_j \equiv 0\,(j = \pm 1, \pm 2, \cdots)$ corresponds to the incoherent state ($Z_0 \equiv 1$ because of the normalization $\int_0^{2\pi} \rho_t(\theta, \omega) d\theta = 1$). In what follows, we consider only the equations for $Z_1, Z_2, \cdots$ because Z_{-j} is the complex conjugate of Z_j.

3 The transition point formula and linear instability

To investigate the stability of the incoherent state, we consider the linearized system. Let $L^2(\mathbb{R}, g(\omega)d\omega)$ be the weighted Lebesgue space with the inner product

$$(\phi, \psi) = \int_{\mathbb{R}} \phi(\omega)\overline{\psi(\omega)} g(\omega) d\omega.$$

We define a one-dimensional integral operator $\mathcal{P}$ on $L^2(\mathbb{R}, g(\omega)d\omega)$ by

$$(\mathcal{P}\phi)(\omega) = \int_{\mathbb{R}} \phi(\omega) g(\omega) d\omega = (\phi, P_0) \cdot P_0(\omega), \tag{3.1}$$

where $P_0(\omega) \equiv 1 \in L^2(\mathbb{R}, g(\omega)d\omega)$ is a constant function. Then, the order parameters are written by

$$\eta_j(t) = \int_{\mathbb{R}} Z_j(t, \omega) g(\omega) d\omega = (Z_j, P_0) \cdot P_0(\omega) = \mathcal{P} Z_j. \tag{3.2}$$

Hence, Eq.(2.3) is expressed as

$$\frac{dZ_j}{dt} = (ij\omega + ijKf_j\mathcal{P})Z_j + ijK \sum_{l \neq j} f_l(\mathcal{P}Z_l) Z_{j-l}. \tag{3.3}$$

The linearized system around the incoherent state is given by

$$\frac{dZ_j}{dt} = T_j Z_j := (ij\omega + ijKf_j\mathcal{P})Z_j, \quad j = 1, 2, \cdots \tag{3.4}$$

where $T_j = ij\omega + ijKf_j\mathcal{P}$ is a linear operator on $L^2(\mathbb{R}, g(\omega)d\omega)$. Let us consider the spectra of T_j. The multiplication operator $\phi(\omega) \mapsto \omega\phi(\omega)$ on $L^2(\mathbb{R}, g(\omega)d\omega)$ is self-adjoint. The spectrum of it consists only of the continuous spectrum given by $\sigma_c(\omega) = \mathrm{supp}(g)$ (the support of g). Therefore, the spectrum of the multiplication by $ij\omega$ lies on the imaginary axis; $\sigma_c(ij\omega) = ij \cdot \mathrm{supp}(g)$ (later we will suppose that g is analytic, so that $\sigma_c(ij\omega)$

is the whole imaginary axis). Since $\mathcal{P}$ is compact, it follows from the perturbation theory of linear operators [8] that the continuous spectrum of T_j is given by $\sigma_c(T_j) = ij \cdot \mathrm{supp}(g)$, and the residual spectrum of T_j is empty.

When $f_j \neq 0$, eigenvalues λ of T_j are given as roots of the equation

$$\int_{\mathbb{R}} \frac{1}{\lambda - ij\omega} g(\omega) d\omega = \frac{1}{ijKf_j}. \tag{3.5}$$

Indeed, the equation $(\lambda - T_j)v = 0$ provides

$$v = ijKf_j(v, P_0)(\lambda - ij\omega)^{-1} P_0.$$

Taking the inner product with P_0, we obtain Eq.(3.5). If λ is an eigenvalue of T_j, the above equality shows that

$$v_\lambda(\omega) = \frac{1}{\lambda - ij\omega} \tag{3.6}$$

is the associated eigenfunction. This is not in $L^2(\mathbb{R}, g(\omega)d\omega)$ when λ is a purely imaginary number. Thus, there are no eigenvalues on the imaginary axis. Putting $\lambda = x + iy$ in Eq.(3.5) provides

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \frac{x}{x^2 + (y - j\omega)^2} g(\omega) d\omega = \frac{-\mathrm{Im}(f_j)}{jK|f_j|^2}, \\ \displaystyle\int_{\mathbb{R}} \frac{y - j\omega}{x^2 + (y - j\omega)^2} g(\omega) d\omega = \frac{\mathrm{Re}(f_j)}{jK|f_j|^2}, \end{cases}$$

which determines eigenvalues of T_j. In what follows, we restrict our problem to the model (1.3), for which the coupling function is given by $f(\theta) = \sin\theta + h\sin 2\theta$. In this case, we have $f_1 = 1/(2i)$, $f_2 = h/(2i)$ and $f_j = 0$ for $j \neq 1, 2$. The spectrum of the operator T_j for $j \neq 1, 2$ consists only of the continuous spectrum on the imaginary axis. T_1 and T_2 also have the continuous spectra on the imaginary axis. Further, they have eigenvalues determined by the equations

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \frac{x}{x^2 + (y - \omega)^2} g(\omega) d\omega = \frac{2}{K}, \\ \displaystyle\int_{\mathbb{R}} \frac{y - \omega}{x^2 + (y - \omega)^2} g(\omega) d\omega = 0, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \frac{x}{x^2 + (y - 2\omega)^2} g(\omega) d\omega = \frac{h}{K}, \\ \displaystyle\int_{\mathbb{R}} \frac{y - 2\omega}{x^2 + (y - 2\omega)^2} g(\omega) d\omega = 0, \end{cases} \tag{3.8}$$

respectively. Eq.(3.5) for $j = 1$ is given by

$$D(\lambda) := \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g(\omega) d\omega = \frac{2}{K}. \tag{3.9}$$

The next lemma follows from formulae of the Poisson integral and the Hilbert transform. See [14] for the proof.

Lemma 3.1. Suppose $g(\omega)$ is continuous. Then, the equality

$$\begin{aligned} \lim_{\lambda \to +0+iy} D^{(n)}(\lambda) &= (-1)^n n! \cdot \lim_{\lambda \to +0+iy} \int_{\mathbb{R}} \frac{1}{(\lambda - i\omega)^{n+1}} g(\omega) d\omega \\ &= \frac{1}{i^n} \cdot \lim_{\lambda \to +0+iy} \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g^{(n)}(\omega) d\omega \\ &= \frac{1}{i^n} \left(\pi g^{(n)}(y) - i\pi H[g^{(n)}](y) \right) \end{aligned}$$

holds for $n = 0, 1, 2, \cdots$, where $\lambda \to +0 + iy$ implies the limit to the point $iy \in i\mathbb{R}$ from the right half plane and $H[g]$ denotes the Hilbert transform defined by

$$\begin{aligned} H[g](y) &= \frac{-1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{1}{\omega} g(\omega + y) d\omega \\ &= \frac{-1}{\pi} \lim_{\varepsilon \to +0} \int_{\varepsilon}^{\infty} \frac{1}{\omega} \left(g(y + \omega) - g(y - \omega) \right) d\omega. \end{aligned}$$

Lemma 3.2. Suppose $K > 0$. Then,
(i) If an eigenvalue λ of T_1 exists, it satisfies $\text{Re}(\lambda) > 0$.
(ii) If $K > 0$ is sufficiently large, there exists at least one eigenvalue λ near infinity on the right half plane.
(iii) If $K > 0$ is sufficiently small, there are no eigenvalues of T_1.

See [3, 5] for the proof.

Eq.(3.7) combined with Lemma 3.1 yields

$$\begin{cases} \displaystyle\lim_{x \to +0} \int_{\mathbb{R}} \frac{x}{x^2 + (y - \omega)^2} g(\omega) d\omega = \pi g(y) = \frac{2}{K}, \\ \displaystyle\lim_{x \to +0} \int_{\mathbb{R}} \frac{y - \omega}{x^2 + (y - \omega)^2} g(\omega) d\omega = \pi H[g](y) = 0. \end{cases}$$

Let $y_1, y_2, \cdots$ be roots of the equation $H[g](y) = 0$, and put $K_j = 2/(\pi g(y_j))$. The pair (y_j, K_j) describes that some eigenvalue $\lambda = \lambda_j(K)$ of T_1 on the right half plane converges to the point iy_j on the imaginary axis as $K \to K_j+0$. Since $\text{Re}(\lambda) > 0$, the eigenvalue $\lambda_j(K)$ is absorbed into the continuous spectrum on the imaginary axis and disappears at $K = K_j$. Suppose that y_c satisfies $\sup_j\{g(y_j)\} = g(y_c)$ and put

$$K_c = \inf_j\{K_j\} = \frac{2}{\pi g(y_c)}. \tag{3.10}$$

In what follows, $\lambda_c(K)$ denotes the eigenvalue of T_1 satisfying $\lambda_c \to +0 + iy_c$ as $K \to K_c + 0$ (y_c and λ_c may not be unique). The following formulae will be used later.

Lemma 3.3. The equalities

$$\begin{aligned} D(iy_c) &:= \lim_{\lambda \to +0+iy_c} D(\lambda) = \frac{2}{K_c}, \\ \frac{d\lambda_c}{dK}\Big|_{K=K_c} &= \frac{-2}{K_c^2 D'(iy_c)} \end{aligned}$$

hold.

Proof. The first one follows from Eq.(3.9) and the definition of (y_c, K_c). The derivative of Eq.(3.9) as a function of λ gives

$$D'(\lambda) = \frac{-2}{iK(\lambda)^2}\frac{dK}{d\lambda}.$$

This proves the second one. □

The eigenvalues of T_2 satisfy the same statement as Lemma 3.2. The limit $x \to +0$ for Eq.(3.8) provides

$$\begin{cases} \displaystyle\lim_{x\to+0}\int_{\mathbb{R}} \frac{x}{x^2+(y-2\omega)^2} g(\omega)d\omega = \frac{1}{2}\pi g(y/2) = \frac{h}{K}, \\ \displaystyle\lim_{x\to+0}\int_{\mathbb{R}} \frac{y-2\omega}{x^2+(y-2\omega)^2} g(\omega)d\omega = \frac{1}{2}\pi H[g](y/2) = 0. \end{cases}$$

Let $y_1, y_2, \cdots$ be roots of the second equation, and define $K_j^{(2)} = 2h/(\pi g(y_j/2))$ and $K_c^{(2)} = \inf_j\{K_j^{(2)}\}$. In what follows, we assume the following;

(A1) $h < 1$.

It is easy to verify that this condition is equivalent to $K_c < K_c^{(2)}$. This implies that the eigenvalue λ_c of T_1 still exists on the right half plane after all eigenvalues of T_2 disappear as K decreases. In other words, as K increases from zero, the eigenvalue λ_c of T_1 first emerges from the imaginary axis before some eigenvalue of T_2 emerges.

Theorem 3.4 (Instability of the incoherent state).
Suppose (A1) and $g(\omega)$ is continuous. If $0 < K \leq K_c$, the spectra of operators $T_1, T_2, \cdots$ consist only of the continuous spectra on the imaginary axis. There exists a small number $\varepsilon > 0$ such that when $K_c < K < K_c + \varepsilon$, the eigenvalue λ_c of T_1 exists on the right half plane. Therefore, the incoherent state is linearly unstable.

This suggests that a first bifurcation occurs at $K = K_c$ and the eigenvalue λ_c of T_1 plays an important role to the bifurcation.

Example 3.5. It is known that if $g(\omega)$ is an even and unimodal function, there exists a unique eigenvalue on the positive real axis for $K > K_c$. Since we are interested in a Hopf bifurcation in this paper, let us consider the following bimodal frequency density defined as the sum of two Lorentzian density [9]

$$g(\omega) = \frac{1}{2\pi}\left(\frac{1}{(\omega-\omega_0)^2+1} + \frac{1}{(\omega+\omega_0)^2+1}\right), \tag{3.11}$$

where $\omega_0 > 0$ is a parameter. When $g''(0) > 0 \Rightarrow \omega_0 \geq 1/\sqrt{3}$, it is a bimodal function. The equation $H[g](y) = 0$ has at most three roots given by

$$y_1 = 0, \quad y_2 = \sqrt{\omega_0^2 - 1}, \quad y_3 = -\sqrt{\omega_0^2 - 1}.$$

Among them, y_2 and y_3 exist only when $\omega_0 > 1$. Otherwise, the eigenvalue uniquely exists on the positive real axis as in the unimodal density case. In what follows, we assume $\omega_0 > 1$. Since $g(0) < g(y_2) = g(y_3) = 1/(2\pi)$, y_c and K_c are given by

$$y_c = \pm\sqrt{\omega_0^2 - 1}, \quad K_c = \frac{2}{\pi g(y_c)} = 4.$$

Eq.(3.9) is calculated as

$$D(\lambda) = \frac{\lambda + 1}{(\lambda+1)^2 + \omega_0^2} = \frac{2}{K}. \tag{3.12}$$

This shows that there are at most two eigenvalues on the right half plane for each K. The motion of the eigenvalues $\lambda = \lambda(K)$ as K increases is represented in Fig.3 (a). When $K < K_c = 4$, there are no eigenvalues. At $K = K_c$, a pair of eigenvalues $\lambda_c(K_c) = \pm i\sqrt{\omega_0^2 - 1}$ pops up from the continuous spectrum on the imaginary axis. At $K = 4\omega_0 > K_c$, two eigenvalues collide with one another on the real axis. For $K > 4\omega_0$, there are two eigenvalues on the positive real axis. One of them goes to the left side as K increases, and it is absorbed into the continuous spectrum and disappears at $K = 2/(\pi g(0)) > 4\omega_0$. The other goes to infinity on the positive real axis as $K \to \infty$. Later we will show that a Hopf bifurcation occurs at $K = K_c$.

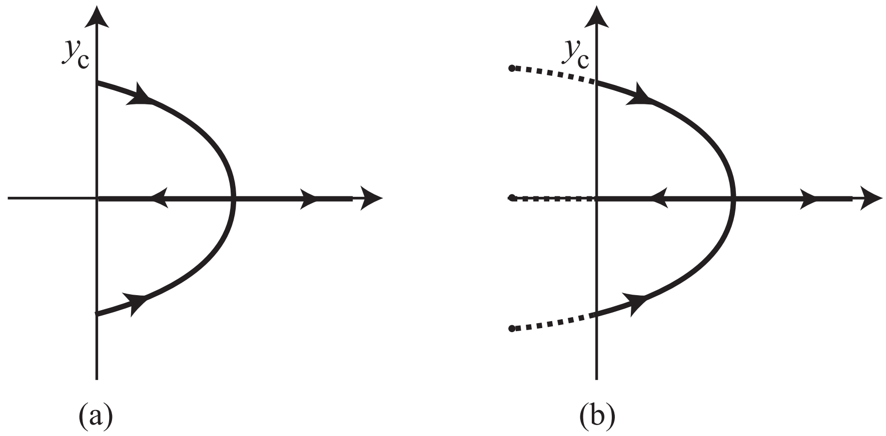

Figure 3: (a) The motion of the eigenvalues as K increases for the density (3.11). The imaginary axis is the continuous spectrum. (b) The motion of the generalized eigenvalues as K increases from zero for (3.11). The imaginary axis is a branch cut of the Riemann surface of the generalized resolvent. The dotted curve denotes the path of the generalized eigenvalue on the second Riemann sheet. See Section 5 for the detail.

4 Linear stability

When $0 < K < K_c$, there are no spectra of operators $T_1, T_2, \cdots$ on the right half plane, while the continuous spectra of them exist on the imaginary axis. Hence, one may expect that the incoherent state is neutrally stable. Nevertheless, we will show that the order parameter is asymptotically stable in a certain sense. For this purpose, we need the following assumption. Let δ be a positive number and define the stripe region on $\mathbb{C}$

$$S(\delta) := \{z \in \mathbb{C} \,|\, 0 \leq \mathrm{Im}(z) \leq \delta\}.$$

We assume that

(A2) The density function $g(\omega)$ has an analytic continuation to the region $S(\delta)$. On $S(\delta)$, there exists a constant $C > 0$ such that the estimate

$$|g(z)| \leq \frac{C}{1 + |z|^2}, \quad z \in S(\delta) \tag{4.1}$$

holds.

Let H_+ be the Hardy space: the set of bounded holomorphic functions on the real axis and the upper half plane. It is a subspace of $L^2(\mathbb{R}, g(\omega)d\omega)$. For $\psi \in H_+$, set $\psi^*(z) := \overline{\psi(\overline{z})}$.

A function $f_t \in L^2(\mathbb{R}, g(\omega)d\omega)$ parameterized by t is said to be convergent to zero in the weak sense if the inner product (f_t, ψ^*) decays to zero as $t \to \infty$ for any $\psi \in H_+$. Note that $P_0 \in H_+$ and the order parameter is written as $\eta_1(t) = (Z_1, P_0) = (Z_1, P_0^*)$. This means that it is sufficient to consider the stability in the weak sense for the stability of the order parameter. The next lemma plays an important role in the generalized spectral theory.

Lemma 4.1. Let $f(z)$ be a holomorphic function on the region $S(\delta)$. Define a function $A[f](\lambda)$ of λ to be

$$A[f](\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} f(\omega) d\omega$$

for $\mathrm{Re}(\lambda) > 0$. It has an analytic continuation $\hat{A}[f](\lambda)$ from the right half plane to the region $-\delta \le \mathrm{Re}(\lambda) \le 0$ given by

$$\hat{A}[f](\lambda) = \begin{cases} A[f](\lambda) & \mathrm{Re}(\lambda) > 0 \\ \lim_{\mathrm{Re}(\lambda)\to +0} A[f](\lambda) & \mathrm{Re}(\lambda) = 0 \\ A[f](\lambda) + 2\pi f(-i\lambda) & -\delta \le \mathrm{Re}(\lambda) < 0. \end{cases} \tag{4.2}$$

See [3, 5] for the proof.

It is known that the semigroup e^{Tt} of an operator T is expressed by the Laplace inversion formula

$$e^{Tt} = \lim_{y\to\infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} e^{\lambda t} (\lambda - T)^{-1} d\lambda, \tag{4.3}$$

for $t > 0$ (under a certain mild condition for T [15]). Here, $x > 0$ is chosen so that the integral path is to the right of the spectrum of T (see Fig.4(a)).

Lemma 4.2. The resolvent of $T_1 = i\omega + iKf_1\mathcal{P}$ is given by

$$(\lambda - T_1)^{-1}\phi = (\lambda - i\omega)^{-1}\phi + \frac{iKf_1}{1 - iKf_1 D(\lambda)} ((\lambda - i\omega)^{-1}\phi, P_0) \frac{1}{\lambda - i\omega}. \tag{4.4}$$

Let λ_c be a simple eigenvalue of T_1. The projection Π_c to the eigenspace of λ_c is given by

$$\Pi_c \phi = \frac{-1}{D'(\lambda_c)} ((\lambda_c - i\omega)^{-1}\phi, P_0) \frac{1}{\lambda_c - i\omega}. \tag{4.5}$$

See [5] for the proof.

Lemma 4.2 provides

$$\begin{aligned} & ((\lambda - T_1)^{-1}\phi, \psi^*) \\ = \; & ((\lambda - i\omega)^{-1}\phi, \psi^*) + \frac{iKf_1}{1 - iKf_1 D(\lambda)}((\lambda - i\omega)^{-1}\phi, P_0) \cdot ((\lambda - i\omega)^{-1}\psi, P_0), \end{aligned}$$

which is meromorphic in λ on the right half plane. Suppose $\phi, \psi \in H_+$. Due to Lemma 4.1, $((\lambda - T_1)^{-1}\phi, \psi^*)$ has an analytic continuation, possibly with new singularities, to the region $-\delta \leq \mathrm{Re}(\lambda) \leq 0$ (Lemma 4.1 is applied to the factors $D(\lambda), ((\lambda - i\omega)^{-1}\phi, \psi^*), ((\lambda - i\omega)^{-1}\phi, P_0)$ and $((\lambda - i\omega)^{-1}\psi, P_0)$). A singularity on the left half plane is a root of the equation

$$1 - iKf_1(D(\lambda) + 2\pi g(-i\lambda)) = 0. \tag{4.6}$$

Such a singularity of the analytic continuation of the resolvent on the left half plane is called the generalized eigenvalue (see Sec.5 for the detail).

Now we can estimate the behavior of the semigroup by using the analytic continuation. We have

$$(e^{T_1 t}\phi, \psi^*) = \lim_{y \to \infty} \frac{1}{2\pi i} \int_{x - iy}^{x + iy} e^{\lambda t}((\lambda - T_1)^{-1}\phi, \psi^*) d\lambda,$$

where the integral path is given as in Fig.4 (a). When $\phi, \psi \in H_+$, the integrand $((\lambda - T_1)^{-1}\phi, \psi^*)$ has an analytic continuation to the region $-\delta \leq \mathrm{Re}(\lambda) \leq 0$ which is denoted by $\mathcal{R}(\lambda)$.

Lemma 4.3. Fix K such that $0 < K < K_c$. Take positive numbers ε, R and consider the rectangle shaped closed path C represented in Fig.4 (b). If $\varepsilon > 0$ is sufficiently small, the analytic continuation of $((\lambda - T_1)^{-1}\phi, \psi^*)$ is holomorphic inside C for any $R > 0$.

Because of this lemma, we have

$$\begin{aligned} 0 \; = \; & \int_{x - iR}^{x + iR} e^{\lambda t}((\lambda - T_1)^{-1}\phi, \psi^*) d\lambda + \int_{-\varepsilon + iR}^{-\varepsilon - iR} e^{\lambda t}\mathcal{R}(\lambda) d\lambda \\ + \; & \int_{x + iR}^{iR} e^{\lambda t}((\lambda - T_1)^{-1}\phi, \psi^*) d\lambda + \int_{iR}^{iR - \varepsilon} e^{\lambda t}\mathcal{R}(\lambda) d\lambda \\ + \; & \int_{-iR}^{-iR + x} e^{\lambda t}((\lambda - T_1)^{-1}\phi, \psi^*) d\lambda + \int_{-\varepsilon - iR}^{-iR} e^{\lambda t}\mathcal{R}(\lambda) d\lambda. \end{aligned}$$

Due to the assumption (A2), we can verify that four integrals in the second and third lines above become zero as $R \to \infty$. Thus, we obtain

$$(e^{T_1 t}\phi, \psi^*) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-\varepsilon - iR}^{-\varepsilon + iR} e^{\lambda t}\mathcal{R}(\lambda) d\lambda.$$

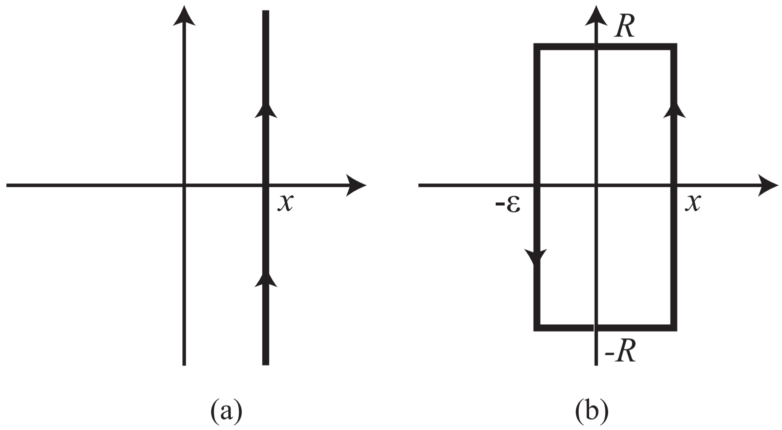

Figure 4: Deformation of the integral path for the Laplace inversion formula.

This proves $|(e^{T_1 t}\phi, \psi^*)| \sim O(e^{-\varepsilon t})$ as $t \to \infty$. We can show the same result for the operators $T_2, T_3, \cdots$.

Theorem 4.4 (Local stability of the incoherent state).
Suppose (A1) and (A2). When $0 < K < K_c$, $(e^{T_j t}\phi, \psi^*)$ decays to zero exponentially as $t \to \infty$ for any $j = 1, 2, \cdots$ and any $\phi, \psi \in H_+$. Thus, the incoherent state is linearly asymptotically stable in the weak sense.

See [5] for the detail of proofs of Lemma 4.3 and Theorem 4.4.

5 The generalized spectral theory

For the study of a bifurcation, we need generalized spectral theory developed in [4] and applied to the Kuramoto model in [3] because the operator T_1 has the continuous spectrum on the imaginary axis (thus, the standard center manifold reduction is not applicable). In this section, a simple review of the generalized spectral theory is given. All proofs are included in [3, 4].

Let H_+ be the Hardy space with the norm

$$||\phi||_{H_+} = \sup_{\mathrm{Im}(z)>0} |\phi(z)|. \tag{5.1}$$

With this norm, H_+ is a Banach space. Let H'_+ be the dual space of H_+; the set of continuous anti-linear functionals on H_+. For $\mu \in H'_+$ and $\phi \in H_+$, $\mu(\phi)$ is denoted by $\langle \mu \,|\, \phi \rangle$. For any $a, b \in \mathbb{C}$, $\phi, \psi \in H_+$ and $\mu, \xi \in H'_+$, the equalities

$$\langle \mu \,|\, a\phi + b\psi \rangle = \overline{a}\langle \mu \,|\, \phi \rangle + \overline{b}\langle \mu \,|\, \psi \rangle,$$
$$\langle a\mu + b\xi \,|\, \phi \rangle = a\langle \mu \,|\, \phi \rangle + b\langle \xi \,|\, \phi \rangle,$$

hold. An element of H'_+ is called a generalized function. The space H_+ is a subspace of $L^2 = L^2(\mathbb{R}, g(\omega)d\omega)$ and the embedding $H_+ \hookrightarrow L^2$ is continuous. Then, we can show that the dual $(L^2)'$ of L^2 is continuously embedded in H'_+. Since L^2 is a Hilbert space satisfying $(L^2)' \simeq L^2$, we have three topological vector spaces called a Gelfand triplet

$$H_+ \subset L^2(\mathbb{R}, g(\omega)d\omega) \subset H'_+.$$

If an element $\phi \in H'_+$ is included in $L^2(\mathbb{R}, g(\omega)d\omega)$, then $\langle \phi \,|\, \psi \rangle$ is given by

$$\langle \phi \,|\, \psi \rangle := (\phi, \psi^*) = \int_{\mathbb{R}} \phi(\omega)\psi(\omega)g(\omega)d\omega.$$

(the conjugate ψ^* is introduced to avoid the complex conjugate $\overline{\psi(\omega)}$ in the integrand). Our operator T_1 and the above triplet satisfy all assumptions given in [4] to develop a generalized spectral theory. Now we give a brief review of the theory. In what follows, we assume (A2).

The multiplication operator $\phi \mapsto i\omega\phi$ has the continuous spectrum on the imaginary axis; its resolvent is given by $(\lambda - i\omega)^{-1}$, and it is not included in $L^2(\mathbb{R}, g(\omega)d\omega)$ when λ is a purely imaginary number. Nevertheless, we show that the resolvent has an analytic continuation from the right half plane to the left half plane in the generalized sense. We define an operator $A(\lambda) : H_+ \to H'_+$, parameterized by $\lambda \in \mathbb{C}$, to be

$$\langle A(\lambda)\phi \,|\, \psi \rangle = \begin{cases} ((\lambda - i\omega)^{-1}\phi, \psi^*), & \mathrm{Re}(\lambda) > 0, \\ \lim_{\mathrm{Re}(\lambda)\to+0} ((\lambda - i\omega)^{-1}\phi, \psi^*) & \mathrm{Re}(\lambda) = 0, \\ ((\lambda - i\omega)^{-1}\phi, \psi^*) & \\ \quad +2\pi\phi(-i\lambda)\psi(-i\lambda)g(-i\lambda) & -\delta \leq \mathrm{Re}(\lambda) < 0, \end{cases}$$

for $\phi, \psi \in H_+$. Due to Lemma 4.1, $\langle A(\lambda)\phi \,|\, \psi \rangle$ is holomorphic. That is, $A(\lambda)\phi$ is a H'_+-valued holomorphic function in λ. In particular, $A(\lambda)$ coincides with $(\lambda - i\omega)^{-1}$ when $\mathrm{Re}(\lambda) > 0$. Since the continuous spectrum of the multiplication operator by $i\omega$ is the whole imaginary axis, $(\lambda - i\omega)^{-1}$ does not have an analytic continuation from the right half plane to the left half plane as an operator on $L^2(\mathbb{R}, g(\omega)d\omega)$, however, it has a continuation $A(\lambda)$ if it is regarded as an operator from H_+ to H'_+. $A(\lambda)$ is called the generalized resolvent of the multiplication operator by $i\omega$.

The next purpose is to define an analytic continuation of the resolvent of T_1 in the generalized sense. Note that $(\lambda - T_1)^{-1}$ is rearranged as

$$(\lambda - i\omega - iKf_1\mathcal{P})^{-1} = (\lambda - i\omega)^{-1} \circ (\mathrm{id} - iKf_1\mathcal{P}(\lambda - i\omega)^{-1})^{-1}.$$

Since the analytic continuation of $(\lambda - i\omega)^{-1}$ in the generalized sense is $A(\lambda)$, we define the generalized resolvent $\mathcal{R}(\lambda) : H_+ \to H'_+$ of T_1 by

$$\mathcal{R}(\lambda) := A(\lambda) \circ \left(\mathrm{id} - iKf_1 \mathcal{P}^\times A(\lambda)\right)^{-1},$$

where $\mathcal{P}^\times : H'_+ \to H'_+$ is the dual operator of $\mathcal{P}$. For each $\phi \in H_+$, $\mathcal{R}(\lambda)\phi$ is a H'_+-valued meromorphic function. It is easy to verify that when $\mathrm{Re}(\lambda) > 0$, it is reduced to the usual resolvent $(\lambda - T_1)^{-1}$. Thus, $\mathcal{R}(\lambda)$ gives a meromorphic continuation of $(\lambda - T_1)^{-1}$ from the right half plane to the left half plane as a H'_+-valued operator. Again, note that T_1 has the continuous spectrum on the imaginary axis, so that it has no continuation as an operator on $L^2(\mathbb{R}, g(\omega)d\omega)$.

A generalized eigenvalue is defined as a singularity of $\mathcal{R}(\lambda)$, namely a singularity of $(\mathrm{id} - iKf_1\mathcal{P}^\times A(\lambda))^{-1}$.

Definition 5.1. If the equation

$$(\mathrm{id} - iKf_1 \mathcal{P}^\times A(\lambda))\mu = 0 \tag{5.2}$$

has a nonzero solution μ in H'_+ for some $\lambda \in \mathbb{C}$, λ is called a generalized eigenvalue and μ is called a generalized eigenfunction.

It is easy to verify that this equation is equivalent to

$$\frac{2}{K} = \begin{cases} D(\lambda) & \mathrm{Re}(\lambda) > 0, \\ \lim\limits_{\mathrm{Re}(\lambda)\to +0} D(\lambda) & \mathrm{Re}(\lambda) = 0, \\ D(\lambda) + 2\pi g(-i\lambda) & -\delta \leq \mathrm{Re}(\lambda) < 0, \end{cases} \tag{5.3}$$

where we use $f_1 = 1/(2i)$. When $\mathrm{Re}(\lambda) > 0$, this is reduced to Eq.(3.9). In this case, μ is included in $L^2(\mathbb{R}, g(\omega)d\omega)$ and a generalized eigenvalue on the right half plane is an eigenvalue in the usual sense. When $\mathrm{Re}(\lambda) \leq 0$, this equation is equivalent to Eq.(4.6). The associated generalized eigenfunction is not included in $L^2(\mathbb{R}, g(\omega)d\omega)$ but an element of the dual space H'_+. Although a generalized eigenvalue is not a true eigenvalue of T_1, it is an eigenvalue of the dual operator:

Theorem 5.2 [3, 4]. Let λ and μ be a generalized eigenvalue and the associated generalized eigenfunction. The equality $T_1^\times \mu = \lambda\mu$ holds.

Let λ_0 be a generalized eigenvalue of T_1 and γ_0 a small simple closed curve enclosing λ_0. The generalized Riesz projection $\Pi_0 : H_+ \to H'_+$ is defined by

$$\Pi_0 = \frac{1}{2\pi i}\int_{\gamma_0} \mathcal{R}(\lambda) d\lambda.$$

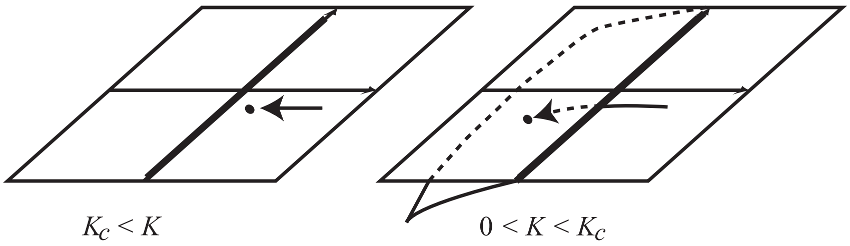

Figure 5: The motion of the (generalized) eigenvalue as K decreases. When $0 < K < K_c$, it lies on the second Riemann sheet of the resolvent and it is not a usual eigenvalue but a generalized eigenvalue.

As in the usual spectral theory, the image of it gives the generalized eigenspace associated with λ_0.

Let $\lambda = \lambda_c(K)$ be an eigenvalue of T_1 defined in Sec.3. Recall that when $K_c < K$, λ_c exists on the right half plane. As K decreases, λ_c goes to the left side, and at $K = K_c$, λ_c is absorbed into the continuous spectrum on the imaginary axis and disappears. However, we can show that even for $0 < K < K_c$, λ_c remains to exist as a root of Eq.(5.3) because the right hand side of Eq.(5.3) is holomorphic. This means that although λ_c disappears from the original complex plane at $K = K_c$, it still exists for $0 < K < K_c$ as a generalized eigenvalue on the Riemann surface of the generalized resolvent $\mathcal{R}(\lambda)$. In the generalized spectral theory, the resolvent $(\lambda - T_1)^{-1}$ is regarded as an operator from H_+ to H_+', not on $L^2(\mathbb{R}, g(\omega)d\omega)$. Then, it has an analytic continuation from the right half plane to the left half plane as H_+'-valued operator. The continuous spectrum on the imaginary axis becomes a branch cut of the Riemann surface of the resolvent. On the Riemann surface, the left half plane is two-sheeted (see Fig.5). We call a singularity of the generalized resolvent on the second Riemann sheet the generalized eigenvalue.

On the dual space H_+', the weak dual topology is equipped; a sequence $\{\mu_n\} \subset H_+'$ is said to be convergent to $\mu \in H_+'$ if $\langle \mu_n \,|\, \psi \rangle \in \mathbb{C}$ is convergent to $\langle \mu \,|\, \psi \rangle$ for each $\psi \in H_+$. Recall that an eigenfunction of a usual eigenvalue λ of T_1 is given by $v_\lambda(\omega) = (\lambda - i\omega)^{-1}$ (Eq.(3.6)). A generalized eigenfunction μ_λ of a generalized eigenvalue iy on the imaginary axis is given by

$$\mu_\lambda = \lim_{\lambda \to +0+iy} \frac{1}{\lambda - i\omega},$$

where the limit is considered with respect to the weak dual topology. This means that $\langle \mu_\lambda \,|\, \psi \rangle$ is defined by

$$\langle \mu_\lambda \,|\, \psi \rangle = \lim_{\lambda \to +0+iy} \langle \frac{1}{\lambda - i\omega} \,|\, \psi \rangle = \lim_{\lambda \to +0+iy} \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} \psi(\omega) g(\omega) d\omega. \qquad (5.4)$$

A generalized eigenfunction μ_λ associated with a generalized eigenvalue λ on the left half plane is given by

$$\langle \mu_\lambda \,|\, \psi \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} \psi(\omega) g(\omega) d\omega + 2\pi \psi(-i\lambda) g(-i\lambda). \tag{5.5}$$

To perform a center manifold reduction, we need the definition of a center subspace. Usually, it is defined to be an eigenspace associated with eigenvalues on the imaginary axis. For our case, the operators $T_1, T_2, \cdots$ have the continuous spectra on the imaginary axis. Therefore, we define a generalized center subspace as a space spanned by generalized eigenfunctions associated with generalized eigenvalues on the imaginary axis. Note that this is a subspace of the dual H'_+, not of $L^2(\mathbb{R}, g(\omega)d\omega)$. As K increases from zero, some of the generalized eigenvalues of T_1 get across the imaginary axis at $K = K_c$, and they become usual eigenvalues on the right half plane (see Fig.5). Hence, there is a nontrivial generalized center subspace at $K = K_c$ given by

$$\mathbf{E}^c := \mathrm{span}\{\mu_\lambda \,|\, \lambda(K_c) \in i\mathbb{R}\},$$

The next purpose is to perform a center manifold reduction.

Example 5.3. Let us consider the density (3.11) given in Example 3.5. The equation (5.3) for generalized eigenvalues is given by (3.12); the left hand side of it already gives an analytic continuation of $D(\lambda)$. By solving it, it turns out that two generalized eigenvalues exist at $\lambda(0) = -1 \pm i\omega_0$ when $K = 0$. As K increases, they go to the right side as is shown in Fig. 3 (b). They get across the imaginary axis when $K = K_c = 4$, and become usual eigenvalues for $K > K_c$. One of them again becomes a generalized eigenvalue at $K = 2/(\pi g(0))$ by getting across the imaginary axis from the right to the left. The generalized center subspace for $K = K_c$ is a two-dimensional space.

6 Center manifold reduction

Recall that $y_c \in \mathbb{R}$ is defined as a number satisfying $\sup_j\{g(y_j)\} = g(y_c)$, where $y_1, y_2, \cdots$ are roots of the equation $H[g](y) = 0$. This gives a point iy_c on the imaginary axis to which some eigenvalue of T_1 approaches as $K \to K_c + 0$. For a Hopf bifurcation, we assume the following:

(A3) There are exactly two nonzero values y_c and $-y_c$ satisfying $\sup_j\{g(y_j)\} = g(\pm y_c)$. Each of the corresponding eigenvalue of T_1 denoted by $\lambda_c^+(K)$ and $\lambda_c^-(K)$, respectively, is simple near K_c (i.e. the eigenspace is one dimensional).

(A4) The real part of $\left.\dfrac{d\lambda_c^\pm}{dK}\right|_{K=K_c}$ is positive.

(A5) $g(\omega)$ is an even function.

The assumption (A3) implies that the generalized center subspace at $K = K_c$ is a two dimensional space given by

$$\mathbf{E}^c = \mathrm{span}\{\mu_+, \mu_-\}, \quad \mu_\pm := \lim_{\lambda \to +0 \pm i y_c} \frac{1}{\lambda - i\omega}. \tag{6.1}$$

The assumption (A4) means that the generalized eigenvalues $\lambda_c^\pm$ of T_1 transversely cross the imaginary axis from the left to the right. Due to (A5), it is easy to verify that the following equalities hold:

$$D(iy_c) = D(-iy_c) = \frac{2}{K_c},\ D'(iy_c) = \overline{D'(-iy_c)},\ D''(iy_c) = \overline{D''(-iy_c)}. \tag{6.2}$$

It seems that (A3) and (A4) are satisfied for a wide class of even and bimodal densities $g(\omega)$ as long as the distance of two peaks are sufficiently far apart, see Example 3.5.

In what follows, we assume (A1) to (A5) and perform the center manifold reduction We put $\varepsilon = K - K_c$, which plays a role of a bifurcation parameter. Our ingredients are;

Equations: The equations (2.3) for $j = 1, 2$ with $f_1 = 1/(2i)$, $f_2 = h/(2i)$ are given by

$$\begin{cases} \dot{Z}_1 = T_c Z_1 + \dfrac{\varepsilon}{2}\mathcal{P} Z_1 + \dfrac{K}{2}(h \cdot \eta_2 Z_{-1} - \overline{\eta_1} Z_2 - h \cdot \overline{\eta_2} Z_3), \\ \dot{Z}_2 = T_2 Z_2 + K(\eta_1 Z_1 - \overline{\eta_1} Z_3 - h \cdot \overline{\eta_2} Z_4), \end{cases} \tag{6.3}$$

where T_c is an operator T_1 estimated at $K = K_c$; that is, K in T_1 is denoted by $K = K_c + \varepsilon$ and accordingly $T_1 = T_c + \varepsilon \mathcal{P}/2$.

Center subspace: As K increases from zero, a pair of the generalized eigenvalues of T_1 denoted by $\lambda_c^\pm(K)$ gets across the imaginary axis at $\pm i y_c$ when $K = K_c$, and they become usual eigenvalues on the right half plane when $K > K_c$. The associated generalized eigenfunctions at $K = K_c$ and the generalized center subspace is given in (6.1).

Projection: The projection to an eigenspace is given in Lemma 4.2. The projection to the generalized center subspace spanned by μ_+ and μ_- is

$$\Pi_c \phi = \frac{-1}{D'(iy_c)} \lim_{\lambda \to i y_c} ((\lambda - i\omega)^{-1}\phi, P_0)\mu_+ + \frac{-1}{D'(-iy_c)} \lim_{\lambda \to -i y_c} ((\lambda - i\omega)^{-1}\phi, P_0)\mu_-. \tag{6.4}$$

We divide our result into two cases, $h = 0$ and $h \neq 0$ because types of bifurcations of them are different.

6.1 Center manifold reduction $(h = 0)$

Assume $h = 0$. Then, $T_2 = 2i\omega$. Since $\Pi_c Z_1$ is a linear combination of μ_+ and μ_-, we suppose $\Pi_c Z_1 = K_c/2 \cdot (\alpha_+(t)\mu_+ + \alpha_-(t)\mu_-)$. The scalar valued functions $\alpha_+(t)$ and $\alpha_-(t)$ denote coordinates on the center subspace, and our purpose is to derive the dynamics of $\alpha_\pm$. Since a solution decays to zero with an exponential rate for $(\mathrm{id} - \Pi_c)Z_1$ direction and $Z_j\,(j = 2, 3, \cdots)$ directions, we assume that $(\mathrm{id} - \Pi_c)Z_1$ and $Z_j\,(j = 2, 3, \cdots)$ are of order $O(\alpha^2)$ which stand for $O(\alpha_+^2, \alpha_+\alpha_-, \alpha_-^2)$. Thus, we write

$$Z_1 = \frac{K_c}{2}(\alpha_+(t)\mu_+ + \alpha_-(t)\mu_-) + O(\alpha^2). \tag{6.5}$$

Then, η_1 is given by

$$\begin{aligned} \eta_1 &= \int_{\mathbb{R}} Z_1 \cdot g(\omega) d\omega \\ &= \frac{K_c}{2}\alpha_+ \lim_{\lambda \to iy_c} \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g(\omega) d\omega + \frac{K_c}{2}\alpha_- \lim_{\lambda \to -iy_c} \int_{\mathbb{R}} \frac{1}{\lambda - i\omega} g(\omega) d\omega + O(\alpha^2) \\ &= \alpha_+ + \alpha_- + O(\alpha^2), \end{aligned} \tag{6.6}$$

where we have used Eq.(6.2). Further, we make the following ansatz

$$\varepsilon \sim O(\alpha^2), \quad \frac{d\alpha_\pm}{dt} = \pm iy_c\alpha_\pm + O(\alpha^2), \tag{6.7}$$

which will be verified if the dynamics on the center manifold is derived.

For $m, n = 0, 1, 2, \cdots$, we define functionals denoted by $\mu_+^m \cdot \mu_-^n \in H_+'$ by

$$\mu_+^m \cdot \mu_-^n := \lim_{\lambda_+ \to +iy_c} \lim_{\lambda_- \to -iy_c} \frac{1}{(\lambda_+ - i\omega)^m} \frac{1}{(\lambda_- - i\omega)^n}, \tag{6.8}$$

where the limit is considered with respect to the weak dual topology.

Lemma 6.1. The following equalities hold.

$$\begin{aligned} &\omega\mu_\pm^2 = i\mu_\pm \pm y_c\mu_\pm^2, \\ &\omega\mu_\pm = i \pm y_c\mu_\pm, \\ &\langle \mu_\pm P_0 \,|\, P_0 \rangle = D(\pm iy_c) = \frac{2}{K_c}, \\ &\langle \mu_\pm^2 P_0 \,|\, P_0 \rangle = -D'(\pm iy_c), \\ &\langle \mu_\pm^3 P_0 \,|\, P_0 \rangle = \frac{1}{2} D''(\pm iy_c), \\ &\langle \mu_+ \cdot \mu_- P_0 \,|\, P_0 \rangle = 0, \\ &\langle \mu_+^2 \cdot \mu_- P_0 \,|\, P_0 \rangle = \frac{1}{2iy_c} D'(iy_c), \\ &\langle \mu_+ \cdot \mu_-^2 P_0 \,|\, P_0 \rangle = \frac{-1}{2iy_c} D'(-iy_c), \end{aligned}$$

where $P_0(\omega) = 1$ is a constant function.

Proof. For the first equality, we have

$$\begin{aligned} \omega\mu_\pm^2 &= \lim_{\lambda\to\pm iy_c} \frac{\omega}{(\lambda - i\omega)^2} = i \cdot \lim_{\lambda\to\pm iy_c} \frac{(\lambda - i\omega) - \lambda}{(\lambda - i\omega)^2} \\ &= i \cdot \lim_{\lambda\to\pm iy_c} \frac{1}{\lambda - i\omega} \pm y_c \lim_{\lambda\to\pm iy_c} \frac{1}{(\lambda - i\omega)^2} \\ &= i\mu_\pm \pm y_c\mu_\pm^2. \end{aligned}$$

The second one is proved in a similar manner. The third one is given in Lemma 3.3. The fourth and fifth equalities are easily shown by the integration by parts, see Lemma 3.1. To prove the sixth equality, we use the partial fraction decomposition as

$$\begin{aligned} \langle \mu_+ \cdot \mu_- P_0 \,|\, P_0 \rangle &= \lim_{\lambda_+\to+iy_c} \lim_{\lambda_-\to-iy_c} \int_{\mathbb{R}} \frac{1}{(\lambda_+ - i\omega)(\lambda_- - i\omega)} g(\omega)d\omega \\ &= \lim_{\lambda_+\to+iy_c} \lim_{\lambda_-\to-iy_c} \frac{1}{\lambda_+ - \lambda_-} \int_{\mathbb{R}} \left(\frac{-1}{\lambda_+ - i\omega} + \frac{1}{\lambda_- - i\omega} \right) g(\omega)d\omega \\ &= \frac{1}{2iy_c} \left(-D(iy_c) + D(-iy_c)\right) = 0. \end{aligned}$$

The last two equalities are also verified by the partial fraction decomposition. □

Lemma 6.2. Define

$$Z_2 = \frac{K_c^2}{4}\alpha_+^2\mu_+^2 + \frac{K_c^2}{4}\alpha_-^2\mu_-^2 - \frac{K_c^2}{4iy_c}\alpha_+\alpha_-(\mu_+ - \mu_-) + O(\alpha^3). \tag{6.9}$$

It satisfies the second differential equation of (6.3) up to the order $O(\alpha^3)$.

Proof. By substituting Eqs.(6.5), (6.6) and (6.9) in the equation, we can confirm with the aid of Lemma 6.1 and (6.7) that

$$\dot{Z}_2 - (T_2 Z_2 + K(\eta_1 Z_1 - \overline{\eta_1} Z_3))$$

is of order $O(\alpha^3)$. □

Let us apply the projection Π_c to the both sides of the first equation of Eq.(6.3) to get

$$\frac{K_c}{2}(\dot{\alpha}_+\mu_+ + \dot{\alpha}_-\mu_-) = T_c^\times \Pi_c Z_1 + \frac{\varepsilon}{2}\eta_1 \Pi_c P_0 - \frac{K}{2}\overline{\eta}_1 \Pi_c Z_2. \tag{6.10}$$

Theorem 5.2 gives

$$T_c^\times \Pi_c Z_1 = \frac{K_c}{2} T_c^\times (\alpha_+\mu_+ + \alpha_-\mu_-) = \frac{K_c}{2} \cdot iy_c \cdot (\alpha_+\mu_+ - \alpha_-\mu_-).$$

The definition of Π_c combined with Lemma 6.1 yields

$$\begin{aligned}
\Pi_c P_0 &= \frac{-2}{K_c D'(iy_c)}\mu_+ + \frac{-2}{K_c D'(-iy_c)}\mu_-, \\
\Pi_c Z_2 &= \frac{-K_c^2}{4D'(iy_c)}\left(\frac{1}{2}\alpha_+^2 D''(iy_c) - \frac{1}{2iy_c}\alpha_-^2 D'(-iy_c) + \frac{1}{iy_c}\alpha_+\alpha_- D'(iy_c)\right)\mu_+ \\
&\quad + \frac{-K_c^2}{4D'(-iy_c)}\left(\frac{1}{2}\alpha_-^2 D''(-iy_c) + \frac{1}{2iy_c}\alpha_+^2 D'(iy_c) - \frac{1}{iy_c}\alpha_+\alpha_- D'(-iy_c)\right)\mu_- \\
&\quad + O(\alpha^3).
\end{aligned}$$

Substituting these equalities into Eq.(6.10) and comparing the coefficients of μ_+ and μ_-, respectively, in the both sides of the equation, we obtain the dynamics on the center manifold

$$\left\{\begin{aligned}
\frac{d\alpha_+}{dt} &= iy_c\alpha_+ + p_1\varepsilon(\alpha_+ + \alpha_-) + (\overline{\alpha}_+ + \overline{\alpha}_-)\left(p_2\alpha_+^2 + p_3\alpha_-^2 + p_4\alpha_+\alpha_-\right) + O(\alpha^4), \\
\frac{d\alpha_-}{dt} &= -iy_c\alpha_- + \overline{p}_1\varepsilon(\alpha_+ + \alpha_-) + (\overline{\alpha}_+ + \overline{\alpha}_-)\left(\overline{p}_2\alpha_-^2 + \overline{p}_3\alpha_+^2 + \overline{p}_4\alpha_+\alpha_-\right) + O(\alpha^4),
\end{aligned}\right. \tag{6.11}$$

where p_1 to p_4 are complex numbers defined by

$$p_1 = \frac{-2}{K_c^2 D'(iy_c)}, \quad p_2 = \frac{K_c^2 D''(iy_c)}{8D'(iy_c)}, \quad p_3 = -\frac{K_c^2\overline{D'(iy_c)}}{8iy_c D'(iy_c)}, \quad p_4 = \frac{K_c^2}{4iy_c}.$$

This is a (real) four dimensional dynamical system. The next purpose is to reduce it. Since the system is invariant under the action $(\alpha_+, \alpha_-) \mapsto (e^{i\beta}\alpha_+, e^{i\beta}\alpha_-)$ for $\beta \in \mathbb{R}$, we can assume without loss of generality that $\arg(\alpha_+) + \arg(\alpha_-) = 0$. Hence, we assume $\alpha_\pm = r_\pm e^{\pm i\psi}$ with $r_\pm, \psi \in \mathbb{R}$. Substituting this into the system, we obtain the three dimensional system

$$\left\{\begin{aligned}
&\dot{\psi} = y_c + O(r_\pm^2) = y_c + O(\varepsilon), \\
&\dot{r}_+ = \varepsilon\mathrm{Re}(p_1)r_+ + \varepsilon\mathrm{Re}(p_1 e^{-2i\psi})r_- + \mathrm{Re}(p_2)r_+^3 + \mathrm{Re}(p_2 e^{2i\psi})r_+^2 r_- \\
&\quad + \mathrm{Re}(p_3 e^{-4i\psi})r_+ r_-^2 + \mathrm{Re}(p_3 e^{-2i\psi})r_-^3 + \mathrm{Re}(p_4 e^{-2i\psi})r_+^2 r_- + O(r_\pm^4), \\
&\dot{r}_- = \varepsilon\mathrm{Re}(p_1)r_- + \varepsilon\mathrm{Re}(p_1 e^{-2i\psi})r_+ + \mathrm{Re}(p_2)r_-^3 + \mathrm{Re}(p_2 e^{2i\psi})r_+ r_-^2 \\
&\quad + \mathrm{Re}(p_3 e^{-4i\psi})r_+^2 r_- + \mathrm{Re}(p_3 e^{-2i\psi})r_+^3 + \mathrm{Re}(p_4 e^{-2i\psi})r_+ r_-^2 + O(r_\pm^4).
\end{aligned}\right. \tag{6.12}$$

To derive this, note that $\mathrm{Re}(p_4) = 0$. Now we apply the averaging method. The right hand sides of the equations of r_+ and r_- are averaged over ψ to obtain the averaging equation

$$\left\{\begin{aligned}
\dot{r}_+ &= \varepsilon\mathrm{Re}(p_1)r_+ + \mathrm{Re}(p_2)r_+^3 + O(r_\pm^4), \\
\dot{r}_- &= \varepsilon\mathrm{Re}(p_1)r_- + \mathrm{Re}(p_2)r_-^3 + O(r_\pm^4).
\end{aligned}\right. \tag{6.13}$$

It is known that the averaging equation provides an approximate solution within the error of order $O(\varepsilon)$. Further, if the averaging equation has a stable fixed point, then the original system has a stable periodic orbit [6]. If $O(r_\pm^4)$-terms are neglected, the averaging equation has at most four fixed points:

$$(r_+, r_-) = (0,0), \quad (r_*, 0), \quad (0, r_*), \quad (r_*, r_*), \quad r_* := \sqrt{\frac{-\varepsilon \mathrm{Re}(p_1)}{\mathrm{Re}(p_2)}}.$$

The last three fixed points exist as long as $-\varepsilon\mathrm{Re}(p_1)/\mathrm{Re}(p_2) > 0$. The Jacobi matrices of the system at the fixed points are given by

$$\varepsilon\mathrm{Re}(p_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \varepsilon\mathrm{Re}(p_1)\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \ \varepsilon\mathrm{Re}(p_1)\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \ \varepsilon\mathrm{Re}(p_1)\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

respectively. Because of the assumption (A4) and Lemma 3.3, we have $\mathrm{Re}(p_1) > 0$. This shows that when $\varepsilon = K - K_c < 0$, the point $(0,0)$ is stable, and when $\varepsilon = K - K_c > 0$ and $\mathrm{Re}(p_2) < 0$, the fixed point (r_*, r_*) exists and is stable. This proves that when $\varepsilon > 0$ and $\mathrm{Re}(p_2) < 0$, the averaging equation (6.13) has a stable fixed point $(r_+, r_-) = (r_*, r_*) + O(\varepsilon)$, and the system (6.11) has a family of stable periodic orbits

$$(\alpha_+, \alpha_-) = \left(r_* e^{i(y_c t + O(\varepsilon)) + i\beta} + O(\varepsilon),\, r_* e^{-i(y_c + O(\varepsilon)) + i\beta} + O(\varepsilon)\right),$$

where $\beta \in \mathbb{R}$ is an arbitrary constant induced by the action $(\alpha_+, \alpha_-) \mapsto (e^{i\beta}\alpha_+, e^{i\beta}\alpha_-)$ and it is specified by an initial condition. Since the order parameter is $\eta_1 = \alpha_+ + \alpha_- + O(\alpha^2)$, we obtain a family of stable solutions

$$\eta_1 = 2r_* e^{i\beta}\cos(y_c t + O(\varepsilon)) + O(\varepsilon). \tag{6.14}$$

This completes the proof of Theorem 1.3 (i).

6.2 Center manifold reduction ($h \neq 0$)

Assume $h \neq 0$. Then, $T_2 = 2i\omega + hK\mathcal{P}$. We again assume (6.5), and make the following ansatz

$$\varepsilon \sim O(\alpha), \quad \frac{d\alpha_\pm}{dt} = \pm i y_c \alpha_\pm + O(\alpha^2). \tag{6.15}$$

Lemma 6.3. Define

$$\begin{aligned} Z_2 \;=\; & \frac{K_c^2}{4}\alpha_+^2\mu_+^2 + \frac{K_c^2}{4}\alpha_-^2\mu_-^2 - \frac{K_c^2}{4iy_c}\alpha_+\alpha_-(\mu_+ - \mu_-) \\ & -\frac{hK_c^3}{8}\frac{D'(iy_c)}{1-h}\alpha_+^2\mu_+ - \frac{hK_c^3}{8}\frac{D'(-iy_c)}{1-h}\alpha_-^2\mu_- + O(\alpha^3). \end{aligned} \tag{6.16}$$

It satisfies the second differential equation of (6.3) up to the order $O(\alpha^3)$.

This is proved in a similar manner to Lemma 6.2. Let us apply the projection Π_c to the both sides of the first equation of Eq.(6.3).

$$\frac{K_c}{2}(\dot{\alpha}_+\mu_+ + \dot{\alpha}_-\mu_-) = T_c^\times \Pi_c Z_1 + \frac{\varepsilon}{2}\eta_1 \Pi_c P_0 + \frac{K_c h}{2}\eta_2 \Pi_c Z_{-1} + O(\alpha^3). \tag{6.17}$$

Lemma 6.3 with Lemma 6.1 gives

$$\begin{aligned}\eta_2 = (Z_2, P_0) &= -\frac{K_c^2}{4}\alpha_+^2 D'(iy_c) - \frac{K_c^2}{4}\alpha_-^2 D'(-iy_c) \\ &\quad -\frac{hK_c^2}{4}\frac{D'(iy_c)}{1-h}\alpha_+^2 - \frac{hK_c^2}{4}\frac{D'(-iy_c)}{1-h}\alpha_-^2 + O(\alpha^3) \\ &= -\frac{K_c^2}{4}\frac{D'(iy_c)}{1-h}\alpha_+^2 - \frac{K_c^2}{4}\frac{D'(-iy_c)}{1-h}\alpha_-^2 + O(\alpha^3).\end{aligned}$$

Lemma 6.4. $\Pi_c Z_{-1}$ is given by

$$\Pi_c Z_{-1} = \frac{-4}{K_c D'(iy_c)} e^{-i\arg(\alpha_+)}\mu_+ + \frac{-4}{K_c D'(-iy_c)} e^{-i\arg(\alpha_-)}\mu_- + O(\alpha).$$

See [2] for the proof.

Substituting these equalities into Eq.(6.17) and comparing the coefficients of μ_+ and μ_-, respectively, in the both sides of the equation, we obtain the dynamics on the center manifold

$$\begin{cases} \dfrac{d\alpha_+}{dt} = iy_c\alpha_+ + q_1\varepsilon(\alpha_+ + \alpha_-) + (q_2\alpha_+^2 + q_3\alpha_-^2)e^{-i\arg(\alpha_+)} + O(\alpha^3), \\ \dfrac{d\alpha_-}{dt} = -iy_c\alpha_- + \overline{q}_1\varepsilon(\alpha_+ + \alpha_-) + (q_2\alpha_-^2 + \overline{q}_3\alpha_+^2)e^{-i\arg(\alpha_-)} + O(\alpha^3), \end{cases} \tag{6.18}$$

where q_1, q_2 and q_3 are complex numbers defined by

$$q_1 = \frac{-2}{K_c^2 D'(iy_c)}, \quad q_2 = \frac{hK_c}{1-h}, \quad q_3 = \frac{hK_c}{1-h}\frac{\overline{D'(iy_c)}}{D'(iy_c)}.$$

(q_1 is the same number as p_1). Note that q_2 is a real number. The next purpose is to reduce this system by the same way as the last section. Since the system is invariant under the action $(\alpha_+, \alpha_-) \mapsto (e^{i\beta}\alpha_+, e^{i\beta}\alpha_-)$ for $\beta \in \mathbb{R}$, we can assume without loss of generality that $\arg(\alpha_+) + \arg(\alpha_-) = 0$. Hence, we put $\alpha_\pm = r_\pm e^{\pm i\psi}$ with $r_\pm, \psi \in \mathbb{R}$. Substituting this into the system, we obtain the three dimensional system

$$\begin{cases} \dot{\psi} = y_c + O(\varepsilon), \\ \dot{r}_+ = \varepsilon \mathrm{Re}(q_1) r_+ + \varepsilon \mathrm{Re}(q_1 e^{-2i\psi}) r_- + q_2 r_+^2 + \mathrm{Re}(q_3 e^{-4i\psi}) r_-^2 + O(r_\pm^3), \\ \dot{r}_- = \varepsilon \mathrm{Re}(q_1) r_- + \varepsilon \mathrm{Re}(q_1 e^{-2i\psi}) r_+ + q_2 r_-^2 + \mathrm{Re}(q_3 e^{-4i\psi}) r_+^2 + O(r_\pm^3). \end{cases} \tag{6.19}$$

Now we apply the averaging method. The right hand sides of the equations of r_+ and r_- are averaged over ψ to obtain the averaging equation

$$\begin{cases} \dot{r}_+ = \varepsilon \mathrm{Re}(q_1) r_+ + q_2 r_+^2 + O(r_\pm^3), \\ \dot{r}_- = \varepsilon \mathrm{Re}(q_1) r_- + q_2 r_-^2 + O(r_\pm^3). \end{cases} \tag{6.20}$$

If $O(r_\pm^3)$-terms are neglected, the averaging equation has at most four fixed points:

$$(r_+, r_-) = (0,0), \quad (r_*, 0), \quad (0, r_*), \quad (r_*, r_*), \quad r_* := \frac{-\varepsilon \mathrm{Re}(q_1)}{q_2}.$$

The last three fixed points exist only when $-\varepsilon \mathrm{Re}(q_1)/q_2 > 0$. The Jacobi matrices of the system at the fixed points are given by

$$\varepsilon \mathrm{Re}(q_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \varepsilon \mathrm{Re}(q_1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \varepsilon \mathrm{Re}(q_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \varepsilon \mathrm{Re}(q_1) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

respectively. Because of the assumption (A4) and Lemma 3.3, we have $\mathrm{Re}(q_1) > 0$. This shows that when $\varepsilon = K - K_c < 0$, the point $(0,0)$ is stable. When $K - K_c < 0$ and $q_2 > 0$, the fixed points $(r_*, 0)$, $(0, r_*)$, (r_*, r_*) exist but they are unstable. When $K - K_c > 0$ and $q_2 < 0$, the fixed point (r_*, r_*) is stable. Since $h < 1$ (the assumption (A1)), $q_2 < 0$ is equivalent to $h < 0$. This proves that when $K - K_c > 0$ and $h < 0$, the averaging equation (6.20) has a stable fixed point $(r_+, r_-) = (r_*, r_*) + O(\varepsilon^2)$, and the system (6.18) has a family of stable periodic orbits

$$(\alpha_+, \alpha_-) = \left(r_* e^{i(y_c t + O(\varepsilon)) + i\beta} + O(\varepsilon^2), \, r_* e^{-i(y_c + O(\varepsilon)) + i\beta} + O(\varepsilon^2) \right),$$

where $\beta \in \mathbb{R}$ is an arbitrary constant induced by the action $(\alpha_+, \alpha_-) \mapsto (e^{i\beta}\alpha_+, e^{i\beta}\alpha_-)$ and it is specified by an initial condition. Since the order parameter is $\eta_1 = \alpha_+ + \alpha_- + O(\alpha^2)$, we obtain a family of stable solutions

$$\eta_1 = 2 r_* e^{i\beta} \cos(y_c t + O(\varepsilon)) + O(\varepsilon^2). \tag{6.21}$$

This completes the proof of Theorem 1.3 (ii).

7 Compliance with Ethical Standards

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