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A compactified Riccati equation of Airy type on a weighted projective space

By

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Abstract

The Riccati equation $dx/dy = x^2 - y$ is investigated from a view point of dynamical systems theory. The equation is realized as a two dimensional vector field on a weighted projective space. The normal form theory and the center manifold theory of vector fields are applied to obtain many properties of the equation.

§ 1. Introduction

In this paper, the Riccati equation

$$\frac{dx}{dy} = x^2 - y \tag{1.1}$$

is investigated via the dynamical systems theory. It is known that putting $x = -u'/u$, u satisfies the linear Airy equation

$$\frac{d^2u}{dy^2} = yu. \tag{1.2}$$

Since the Airy equation is well studied, many properties of the Riccati equation can be easily obtained. Our purpose in this paper is to study the Riccati equation *without* using the linear equation. Since Eq.(1.2) is not used, in what follows, we call Eq.(1.1) the Airy equation.

The Airy equation is regarded as a two dimensional vector field $dx/dt = x^2 - y$, $dy/dt = 1$, where $t \in \mathbb{C}$ is an additional parameter. In order to investigate behavior of solutions near infinity, we will propose a compactification of the vector field defined on

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a compact manifold $\mathbb{C}P^2(1,2,3)$ called a weighted projective space. Roughly speaking, a vector field on $\mathbb{C}P^2(1,2,3)$ is given as a projectivization of the vector field $dx/dt = x^2 - y,\ dy/dt = 1$ in some weighted manner, where the weight reflects a symmetry of the Airy equation. The vector field on $\mathbb{C}P^2(1,2,3)$ has three fixed points. One of them corresponds to a pole of a solution (i.e. $x = \infty$), and the other fixed points correspond to an irregular singular point of the equation (i.e. $y = \infty$). The dynamical systems theory, in particular, local theory near fixed points are applied to investigate behavior of solutions near poles and the irregular singular point. By using this setting, we will prove that

- the Airy equation is locally integrable near poles,
- any solutions are meromorphic functions,
- any solutions have infinitely many poles,
- the equation has a holomorphic first integral,
- the existence of a solution without poles on a certain sector,
- the Airy equation is uniquely characterized by the geometry of $\mathbb{C}P^2(1,2,3)$ and a certain local condition.

Although some of them are easily proved if we use the linear equation, our proofs without using the linear equation are applicable to any nonlinear differential equations. For example, we can prove that the Painlevé equations can be transformed into certain integrable systems near each poles. An application to the Painlevé equations will appear in a forthcoming paper.

§ 2. A weighted projective space

In this section, we give a definition of a weighted projective space, on which a compactified Airy equation is defined.

Let $\widetilde{U}$ be a complex manifold and Γ a finite group acting analytically and effectively on $\widetilde{U}$. In general, the quotient space $\widetilde{U}/\Gamma$ is not a smooth manifold if the action has fixed points. Roughly speaking, a (complex) orbifold M is defined by glueing a family of such spaces $\widetilde{U}_\alpha/\Gamma_\alpha$; a Hausdorff space M is called an orbifold if there exist an open covering $\{U_\alpha\}$ of M and homeomorphisms $\varphi_\alpha : U_\alpha \simeq \widetilde{U}_\alpha/\Gamma_\alpha$. See [3] for more details. In this article, we will consider the quotient space of the form $\mathbb{C}^n/\mathbb{Z}_p$, an algebraic variety having a unique conical singularity.

Let $\widetilde{\omega}$ be a holomorphic differential form on a complex manifold $\widetilde{U}$. If $\widetilde{\omega}$ is invariant under an analytic action of Γ, it induces a holomorphic differential form on $\widetilde{U}/\Gamma$ outside the set of singularities. A holomorphic differential form on a complex orbifold $M = \bigcup U_\alpha \simeq \bigcup \widetilde{U}_\alpha/\Gamma_\alpha$ is defined to be a family $\widetilde{\omega}_\alpha$ of Γ_α-invariant holomorphic forms on $\widetilde{U}_\alpha$, which is consistent on intersections $U_\alpha \cap U_\beta$. More formally, let $\{\widetilde{\omega}_\alpha\}$ be a family of Γ_α-invariant holomorphic forms on $\widetilde{U}_\alpha$. If there is an open set $U_3 \simeq \widetilde{U}_3/\Gamma_3$ such that

$U_3 \subset U_1 \cap U_2$, we suppose that there are injections $\lambda_j : \widetilde{U}_3 \to \widetilde{U}_j$ such that $\lambda_1^* \widetilde{\omega}_1 = \lambda_2^* \widetilde{\omega}_2$. Then, the family $\{\widetilde{\omega}_\alpha\}$ is called a holomorphic differential form on the orbifold M. A meromorphic form on M is defined in a similar manner. We also define a holomorphic (meromorphic) differential equation on an orbifold by regarding differential equations as Pfaffian forms.

Consider the weighted $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by

$$(x_0, \cdots, x_n) \mapsto (\lambda^{p_0} x_0, \cdots, \lambda^{p_n} x_n), \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}, \tag{2.1}$$

with the weight $(p_0, \cdots, p_n) \in \mathbb{Z}^n$. The quotient space

$$\mathbb{C}P^n(p_0, \cdots, p_n) := \mathbb{C}^{n+1}/\mathbb{C}^* \tag{2.2}$$

is called the weighted projective space. Note that $\mathbb{C}P^n(1, \cdots, 1)$ is a usual projective space, while otherwise a weighted projective space is not a complex manifold but an orbifold with several singularities.

Example 2.1. $\mathbb{C}P^2(1,2,3)$.

This space is defined by the relation $[x, y, z] \sim [\lambda x, \lambda^2 y, \lambda^3 z],\ \lambda \in \mathbb{C}$.

(i) When $x \neq 0$,

$$[x, y, z] \sim [1, \frac{y}{x^2}, \frac{z}{x^3}] := [1, Y_1, Z_1].$$

This implies that the set of points on $\mathbb{C}P^2(1,2,3)$ with $x \neq 0$ is homeomorphic to $\mathbb{C}^2 = \{(Y_1, Z_1)\}$.

(ii) When $y \neq 0$,

$$[x, y, z] \sim [y^{-1/2} x, 1, y^{-3/2} z] := [X_2, 1, Z_2].$$

On the other hand, putting $y = e^{2\pi i} y$ yields

$$[x, y, z] \sim [-y^{-1/2} x, 1, -y^{-3/2} z] = [-X_2, 1, -Z_2].$$

This means that two points (X_2, Z_2) and $(-X_2, -Z_2)$ should be identified. Hence, the set of points on $\mathbb{C}P^2(1,2,3)$ with $y \neq 0$ is homeomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.

(iii) When $z \neq 0$,

$$[x, y, z] \sim [z^{-1/3} x, z^{-2/3} y, 1] := [X_3, Y_3, 1].$$

As above, the set of points on $\mathbb{C}P^2(1,2,3)$ with $z \neq 0$ is homeomorphic to $\mathbb{C}^2/\mathbb{Z}_3$, where the $\mathbb{Z}_3$-action is defined by $(X_3, Y_3) \mapsto (e^{2\pi i/3} X_3, e^{4\pi i/3} Y_3)$.

This proves that

$$\mathbb{C}P^2(1,2,3) \simeq \mathbb{C}^2 \ \cup\ \mathbb{C}^2/\mathbb{Z}_2 \ \cup\ \mathbb{C}^2/\mathbb{Z}_3 \tag{2.3}$$

and thus $\mathbb{C}P^2(1,2,3)$ is an orbifold with two singularities.

We call local coordinates $(Y_1, Z_1), (X_2, Z_2), (X_3, Y_3)$ an inhomogeneous coordinates system on $\mathbb{C}P^2(1,2,3)$. Note that they are not actual local coordinates on $\mathbb{C}P^2(1,2,3)$ but coordinates on the covering spaces $\widetilde{U}_\alpha$. They are related through

$$(2.4)\quad \begin{cases} X_3 = X_2 Z_2^{-1/3} \\ Y_3 = Z_2^{-2/3} \end{cases}, \quad \begin{cases} X_3 = Z_1^{-1/3} \\ Y_3 = Y_1 Z_1^{-2/3} \end{cases}, \quad \begin{cases} X_2 = X_3 Y_3^{-1/2} \\ Z_2 = Y_3^{-3/2} \end{cases}, \quad \begin{cases} Y_1 = Y_3 X_3^{-2} \\ Z_1 = X_3^{-3} \end{cases},$$

which are often used throughout the paper.

Recall that a meromorphic differential equation on $\mathbb{C}P^2(1,2,3)$ is a family of Γ_α-invariant meromorphic equations on the covering spaces $\widetilde{U}_\alpha$. In the inhomogeneous coordinates, they are expressed as meromorphic equations

$$(2.5)\quad \frac{dY_1}{dZ_1} = f_1(Y_1, Z_1), \quad \frac{dX_2}{dZ_2} = f_2(X_2, Z_2), \quad \frac{dX_3}{dY_3} = f_3(X_3, Y_3),$$

which are invariant under the actions of $\mathrm{id}, \mathbb{Z}_2, \mathbb{Z}_3$, respectively. The next lemma shows that if we use the inhomogeneous coordinates with the relation (2.4), meromorphy of f_1, f_2, f_3 implies $\mathrm{id}, \mathbb{Z}_2, \mathbb{Z}_3$-invariance of Eq.(2.5).

Lemma 2.2. *Suppose that differential equations on the coordinates* $(Y_1, Z_1), (X_2, Z_2)$ *and* (X_3, Y_3) *are given as (2.5). They define a meromorphic differential equation on* $\mathbb{C}P^2(1,2,3)$ *if and only if* f_1, f_2, f_3 *are meromorphic.*

Proof. If Eq.(2.5) defines a meromorphic differential equation on $\mathbb{C}P^2(1,2,3)$, then f_1, f_2, f_3 are meromorphic by the definition. Conversely, suppose that f_1, f_2, f_3 are meromorphic. We should prove that equations (2.5) are $\mathrm{id}, \mathbb{Z}_2, \mathbb{Z}_3$-invariant. Due to the relation (2.4), the third equation $dX_3/dY_3 = f_3(X_3, Y_3)$ is transformed into

$$(2.6)\quad \frac{dY_1}{dZ_1} = \frac{Z_1^{1/3} - 2Y_1 f_3(Z_1^{-1/3}, Y_1 Z_1^{-2/3})}{-3Z_1 f_3(Z_1^{-1/3}, Y_1 Z_1^{-2/3})}.$$

For simplicity, suppose that f_3 is holomorphic and expressed as $f_3(X_3, Y_3) = \sum a_{ij} X_3^i Y_3^j$ (even if f_3 is meromorphic, the proof is done in the same way by expressing it as a quotient of two holomorphic functions). This provides

$$(2.7)\quad \frac{dY_1}{dZ_1} = \frac{Z_1^{1/3} - 2Y_1 \sum a_{ij} Z_1^{-(i+2j)/3} Y_1^j}{-3Z_1 \sum a_{ij} Z_1^{-(i+2j)/3} Y_1^j}.$$

Since the right hand side is meromorphic, $a_{ij} \neq 0$ only when $i + 2j \in 3\mathbb{Z} - 1$. Then,

$$(2.8)\quad \frac{dY_1}{dZ_1} = \frac{1 - 2Y_1 \sum_{i+2j=3n-1} a_{ij} Z_1^{-n} Y_1^j}{-3Z_1 \sum_{i+2j=3n-1} a_{ij} Z_1^{-n} Y_1^j}.$$

Hence, the third equation is of the form

$$\frac{dX_3}{dY_3} = \sum_{j,n} a_{3n-1-2j,j} X_3^{3n-1-2j} Y_3^j. \tag{2.9}$$

It is easy to verify that this equation is invariant under the $\mathbb{Z}_3$-action $(X_3, Y_3) \mapsto (e^{2\pi i/3}X_3, e^{4\pi i/3}Y_3)$. Similarly, we can show that the second equation is invariant under the $\mathbb{Z}_2$-action $(X_2, Z_2) \mapsto (-X_2, -Z_2)$. This proves the lemma. □

Although we use only $\mathbb{C}P^2(1,2,3)$ in this paper, the above properties are common among any weighted projective spaces.

§ 3. A compactified Airy equation

Let us consider the weighted projective space $\mathbb{C}P^2(1,2,3)$ with the inhomogeneous coordinates $(Y_1, Z_1), (X_2, Z_2), (X_3, Y_3)$ satisfying the relation (2.4). On the third coordinate, we give the Airy equation $dX_3/dY_3 = X_3^2 - Y_3$. This induces a well-defined meromorphic differential equation on $\mathbb{C}P^2(1,2,3)$. Indeed, the relation (2.4) transforms the Airy equation into

$$\frac{dY_1}{dZ_1} = \frac{Z_1 + 2Y_1(Y_1 - 1)}{3Z_1(Y_1 - 1)}, \quad \frac{dX_2}{dZ_2} = \frac{2 - 2X_2^2 + X_2Z_2}{3Z_2^2}, \quad \frac{dX_3}{dY_3} = X_3^2 - Y_3. \tag{3.1}$$

Since they are meromorphic, they define a meromorphic differential equation on $\mathbb{C}P^2(1,2,3)$ due to Lemma 2.2. Note that the sets $\{Z_1 = 0\}$ and $\{Z_2 = 0\}$ correspond to $\{X_3 = \infty\}$ and $\{Y_3 = \infty\}$, respectively. Hence, the first two equations of (3.1) describe behavior of the Airy equation near infinity. In this sense, we call the system (3.1) a compactified Airy equation on $\mathbb{C}P^2(1,2,3)$.

Remark. The relation (2.4) shows that X_2 and Y_1 satisfy $X_2^2 = Y_1^{-1}$. We can see that this is a coordinate transformation between inhomogeneous coordinates on the weighted projective space $\mathbb{C}P^1(1,2)$. Thus, we have a cellular decomposition of $\mathbb{C}P^2(1,2,3)$ as

$$\mathbb{C}P^2(1,2,3) = \mathbb{C}^2/\mathbb{Z}_3 \ \cup \ \mathbb{C}P^1(1,2), \quad \text{(disjoint)}, \tag{3.2}$$

where $\mathbb{C}^2 = \{(X_3, Y_3)\}$ and $\mathbb{C}P^1(1,2) = \{(Y_1, 0)\} \cup \{(X_2, 0)\}$ related by $X_2^2 = Y_1^{-1}$. This implies that $\mathbb{C}P^2(1,2,3)$ is obtained by attaching $\mathbb{C}P^1(1,2)$ to $\mathbb{C}^2/\mathbb{Z}_3$ at "infinity".

We will use several theorems on dynamical systems (vector fields). For this purpose, it is convenient to regard Eq.(3.1) as 2-dim dynamical systems

$$\begin{cases} \dot{Y}_1 = 2Y_1 + \dfrac{Z_1}{Y_1 - 1} \\ \dot{Z}_1 = 3Z_1, \end{cases} \quad \begin{cases} \dot{X}_2 = 2 - 2X_2^2 + X_2Z_2 \\ \dot{Z}_2 = 3Z_2^2, \end{cases} \quad \begin{cases} \dot{X}_3 = X_3^2 - Y_3 \\ \dot{Y}_3 = 1, \end{cases} \tag{3.3}$$

where ($\dot{}$) denotes the derivative d/dt and $t \in \mathbb{C}$ is an additional parameter. Fixed points of the vector fields are given by $(Y_1, Z_1) = (0, 0)$ and $(X_2, Z_2) = (\pm 1, 0)$, which play an important role. We can show that any solutions (X_3, Y_3) of the Airy equation satisfying $X_3 \to \infty$ or $Y_3 \to \infty$ approach to one of the fixed points. Hence, local analysis near the fixed points based on the dynamical systems theory gives much information on the asymptotic behavior of the Airy equation.

§ 4. Meromorphy of solutions of the Airy equation

Now we prove that any solutions $X_3 = X_3(Y_3)$ of the Airy equation are meromorphic functions by using the above setting. Of course, it is very easy to prove it if we use the fact that the Airy equation comes from the linear equation $u'' = yu$. Nevertheless, our proof without using a linear equation is significant because it is also applicable to more higher order equations such as the Painlevé equations. Since our proof is based on Poincaré's linearization theorem of vector fields, we give a simple review of it.

Let $Ax + f(x)$ be a holomorphic vector field on $\mathbb{C}^n$ with a fixed point $x = 0$, where A is an $n \times n$ constant matrix and $f(x) \sim O(|x|^2)$ is a nonlinearity. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of A. We consider the following two conditions:

(Nonresonance) There are no $j \in \{1, \cdots, n\}$ and non-negative integers $m_1, \cdots, m_n$ satisfying the resonant condition

$$m_1\lambda_1 + \cdots + m_n\lambda_n = \lambda_j, \quad (m_1 + \cdots + m_n \geq 2). \tag{4.1}$$

(Poincaré domain) The convex hull of $\{\lambda_1, \cdots, \lambda_n\}$ in $\mathbb{C}$ does not include the origin.

Theorem 4.1 (Poincaré. See [1] for the proof)**.** *Suppose that A is diagonal and eigenvalues satisfy the above two conditions. Then, there exists a local analytic transformation $y = x + \varphi(x)$, $\varphi(x) \sim O(|x|^2)$ defined near the origin such that the equation $dx/dt = Ax + f(x)$ is transformed into the linear system $dy/dt = Ay$.*

We will give an idea of the proof later.

Theorem 4.2. *There exists a local holomorphic function $\varphi(Y_1, Z_1)$ defined near $(Y_1, Z_1) = (0, 0)$ such that $\varphi(0, 0) = 0$ and the Airy equation $dX_3/dY_3 = X_3^2 - Y_3$ is transformed into the integrable equation $dx/dy = x^2$ by the local transformation*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_3 \\ Y_3 + X_3^{-1}\varphi(Y_3 X_3^{-2}, X_3^{-3}) \end{pmatrix}. \tag{4.2}$$

Since $\varphi(Y_3 X_3^{-2}, X_3^{-3})$ is holomorphic near $X_3 = \infty$, we can say that the Airy equation is locally integrable near each singularities of solutions.

Proof. Suppose that a solution $X_3 = X_3(Y_3)$ of the Airy equation is not holomorphic at some finite $Y_3 = Y_*$. If $X_3(Y_*)$ is finite, a fundamental theorem on ODEs proves that $X_3(Y_3)$ is holomorphic near Y_*. Thus, we consider a solution such that $X_3 \to \infty$ as $Y_3 \to Y_*$. Because of (2.4), $(Y_1, Z_1) \to (0, 0)$ as $Y_3 \to Y_*$, which is a fixed point of the first vector field of (3.3). Eigenvalues of the Jacobian matrix of this vector field are $\lambda = 2, 3$, and they satisfy the conditions for Poincaré's theorem. To apply it, put $\hat{Y}_1 = Y_1 + Z_1$. Then, the first equation of (3.3) is transformed into

$$\frac{d\hat{Y}_1}{dt} = 2\hat{Y}_1 + \frac{Z_1\hat{Y}_1 - Z_1^2}{\hat{Y}_1 - Z_1 - 1}, \quad \frac{dZ_1}{dt} = 3Z_1, \tag{4.3}$$

and the linear part becomes diagonal. Now Poincaré's theorem proves that there is a local analytic transformation

$$\begin{pmatrix} \hat{u} \\ v \end{pmatrix} = \begin{pmatrix} \hat{Y}_1 + \phi_1(\hat{Y}_1, Z_1) \\ Z_1 + \phi_2(\hat{Y}_1, Z_1) \end{pmatrix}, \quad \phi_1, \phi_2 \sim O(|x|^2),$$

such that Eq.(4.3) is linearized as $d\hat{u}/dt = 2\hat{u},\ dv/dt = 3v$. We can prove that $\phi_2 \equiv 0$ because the equation $\dot{Z}_1 = 3Z_1$ is already linear (thus, we need not change Z_1). Further, the function ϕ_1 can be written as $\phi_1(\hat{Y}_1, Z_1) = Z_1\hat{\varphi}(\hat{Y}_1, Z_1)$, where $\hat{\varphi} \sim O(\hat{Y}_1, Z_1)$ is a local holomorphic function. This follows from the fact that when $Z_1 = 0$, then Eq.(4.3) is already linear, so that $\phi_1(\hat{Y}_1, 0) = 0$. Hence,

$$\begin{pmatrix} \hat{u} \\ v \end{pmatrix} = \begin{pmatrix} \hat{Y}_1 + Z_1\hat{\varphi}(\hat{Y}_1, Z_1) \\ Z_1 \end{pmatrix}, \quad \hat{\varphi} \sim O(\hat{Y}_1, Z_1). \tag{4.4}$$

Now we have performed the series of transformations

$$\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} \mapsto \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} Y_3X_3^{-2} \\ X_3^{-3} \end{pmatrix} \mapsto \begin{pmatrix} \hat{Y}_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} Y_1 + Z_1 \\ Z_1 \end{pmatrix} \mapsto \begin{pmatrix} \hat{u} \\ Z_1 \end{pmatrix} \tag{4.5}$$

to obtain the linear system $d\hat{u}/dt = 2\hat{u},\ dZ_1/dt = 3Z_1$. Next, we are back to the original coordinate by the inverse transformations given by

$$\begin{pmatrix} \hat{u} \\ Z_1 \end{pmatrix} \mapsto \begin{pmatrix} u \\ Z_1 \end{pmatrix} := \begin{pmatrix} \hat{u} - Z_1 \\ Z_1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} Z_1^{-1/3} \\ uZ_1^{-2/3} \end{pmatrix}.$$

Then, the system $d\hat{u}/dt = 2\hat{u},\ dZ_1/dt = 3Z_1$ is transformed into the equation $dx/dy = x^2$. Eq.(4.2) is obtained by combining all transformations above if we put $\hat{\varphi}(\hat{Y}_1, Z_1) = \hat{\varphi}(Y_1 + Z_1, Z_1) := \varphi(Y_1, Z_1)$. □

The equation $dx/dy = x^2$ is solved as $x = (C - y)^{-1}$, where $C \in \mathbb{C}$ is an integral constant. By the transformation (4.2), we obtain the local first integral of the Airy equation as

$$Y_3 + X_3^{-1} + X_3^{-1}\varphi(Y_3X_3^{-2}, X_3^{-3}) = C. \tag{4.6}$$

We will show later that this is actually a global first integral.

Corollary 4.3. *Any solutions of the Airy equation $dX_3/dY_3 = X_3^2 - Y_3$ are meromorphic.*

Proof. Suppose that a solution $X_3 = X_3(Y_3)$ of the Airy equation is not holomorphic at some finite $Y_3 = Y_*$. As was explained in the above proof, we assume that $X_3 \to \infty$ as $Y_3 \to Y_*$. Near the point $(X_3, Y_3) = (\infty, Y_*)$, we have the first integral (4.6). Since $X_3 \to \infty$ as $Y_3 \to Y_*$, it turns out that $C = Y_*$. Put $X_3^{-1} = \xi$;

$$Y_3 + \xi + \xi\varphi(Y_3\xi^2, \xi^3) - Y_* = 0.$$

Since $\varphi(Y_3\xi^2, \xi^3) \sim O(\xi^2)$, it is easy to verify that the derivative of the above with respect to both of ξ and Y_3 at $(\xi, Y_3) = (0, Y_*)$ are not zero. Hence, the implicit function theorem proves that the above relation is locally solved as $\xi = g(Y_3)$, where $g(Y_3)$ is holomorphic near Y_* and $g(Y_*) = 0, g'(Y_*) \neq 0$. Therefore, $X_3 = 1/g(Y_3)$ has a pole of first order at Y_*. □

The next purpose is to show that the local holomorphic function φ in Theorem 4.2 has an analytic continuation to a sufficiently large domain. In general, the transformation $y = x + \varphi(x)$ in Thm.4.1 is biholomorphic from a small neighborhood U of the origin onto a small neighborhood V of the origin. However, the function $x + \varphi(x)$ may have an analytic continuation to a larger domain, although it is not *bi*holomorphic (in particular, it is not injective) outside U. To explain it, recall a proof of Poincaré's theorem.

Suppose that a vector field $Ax + f(x)$ satisfying the conditions for Poincaré's theorem is linearized by the transformation $y = x + \varphi(x)$, $\varphi(x) \sim O(|x|^2)$. Substituting $y = x + \varphi(x)$ into $\dot{y} = Ay$ yields

$$\dot{x} + \frac{\partial \varphi}{\partial x}(x)\dot{x} = Ax + A\varphi(x).$$

Since $\dot{x} = Ax + f(x)$, φ satisfies the partial differential equation

$$\frac{\partial \varphi}{\partial x}(x)(Ax + f(x)) = A\varphi(x) - f(x) \tag{4.7}$$

$$\Big(\sum_{j=1}^{n} \frac{\partial \varphi_k}{\partial x_j}(x)(\lambda_j x_j + f_j(x)) = \lambda_k \varphi_k(x) - f_k(x),\ k = 1, \cdots, n\Big).$$

The existence of a solution $\varphi(x)$ can be proved by the contraction mapping principle on a certain Banach space of local holomorphic functions $h(x)$ such that $h \sim O(|x|^2)$. See [1] for the details.

Let U be a neighborhood of the origin on which $\varphi(x)$ is defined and holomorphic. Our purpose is to construct an analytic continuation of φ. Let $\phi_t(x_0)$ be the flow of the vector field $Ax+f(x)$ (i.e. a solution of $\dot{x}=Ax+f(x)$ satisfying the initial condition $x(0)=x_0$). We will show that φ is analytically continued along the flow ϕ_t.

Proposition 4.4. *Let S be an analytic hypersurface ($(n-1)$-dim complex manifold) in $U\subset\mathbb{C}^n$. Suppose that at each point $x_0\in S$, an integral curve of the vector field $Ax+f(x)$ transversely intersects S. Then, the function $\varphi(x)$ has an analytic continuation from U to the region $\{\phi_t(x_0)\,|\,t\in\mathbb{C},\, x_0\in S\}\cap\mathbb{C}^n$.*

Proof. Since Eq.(4.7) is a first order linear PDE of φ, it is integrated by the characteristic curve method; that is, we assume that along a characteristic curve $x(t)$, $\varphi(x(t))$ satisfies an ODE

$$\frac{d}{dt}\varphi(x(t))=A\varphi(x(t))-f(x(t)), \tag{4.8}$$

where a characteristic curve is given by an integral curve of $\dot{x}=Ax+f(x)$ due to Eq.(4.7). Denote the curve $x(t)=\phi_t(x_0)$ by using the flow. Along this curve, Eq.(4.8) is integrated as

$$\varphi(\phi_t(x_0))=e^{At}\Big[-\int_0^t e^{-As}f(\phi_s(x_0))ds+C\Big],\quad C=\varphi(x_0).$$

Now we take an analytic hypersurface S. We locally express S as a graph of a holomorphic function $x=h(\tau)$, $\tau\in\mathbb{C}^{n-1}$. Put $x_0=h(\tau)\in S\subset U$. Then, $\varphi(h(\tau))$ is holomorphic and

$$\varphi(\phi_t(h(\tau)))=e^{At}\Big[-\int_0^t e^{-As}f(\phi_s(h(\tau)))ds+\varphi(h(\tau))\Big].$$

This shows that $\varphi(\phi_t(h(\tau)))$ is holomorphic in $(t,\tau)\in\mathbb{C}^n$ as long as $\varphi(\phi_t(h(\tau)))$ is bounded. To prove that $\varphi(x)$ is holomorphic at a point $x=\phi_t(h(\tau))$, it is sufficient to show that the Jacobian matrix of $\phi_t(h(\tau))$ with respect to (t,τ) is nonsingular.

Since ϕ_t is a flow of the vector field $g(x):=Ax+f(x)$, the Jacobian matrix of $\phi_t(h(\tau))$ is given by

$$J=\left(g(\phi_t(h(\tau))),\ \frac{\partial\phi_t}{\partial x}(h(\tau))\frac{\partial h}{\partial\tau}\right)=\frac{\partial\phi_t}{\partial x}(h(\tau))\left(\frac{\partial\phi_t}{\partial x}(h(\tau))^{-1}g(\phi_t(h(\tau))),\ \frac{\partial h}{\partial\tau}\right). \tag{4.9}$$

It is well known that the derivative $\partial\phi_t/\partial x$ of the flow is nonsingular because it is a fundamental solution of the variational equation

$$\frac{d}{dt}\left(\frac{\partial\phi_t}{\partial x}\right)=\frac{\partial g}{\partial x}(\phi_t)\cdot\left(\frac{\partial\phi_t}{\partial x}\right) \tag{4.10}$$

Next, we have

$$\frac{d}{dt}\left(\frac{\partial \phi_t}{\partial x}(h(\tau))\right)^{-1} g(\phi_t(h(\tau)))$$
$$= -\left(\frac{\partial \phi_t}{\partial x}\right)^{-1} \cdot \frac{d}{dt}\left(\frac{\partial \phi_t}{\partial x}\right) \cdot \left(\frac{\partial \phi_t}{\partial x}\right)^{-1} g(\phi_t) + \left(\frac{\partial \phi_t}{\partial x}\right)^{-1} \cdot \frac{\partial g}{\partial x}(\phi_t) g(\phi_t).$$

Substituting Eq.(4.10) provides

$$\frac{d}{dt}\left(\frac{\partial \phi_t}{\partial x}(h(\tau))\right)^{-1} g(\phi_t(h(\tau))) = 0.$$

Hence,

$$\left(\frac{\partial \phi_t}{\partial x}(h(\tau))\right)^{-1} g(\phi_t(h(\tau))) = g(h(\tau)).$$

Therefore,

$$J = \frac{\partial \phi_t}{\partial x}(h(\tau)) \left(g(h(\tau)),\ \frac{\partial h}{\partial \tau}\right). \tag{4.11}$$

By the assumption for the surface S, the above matrix is nonsingular. □

Now we are back to the Airy equation. Let $\varphi(Y_1, Z_1)$ be a local holomorphic function defined near $(Y_1, Z_1) = (0, 0)$ described in Thm.4.2.

Theorem 4.5. *The function $\varphi(Y_1, Z_1)$ has a (multi-valued) analytic continuation to the region $\{(Y_1, Z_1) \,|\, Z_1 \neq 0\}$. In particular, Eq.(4.6) gives a global first integral which is holomorphic on the region $\{(X_3, Y_3) \,|\, X_3 \neq 0\}$.*

Proof. Recall that the function $\varphi(Y_1, Z_1)$ is obtained by applying the Poincaré's theorem to the first vector field of (3.3). Let $U \subset \mathbb{C}^2$ be a neighborhood of $(Y_1, Z_1) = (0, 0)$ on which φ is holomorphic.

Let $\delta > 0$ be a sufficiently small number and take an analytic hypersurface (curve) S in U defined by $(Y_1, Z_1) = (\tau, \delta)$, $\tau \in \mathbb{C}$. The tangent vector of S is $(1, 0)$, which is transverse to the first vector field of (3.3) when $Z_1 \neq 0$. Hence, $\varphi(Y_1, Z_1)$ has an analytic continuation along integral curves of the vector field starting at points on $S \subset U$.

Now we use the well known fact that any solutions of the Airy equation $X_3' = X_3^2 - Y_3$ have poles (see the next proposition). Moving to the (Y_1, Z_1) coordinate, this implies that for any initial point (Y_0, Z_0) such that $Z_0 \neq 0$, a solution of the first equation of (3.3) can approach to $(Y_1, Z_1) = (0, 0)$ and intersects with S if $\delta > 0$ is sufficiently small. In other words, integral curves starting at points on S can reach any points (Y_0, Z_0), $Z_0 \neq 0$. This fact and Prop.4.4 complete the proof. □

As an application of the dynamical systems theory, let us show the following known result without using a linear equation.

Proposition 4.6. *(i) Any solutions of the Airy equation $X_3' = X_3^2 - Y_3$ have infinitely many poles. (ii) A position of each pole analytically depends on an initial condition.*

Proof. Fix a solution $X_3 = h(Y_3)$ of the Airy equation. It is sufficient to show the existence of poles for large Y_3 (actually they accumulate at $Y_3 = \infty$). When Y_3 is large, then $Z_2 = Y_3^{-3/2}$ is small. Thus it is convenient to use the second system of (3.3), say E_2, with small Z_2.

Give an initial condition $(X_2, Z_2) = (u, v)$ for E_2, which lies on the solution $X_3 = h(Y_3)$. Let us consider the approximate dynamical system

$$\dot{X}_2 = 2 - 2X_2^2, \quad \dot{Z}_2 = 3Z_2^2. \tag{4.12}$$

This is solved as

$$X_2(t) = \frac{1 + X_0 e^{-4t}}{1 - X_0 e^{-4t}},\ Z_2(t) = \frac{v}{1 - 3tv}, \quad \left(X_0 = \frac{u-1}{u+1}\right). \tag{4.13}$$

There is a path $\{\tau e^{i\theta} \,|\, 0 \le \tau < \infty\}$ in the t-plane such that when $|Z_2(0)| = |v| < \varepsilon_1$, then $|Z_2(t)| < \varepsilon_1$ for any $t > 0$ and $|X_2(t)| \to \infty$ along the path. Now we regard the system E_2 as a perturbation of Eq.(4.12). Since solutions are continuous with respect to a small perturbation of a vector field, for any positive number M, there is $\varepsilon_1 > 0$ and a time t_0 such that when $|v| < \varepsilon_1$, then $|Z_2(t_0)| < \varepsilon_1$ and $|X_2(t_0)| > M$. Since $Y_1 = X_2^{-2}$ and $Z_1 = X_2^{-3} Z_2$, it follows that if $\varepsilon_1 > 0$ is sufficiently small, then the solution of E_2 written in the (Y_1, Z_1) coordinate passes through inside of U, where U is a neighborhood of $(Y_1, Z_1) = (0, 0)$, on which Thm.4.2 is valid. Then, the equation is transformed into $x' = x^2$, and the solution has a pole. Let $Y_3 = \zeta$ be the position of the pole.

Next, take a different initial value (u, v) for the system E_2, which lies on the solution $X_3 = h(Y_3)$, such that $|v| < \varepsilon_2 << |\zeta|^{-3/2}$. By the same argument as above, we have $|Z_2(t_0)| < \varepsilon_2$ and $|X_2(t_0)| > M$ for some t_0. Thus we find a pole of the solution again.

Let us estimate the position of the latter pole. Inside U, we have the local first integral (4.6). The number C gives a position of a pole because $Y_3 \to C$ as $X_3 \to \infty$ in (4.6). In the (X_2, Z_2) coordinate, (4.6) is rewritten as

$$Z_2^{-2/3} + X_2^{-1} Z_2^{1/3} + X_2^{-1} Z_2^{1/3} \varphi(X_2^{-2}, X_2^{-3} Z_2) = C. \tag{4.14}$$

Therefore, the position of the latter pole $Y_3 = Y_*$ is estimated as

$$Y_* = Z_2(t_0)^{-2/3} + O(1/M), \quad |Y_*| > \varepsilon_2^{-2/3} + O(1/M) >> |\zeta| + O(1/M).$$

Hence, the latter pole Y_* is different from the first one ζ. Repeating this procedure, we can find infinitely many poles. Part (ii) of the proposition immediately follows from (4.6). □

§ 5. A characterization of the Airy equation

In the previous section, we have shown for the Airy equation $X_3' = X_3^2 - Y_3$ that
(i) it induces a meromorphic equation on $\mathbb{C}P^2(1,2,3)$; the Airy equation is also meromorphic in (Y_1, Z_1) and (X_2, Z_2) coordinates.
(ii) there is a hyperbolic fixed point $(Y_1, Z_1) = (0,0)$ of the corresponding dynamical system (3.3), whose Jacobian matrix is given by

$$J = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}. \tag{5.1}$$

The eigenvalues $\lambda = 2, 3$ allow us to apply Poincaré's linearization theorem. Here, let us observe that the $(1,2)$-component of J $(= -1)$ also plays an important role. If the $(1,2)$-component were zero, that is, if an equation on (Y_1, Z_1) coordinate were of the form

$$\begin{cases} \dot{Y}_1 = 2Y_1 + O(Y_1^2, Y_1 Z_1, Z_1^2) \\ \dot{Z}_1 = 3Z_1 \end{cases},$$

then, we can show the following by the same way as Thm.4.2; by the coordinate transformation of the form (4.2), $X_3' = X_3^2 - Y_3$ is transformed into the equation $\dot{y} = 0$. Since $y = C =$ constant, we obtain the first integral

$$Y_3 + X_3^{-1}\varphi(Y_3 X_3^{-2}, X_3^{-3}) = C, \tag{5.2}$$

(compare with Eq.(4.6)). In this case, Cor.4.3 is not true because the implicit function theorem is not applicable (ξ-derivative vanishes).

In this section, we prove that the above properties (i),(ii) uniquely determine the Airy equation.

Theorem 5.1. *Consider the space $\mathbb{C}P^2(1,2,3)$ with the inhomogeneous coordinates $(Y_1, Z_1), (X_2, Z_2), (X_3, Y_3)$. Give a differential equation*

$$\frac{dX_3}{dY_3} = f(X_3, Y_3), \tag{5.3}$$

on the third coordinate, where f is holomorphic in X_3 and meromorphic in Y_3. For this equation, suppose that
(i) *it is also a meromorphic equation in (Y_1, Z_1) and (X_2, Z_2) coordinates.*
(ii) *the corresponding 2-dim vector field has a hyperbolic fixed point $(Y_1, Z_1) = (0,0)$. The $(1,2)$-component of its Jacobian matrix is not zero.*

Then, Eq.(5.3) is of the form

$$\frac{dX_3}{dY_3} = a_2 X_3^2 + a_1 Y_3, \quad a_2, a_1 \in \mathbb{C},\ a_2 \neq 0. \tag{5.4}$$

In particular, when $a_1 = 0$*, it is equivalent to the integrable equation* $X_3' = X_3^2$*, and when* $a_1 \neq 0$*, it is equivalent to the Airy equation.*

The condition (i) means that a given equation is meromorphic on $\mathbb{C}P^2(1,2,3)$ due to Lemma 2.2. Hence, the condition (i) is a global condition which reflects a structure of $\mathbb{C}P^2(1,2,3)$. On the other hand, the condition (ii) is an assumption only for one point. Thus, we may say that

$$\begin{pmatrix} \text{Airy equation} \\ x' = x^2 \end{pmatrix} = (\text{structure of } \mathbb{C}P^2(1,2,3)) + (\text{local behavior at one point}).$$

Actually, the global first integral (4.6) was constructed by the analysis at one point $(Y_1, Z_1) = (0,0)$. Note that we can not distinguish the Airy and $x' = x^2$ by the condition (ii) because of Thm.4.2. The proof of this theorem will be given in the end of this section.

The next theorem is motivated by the following fact. Recall that $\mathbb{C}P^2(1,2,3)$ admits the decomposition (3.2). The set $\mathbb{C}^2/\mathbb{Z}_3$ corresponds to the (X_3, Y_3)-space, and the set $\mathbb{C}P^1(1,2)$ corresponds to the region $\{Z_1 = 0\} \cup \{Z_2 = 0\}$ (i.e. $\{X_3 = \infty\} \cup \{Y_3 = \infty\}$). It is remarkable that the set $\mathbb{C}P^1(1,2)$ is an invariant manifold of the dynamical system (3.3); if $Z_1 = 0$ (resp. $Z_2 = 0$) at an initial time, then $Z_1 = 0$ (resp. $Z_2 = 0$) for all time. On the invariant manifold, the dynamical system is reduced to

$$\dot{Y}_1 = 2Y_1(Y_1 - 1), \quad \dot{X}_2 = 2 - 2X_2^2, \tag{5.5}$$

which governs the behavior of the Airy equation at "infinity" (here, we rewrite the first equation of (3.3) as a polynomial vector field $\dot{Y}_1 = 2Y_1(Y_1 - 1) + Z_1$, $\dot{Z}_1 = 3Z_1(Y_1 - 1)$ to avoid the singularity $Y_1 = 1$). Now we show that the dynamics at infinity uniquely determines the Airy equation.

Theorem 5.2. *Consider the space* $\mathbb{C}P^2(1,2,3)$ *with the inhomogeneous coordinates* $(Y_1, Z_1), (X_2, Z_2), (X_3, Y_3)$*. Give a differential equation*

$$\frac{dX_3}{dY_3} = f(X_3, Y_3), \tag{5.6}$$

on the third coordinate, where f *is holomorphic in* X_3 *and* Y_3*. For this equation, suppose that*

(i) *it is also a meromorphic equation in* (Y_1, Z_1) *and* (X_2, Z_2) *coordinates.*

(ii) *when* $Z_1 = 0$ *and* $Z_2 = 0$*, the corresponding* 2*-dim polynomial vector field is reduced to (5.5).*

Then, Eq.(5.6) is the Airy equation.

Since $\mathbb{C}P^1(1,2)$ is a codimension 1 submanifold, again the Airy equation is characterized by a structure of $\mathbb{C}P^2(1,2,3)$ and a local condition. The proof of this theorem is similar to that of Thm.5.1 and omitted.

Proof of Thm.5.1. At first, we show that $f(X_3, Y_3)$ is polynomial in X_3 and rational in Y_3. In the (X_2, Z_2) coordinate, the equation $X_3' = f(X_3, Y_3)$ is written as

$$\frac{dX_2}{dZ_2} = \frac{X_2 Z_2 - 2Z_2^{2/3} f(X_2 Z_2^{-1/3}, Z_2^{-2/3})}{3Z_2^2}.$$

Due to the assumption (i), $Z_2^{2/3} f(X_2 Z_2^{-1/3}, Z_2^{-2/3})$ is meromorphic. Putting $u_2 = Z_2^{1/3}$ shows that $f(X_2 u_2^{-1}, u_2^{-2})$ is meromorphic in u_2. By the assumption for f, $f(X_2 u_2, u_2^2)$ is also meromorphic in u_2. Since a meromorphic function on $\mathbb{C}P^1$ is a rational function, it turns out that $f(X_2 u_2, u_2^2)$ is rational in u_2. Thus $f(X_2 u_2, u_2^2)$ is expressed as

$$f(X_2 u_2, u_2^2) = \frac{\sum a_j(X_2) u_2^j}{\sum b_j(X_2) u_2^j}, \quad \text{(finite sum)}, \tag{5.7}$$

where a_j and b_j are meromorphic. Similarly, considering in (Y_1, Z_1) coordinate shows that $f(u_1, Y_1^2 u_1^2)$ is rational in u_1 and meromorphic in Y_1. Putting $u_2 = Y_1 u_1,\ X_2 = Y_1^{-1}$ in Eq.(5.7) yields

$$f(u_1, Y_1^2 u_1^2) = \frac{\sum a_j(Y_1^{-1}) Y_1^j u_1^j}{\sum b_j(Y_1^{-1}) Y_1^j u_1^j}.$$

Thus, $a_j(Y_1^{-1}), b_j(Y_1^{-1})$ are meromorphic in Y_1. Since both of $a_j(X)$ and $a_j(X^{-1})$ are meromorphic, a_j is rational, and so is b_j. Hence, $f(X_3, Y_3)$ is rational in X_3 and Y_3. By the assumption for f, it is polynomial in X_3.

Therefore, we assume that f is written as a quotient of polynomials as

$$f(X, Y) = \frac{\sum_{i,j=0} a_{ij} X^i Y^j}{\sum_{j=0} b_j Y^j}, \tag{5.8}$$

where the right hand side is a finite sum. Then, the the equation $X_3' = f(X_3, Y_3)$ is written as

$$\frac{dY_1}{dZ_1} = \frac{1}{3Z_1}\left(2Y_1 - \frac{\sum b_j Y_1^j Z_1^{-(2j-1)/3}}{\sum a_{ij} Y_1^j Z_1^{-(i+2j)/3}}\right), \tag{5.9}$$

$$\frac{dX_2}{dZ_2} = \frac{1}{3Z_2^2}\left(X_2 Z_2 - 2\frac{\sum a_{ij} X_2^i Z_2^{-(i+2j)/3}}{\sum b_j Z_2^{-(2j+2)/3}}\right), \tag{5.10}$$

in (Y_1, Z_1) and (X_2, Z_2) coordinates, respectively. Since they are meromorphic, they have to satisfy

$$\begin{cases} a_{ij} \neq 0 \ \text{ only if } \ i + 2j = 3m + \delta \ (m = 0, \cdots, M), \\ b_j \neq 0 \ \text{ only if } \ 2j = 3n - 2 + \delta \ (n = 0, \cdots, N), \end{cases} \tag{5.11}$$

where $\delta \in \{0, 1, 2\}$ and M, N are maximum integers satisfying the above relations, which exist because f is rational. Substituting them into Eq.(5.9) yields

$$\frac{dY_1}{dZ_1} = \frac{1}{3Z_1}\left(2Y_1 - \frac{\sum b_j Y_1^j Z_1^{-n+1}}{\sum a_{ij} Y_1^j Z_1^{-m}}\right). \tag{5.12}$$

We regard it as a dynamical system

$$\begin{cases} \dot{Y}_1 = 2Y_1 - \dfrac{\sum b_j Y_1^j Z_1^{-n+1}}{\sum a_{ij} Y_1^j Z_1^{-m}}, \\ \dot{Z}_1 = 3Z_1. \end{cases} \tag{5.13}$$

(I) When $M \geq N$, we obtain

$$\begin{cases} \dot{Y}_1 = 2Y_1 - \dfrac{\sum b_j Y_1^j Z_1^{M-n+1}}{\sum a_{ij} Y_1^j Z_1^{M-m}}, \\ \dot{Z}_1 = 3Z_1. \end{cases} \tag{5.14}$$

The constant term $a_{3M+\delta,0}$ of $\sum a_{ij} Y_1^j Z_1^{M-m}$ has to be not zero so that $(Y_1, Z_1) = (0, 0)$ is a fixed point. The $(1, 2)$-component of the Jacobian matrix of the fixed point arises from a monomial Z_1 in the polynomial $\sum b_j Y_1^j Z_1^{M-n+1}$. In the polynomial, a monomial Z_1 exists only if $j = 0$ when $n = M$. The condition (5.11) provides $0 = 3M - 2 + \delta$. This yields $M = N = 0, \delta = 2$. Therefore, we obtain

$$\begin{cases} a_{ij} \neq 0 \;\text{ only if }\; i + 2j = 2, \\ b_j \neq 0 \;\text{ only if }\; 2j = 0. \end{cases} \tag{5.15}$$

This proves that nonzero numbers among a_{ij}, b_j are only a_{20}, a_{01} and b_0, and the equation is $X_3' = (a_{20}X_3^2 + a_{01}Y_3)/b_0$. In particular, the Jacobian matrix at the fixed point $(0, 0)$ of Eq.(5.14) is given by

$$J = \begin{pmatrix} 2 & -b_0/a_{20} \\ 0 & 3 \end{pmatrix}, \quad b_0 \neq 0, a_{20} \neq 0 \tag{5.16}$$

(II) When $M < N$, we obtain

$$\begin{cases} \dot{Y}_1 = 2Y_1 - \dfrac{\sum b_j Y_1^j Z_1^{N-n}}{\sum a_{ij} Y_1^j Z_1^{N-m-1}}, \\ \dot{Z}_1 = 3Z_1. \end{cases} \tag{5.17}$$

The constant term $a_{3(N-1)+\delta,0}$ of $\sum a_{ij} Y_1^j Z_1^{N-m-1}$ has to be not zero so that $(Y_1, Z_1) = (0, 0)$ is a fixed point. This proves $M = N - 1$. The $(1, 2)$-component of the Jacobian matrix of the fixed point arises from a monomial Z_1 in the polynomial $\sum b_j Y_1^j Z_1^{N-n}$.

In the polynomial, a monomial Z_1 exists only if $j = 0$ when $n = N - 1$. The condition (5.11) provides $0 = 3(N-1) - 2 + \delta$. This yields $N = 1, M = 0, \delta = 2$. Therefore, we obtain

$$\begin{cases} a_{ij} \neq 0 \;\; \text{only if} \;\; i + 2j = 2, \\ b_j \neq 0 \;\; \text{only if} \;\; 2j = 0, 3. \end{cases} \tag{5.18}$$

This proves that nonzero numbers are only a_{20}, a_{01} and b_0 as before. □

§ 6. Asymptotics at an irregular singular point

In the previous sections, we have investigated the fixed point $(Y_1, Z_1) = (0, 0)$ of the system (3.3). In this section, we investigate another fixed point $(X_2, Z_2) = (\pm 1, 0)$ (it is sufficient to consider one of them because they are related by the $\mathbb{Z}_2$ symmetry). At this fixed point, the Jacobian matrix of the second system of (3.3) has eigenvalues $\lambda = \mp 4, 0$. Since it has a zero eigenvalue, the system is not approximated by the linearized one. Instead, the system has one dimensional center manifold. This is related to the fact that $Y_3 = \infty$ is an irregular singular point of the Airy equation, at which there are no holomorphic solutions. In this section, we will investigate the irregular singular point via the center manifold theory. There are a lot of references on the center manifold theory. For analytic properties of center manifolds, [2] is well written and it provides all ingredients for our purpose.

In order to give a brief review of the center manifold theory, let us consider the general system

$$\begin{cases} \dot{x} = Ax + f_1(x, y), & x \in \mathbb{R}^m, \\ \dot{y} = By + f_2(x, y), & y \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

where A and B are matrices and $f_1, f_2 \sim (|x + y|^2)$ are C^r nonlinearities. We suppose that eigenvalues of A have nonzero real parts, and eigenvalues of B lie on the imaginary axis.

Theorem 6.1 (Center manifold theorem). *There is a neighborhood $V \subset \mathbb{R}^n$ of the origin and a C^r function $\varphi : V \to \mathbb{R}^m$ such that*
(i) $\varphi(0) = \varphi'(0) = 0$,
(ii) $(\varphi(y), y)$ is a local invariant manifold, which is called the center manifold.

A center manifold is exponentially attracting or repelling. Hence, long time behavior of the system is governed by the reduced system $\dot{y} = By + f_2(\varphi(y), y)$. This is a basic strategy to detect bifurcations of dynamical systems.

Remark. (i) Center manifolds are not unique.
(ii) Even if a given system is analytic, a center manifold is neither C^ω nor C^∞. Indeed, a neighborhood $V = V(r)$ in the theorem may shrink as $r \to \infty$. However, if we restrict the domain of φ to some sector, then φ may become analytic.
(iii) A formal expansion $\varphi(y) = \sum_{n=2}^{N} a_n y^n + o(y^N)$ of a center manifold is obtained by substituting this to the system. It is easy to see that coefficients $a_2, a_3, \cdots$ are uniquely determined. Thus, nonunique center manifolds have the common asymptotic expansion.

The existence of a center manifold is proved as follows: For the sake of simplicity, we suppose that all eigenvalues of A have negative real parts. In this case, a center manifold is attracting.

Let $(x(t), y(t))$ be a solution of Eq.(6.1) with the initial condition (x_0, y_0) at $t = 0$. We suppose that (x_0, y_0) lies on a (unknown) center manifold $x = \varphi(y)$. The first equation of (6.1) is integrated as

$$\varphi(y(t)) = e^{A(t-t_0)}\varphi(y(t_0)) + \int_{t_0}^{t} e^{A(t-s)} f_1(\varphi(y(s)), y(s))ds.$$

If $\varphi(y(t))$ moves slowly, putting $t = 0$ and $t_0 \to -\infty$ yields

$$\varphi(y_0) = \int_{-\infty}^{0} e^{-As} f_1(\varphi(y(s)), y(s))ds. \tag{6.2}$$

We want to obtain a center manifold as a solution of this integral equation. Unfortunately, the integral of the right hand side does not exist in general because $y(s)$ may diverge too rapidly as $s \to -\infty$. To handle this difficulty, we introduce a modified system.

Let $V_1 \subset V_2 \subset \mathbb{R}^n$ be small neighborhoods of the origin. Let $\chi(y)$ be a C^∞ function such that $\chi \equiv 1$ on V_1 and $\chi \equiv 0$ outside V_2. We consider the modified system $\dot{x} = Ax + f_1(x, y)\chi(y)$. The equation for φ becomes

$$\varphi(y_0) = \int_{-\infty}^{0} e^{-As} f_1(\varphi(y(s)), y(s))\chi(y(s))ds. \tag{6.3}$$

In this case, the integral is well defined and we can prove the existence of a unique solution $\varphi(y)$ by the contraction mapping principle on a certain Banach space when V_1 and V_2 are sufficiently small. This gives a global center manifold for the modified system. Since $\chi(y) \equiv 1$ on V_1, this gives a local center manifold on V_1 for the original system.

Since χ is not analytic and a choice is not unique, a local center manifold is not analytic and not unique. However, if y_0 is restricted to the region such that the negative orbit $y(s)$, $-\infty < s < 0$ is included in V_1, then $\chi(y(s)) \equiv 1$ on the orbit and (6.3) is

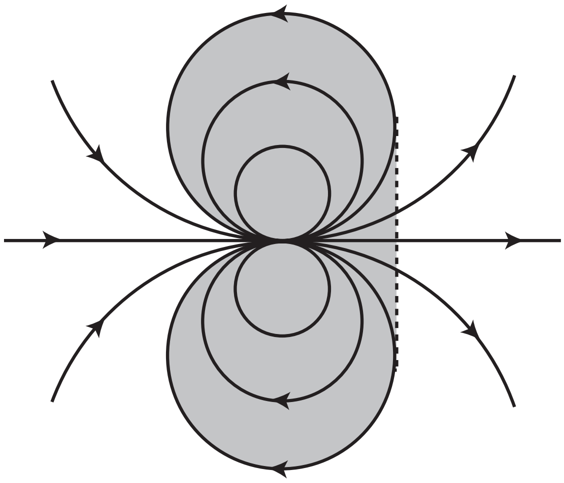

Figure 1. Flow of the equation $\dot{y} = 3y^2$. The gray region is W.

reduced to (6.2). Hence, if y_0 is restricted to such a region, then we can show that $\varphi(y_0)$ is uniquely determined and analytic (if a given system is analytic).

Now we are back to the Airy equation. For the second equation of (3.3), put $X_2 = 1 + x + y/4$ and $Z_2 = y$. Then, we obtain

$$\begin{cases} \dot{x} = -4x - 2x^2 - \dfrac{5}{8}y^2, \\ \dot{y} = 3y^2, \end{cases} \tag{6.4}$$

so that the origin is a fixed point and the Jacobian matrix is diagonal. This system has a center manifold $x = \varphi(y)$ near $y = 0$. Let us determine a sector on which φ is holomorphic. The equation $\dot{y} = 3y^2$ is solved as $y(t) = y_0/(1 - 3ty_0)$. Orbits $(\mathrm{Re}(y(t)), \mathrm{Im}(y(t)))$ as $t \in \mathbb{R}$ increases are shown in Fig.1. Let $V \subset \mathbb{C}$ be a neighborhood of $y = 0$, on which a local center manifold $x = \varphi(y)$ exists by the general theory. In V, define a region W so that if $y_0 \in W$, then $y(t) \in W$ for $-\infty < t < 0$, see Fig.1.

Note that for any $\varepsilon > 0$, there is an open neighborhood U of the origin such that $W \supset U \cap S(-\pi + \varepsilon, \pi - \varepsilon)$, where $S(-\pi + \varepsilon, \pi - \varepsilon)$ denotes the sector $-\pi + \varepsilon < \arg(y) < \pi - \varepsilon$. When $y_0 \in W$, a holomorphic center manifold is given as a unique solution of

$$\varphi(y_0) = \int_{-\infty}^{0} e^{4s} \left(-2\varphi(y(s))^2 - \frac{5}{8} y(s)^2 \right) ds, \quad y(s) = \frac{y_0}{1 - 3sy_0}, \ y_0 \in W. \tag{6.5}$$

It is easy to see that $x = \varphi(y)$ is a solution of the ODE

$$\frac{dx}{dy} = \frac{1}{3y^2} \left(-4x - 2x^2 - \frac{5}{8} y^2 \right) \tag{6.6}$$

corresponding to Eq.(6.4). Let us construct an analytic continuation of $\varphi(y_0)$, $y_0 \in W$. Our purpose is to show that

Theorem 6.2. *For the Airy equation $X_3' = X_3^2 - Y_3$, there is a unique solution $X_3 = \hat{A}(Y_3)$ satisfying the following.*
(i) For any $\varepsilon > 0$, there exists a positive number R such that $\hat{A}(Y_3)$ is holomorphic on the region

$$\{Y_3 \in \mathbb{C} \,|\, |Y_3| > R, \ -\pi + \varepsilon < \arg(Y_3) < \pi - \varepsilon\}. \tag{6.7}$$

(ii) Around $Y = \infty$, $\hat{A}(Y)$ has an asymptotic expansion

$$\hat{A}(Y) = Y^{1/2} + \frac{1}{4}Y^{-1} + \sum_{n=2}^{\infty} a_n Y^{1/2-3n/2}, \quad (Y \to \infty) \tag{6.8}$$

where a_n is determined by

$$a_2 = -\frac{5}{32}, \quad a_{k+1} = -\frac{3}{4}ka_k - \frac{1}{2}\sum_m a_m a_{k+1-m}. \tag{6.9}$$

Remark. (i) For the *linear* Airy equation, $d^2u/dy^2 = yu$, there is a unique solution such that zeros exist only on the negative real axis. Such a solution is well known as the Airy function $u = \mathrm{Ai}(y)$. Our solution is related with the Airy function by $\hat{A}(Y) = -\mathrm{Ai}'(Y)/\mathrm{Ai}(Y)$. Hence, $\hat{A}(Y)$ has poles only on the negative real axis. In this sense, our result is weaker than the known result because the region above is smaller than 2π by 2ε. It seems that it is very difficult to obtain such a strict result without using a linear equation.
(ii) Because of the symmetry $(X_3, Y_3) \mapsto (e^{2\pi i/3}X_3, e^{4\pi i/3}Y_3)$ of the equation $X_3' = X_3^2 - Y_3$, there are exactly three solutions having similar properties.
(iii) Let $\hat{A}_+(Y_3) = e^{-2\pi i/3}\hat{A}(e^{4\pi i/3}Y_3)$ be a solution obtained by the symmetry. It has the same asymptotic expansion as $\hat{A}(Y_3)$. Hence, it is expected that they are exponentially close to each other. Indeed, we can prove (without using a linear equation) that

$$\hat{A}(Y) - \hat{A}_+(Y) = C \cdot \exp[-\frac{4}{3}Y^{3/2}] \cdot (1 + o(Y)), \quad (Y \to \infty) \tag{6.10}$$

on a suitable sector. The constant C (so called the Stokes multiplier) can be obtained as

$$C = 2\pi i \cdot \left[\sum_{n=2}^{\infty} \frac{1}{(n-1)!}\left(\frac{4}{3}\right)^{n-1}\left((n-1)a_{n-1} - \frac{4}{3}a_n\right)\right], \tag{6.11}$$

(we omit the proof). A numerical simulation suggests that $C = 2i$, and we can confirm it if we use the linear equation. Again, in my opinion, it is very difficult to obtain the exact value of C without using the linear equation.
(iv) Our strategy based on the center manifold theory is applicable to any nonlinear

systems. For the first Painlevé equation, a solution obtained by the center manifold theory is called Boutroux's tritronquee solution.

Proof. Instead of Eq.(6.4), let us consider the system

$$\begin{cases} \dot{x} = e^{i\theta}(-4x - 2x^2 - \frac{5}{8}y^2), \\ \dot{y} = e^{i\theta}(3y^2), \end{cases} \tag{6.12}$$

for a fixed $-\pi/2 < \theta < \pi/2$. Note that the corresponding ODE is the same as Eq.(6.6). Define the region W_θ by rotating W by $-\theta$. Then, by the center manifold theory, a holomorphic center manifold of (6.12) defined on W_θ is uniquely given by a solution of the integral equation

(6.13)

$$\varphi_\theta(y_0) = \int_{-\infty}^{0} e^{4e^{i\theta}s} e^{i\theta} \left(-2\varphi_\theta(y(s))^2 - \frac{5}{8}y(s)^2 \right) ds, \quad y(s) = \frac{y_0}{1 - 3se^{i\theta}y_0},\ y_0 \in W_\theta.$$

Putting $e^{i\theta}s \mapsto s$ proves that $\varphi_\theta(y_0) = \varphi(y_0)$ when $y_0 \in W \cap W_\theta$. Hence, $\varphi_\theta(y_0)$ is an analytic continuation of $\varphi(y_0)$ to W_θ. This argument is valid as long as $-\pi/2 < \theta < \pi/2$; when $\theta = \pm\pi/2$, the stable eigenvalue $-4e^{i\theta}$ lies on the imaginary axis and the dimension of a center manifold changes. Therefore, for any $\varepsilon > 0$, there exists a positive number r such that

(i) $\varphi(y)$ is holomorphic in $\{y \,|\, 0 < |y| < r,\ -3\pi/2 + \varepsilon < \arg(y) < 3\pi/2 - \varepsilon\}$.

(ii) $x = \varphi(y)$ is a solution of Eq.(6.6).

(iii) $\varphi(y)$ has an asymptotic expansion of the form $\varphi(y) = \sum_{n=2}^{\infty} a_n y^n$, which is uniquely determined by (6.9).

(iv) A solution of (6.6) satisfying (i), (iii) is unique.

Changing to the (X_3, Y_3) coordinate, we obtain the theorem.

□

References

[1] S. N. Chow, C. Li, D. Wang, Normal forms and bifurcation of planar vector fields, Cambridge University Press, Cambridge, (1994).

[2] J. Sijbrand, Properties of center manifolds. Trans. Amer. Math. Soc. 289 (1985), no. 2, 431-469.

[3] W. Thurston, The geometry and topology of three-manifolds (chapter 13), 1978-1981. Lecture notes. http://msri.org/publications/books/gt3m/