# Spontaneous motion and deformation of a self-propelled droplet: Supplementary information

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## S1. PERTURBED CONCENTRATION

Here we show the explicit forms appearing in the expansion (41) of a concentration field close to the critical point of drift bifurcation. The second-order in the expansion is expressed in terms of a normal vector and a surface gradient derivative as

$$
\begin{aligned}
&c_I^{(2)}(\mathbf{r}_G+\mathbf{s})\\
=&\frac{A}{D^3}\int_{\mathbf{q}}(i\mathbf{q}\cdot\dot{\mathbf{u}})G_q^3S_qe^{-i\mathbf{q}\cdot\mathbf{s}}+\frac{A}{D^3}\int_{\mathbf{q}}(i\mathbf{q}\cdot\mathbf{u})^2G_q^3S_qe^{-i\mathbf{q}\cdot\mathbf{s}}\\
=&-\dot{u}_i\frac{A}{D^3}\left[n_i\left(\frac{\partial Q_3^{(0)}(s)}{\partial s}+\frac{\partial Q_3^{(1)}(\theta,\varphi)}{\partial s}\right)+\nabla_{s,i}Q_3^{(1)}(\theta,\varphi)\right]\\
&+u_iu_j\frac{A}{D^3}\left[\frac{1}{R_0}\mathcal{P}_{ij}\frac{\partial}{\partial s}+n_in_j\frac{\partial^2}{\partial s^2}\right]\left(Q_3^{(0)}(s)+Q_3^{(1)}\right)+u_iu_j\frac{A}{D^3}\left[(n_j\nabla_{s,i}+n_i\nabla_{s,j})\frac{\partial Q_3^{(1)}}{\partial s}+\nabla_{s,i}\nabla_{s,j}Q_3^{(1)}\right].
\end{aligned}
\tag{S1.1}
$$

Similarly, the third-order in the expansion is expressed as

$$
\begin{aligned}
&c_I^{(3)}(\mathbf{r}_G+\mathbf{s})\\
=&-\frac{A}{D^4}\int_{\mathbf{q}}(i\mathbf{q}\cdot\mathbf{u})^3G_q^4S_qe^{-i\mathbf{q}\cdot\mathbf{s}}\\
=&u_iu_ju_k\frac{A}{D^4}\left[\left(-\frac{1}{R_0^2}\left(⟦\delta_{ij}n_k⟧-3n_in_jn_k\right)\frac{\partial}{\partial s}+\frac{1}{R_0}\left(⟦\delta_{jk}n_i⟧-3n_in_jn_k\right)\frac{\partial^2}{\partial s^2}+n_in_jn_k\frac{\partial^3}{\partial s^3}\right)\left(Q_4^{(0)}(s)+Q_4^{(1)}(\theta,\varphi)\right)\right.\\
&\left.+\frac{1}{R_0}⟦\mathcal{P}_{ij}\nabla_{s,k}⟧\frac{\partial Q_4^{(1)}}{\partial s}+⟦n_in_j\nabla_{s,k}⟧\frac{\partial^2Q_4^{(1)}}{\partial s^2}+⟦n_j\nabla_{s,i}\nabla_{s,k}⟧\frac{\partial Q_4^{(1)}}{\partial s}+⟦\nabla_{s,i}\nabla_{s,j}\nabla_{s,k}⟧Q_4^{(1)}\right],
\end{aligned}
\tag{S1.2}
$$

where $⟦⟧$ shows two other permutations among $i,j,k$.

## S2. COEFFICIENTS OF THE AMPLITUDE EQUATIONS

### A. first mode

Using the concentration field obtained in the main text (41), the velocity resulting from the normal force becomes

$$
u_{i,1}^{(\alpha)}=\sum_{\alpha=0}^{5}\sum_{\beta=0}^{3}u_{i,1}^{(\alpha,\beta)}
\tag{S2.1}
$$

where $\alpha=0$ is the isotropic part in (28) and (31). For any $\alpha$, the zeroth-order contribution in the expansion of the concentration field vanishes. This is because there is no $l=1$ mode in deformation.

$$
u_{i,1}^{(\alpha,0)}=u_{i,2}^{(\alpha,0)}=0.
\tag{S2.2}
$$

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For $\alpha \geq 1$

$$u_{i,1}^{(\alpha,2)}, u_{i,2}^{(\alpha,2)} = \mathcal{O}\left(\dot{u}_j S_{ij}, u_j u_k T_{ijk}\right), \tag{S2.3}$$

$$u_{i,1}^{(\sigma,3)}, u_{i,2}^{(\sigma,3)} = \mathcal{O}\left(u_i u_j u_k S_{jk}, u_j u_k u_k D_{ijkl}\right), \tag{S2.4}$$

and we may neglect them. The results are summarized.

$$u_{i,1}^{(0,1)} = -\frac{8\gamma_c A}{15D^2\eta}\left[u_i \frac{\partial Q_2^{(0)}}{\partial s} + u_j S_{ij}\left(a_{2,0}^{(2)} \frac{\partial \bar{Q}_2^{(1)}}{\partial s} + a_{1,1}^{(2)} \bar{Q}_2^{(1)}\right)\right]. \tag{S2.5}$$

using (D2) and (D6).

$$u_{i,1}^{(0,2)} = \frac{8\gamma_c A}{15D^3\eta}\left[\dot{u}_i \frac{\partial Q_3^{(0)}}{\partial s} + \dot{u}_j S_{ij}\left(a_{2,0}^{(2)} \frac{\partial \bar{Q}_{3,2}^{(1)}}{\partial s} + a_{1,1}^{(2)} \bar{Q}_{3,2}^{(1)}\right)\right], \tag{S2.6}$$

$$u_{i,1}^{(0,3)} = \frac{8\gamma_c A}{15D^4\eta} u_i u^2 \left[-\frac{6}{5R_0^2}\frac{\partial}{\partial s} + \frac{6}{5R_0}\frac{\partial^2}{\partial s^2} + \frac{3}{5}\frac{\partial^3}{\partial s^3}\right] Q_4^{(0)}(s). \tag{S2.7}$$

Here we have used (D2) and (D3). At linear order in deformation, the other terms have only isotropic part of $Q_n$. The results of $u_{i,1}^{(\alpha,1)}$ are listed:

$$u_{i,1}^{(1,1)} = -\frac{8\gamma_c A}{5R_0\eta D^2} u_j a_{2,0}^{(2)} \frac{\partial Q_2^{(0)}(s)}{\partial s} S_{ij}. \tag{S2.8}$$

$$u_{i,1}^{(2,1)} \simeq \frac{2\gamma_c A}{5\eta D^2} a_{1,1}^{(2)} u_j S_{ij} \frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.9}$$

$$u_{\alpha,1}^{(3,1)} = \frac{6\gamma_c A}{15\eta D^2} a_{1,1}^{(2)} u_j S_{ij} \frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.10}$$

$$u_{i,1}^{(4,1)} = -\frac{16\gamma_c A}{15\eta R_0 D^2} a_{2,0}^{(2)} u_j S_{ij} \frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.11}$$

$$u_{i,1}^{(5,1)} = -\frac{16A\gamma_c}{15D^2 R_0\eta} a_{2,0}^{(2)} \frac{\partial Q_2^{(0)}}{\partial s} u_j S_{ij}. \tag{S2.12}$$

The velocity originated from tangential force is decomposed with

$$\begin{Bmatrix} u_{i,2}^{(0)} \\ u_{i,2}^{(1)} \\ u_{i,2}^{(2)} \\ u_{i,2}^{(3)} \\ u_{i,2}^{(4)} \end{Bmatrix} = \frac{\gamma_c R_0}{\Omega}\int da \int da' n_i(a) \mathsf{T}_{jk}(\mathbf{r}, \mathbf{r}(a')) \begin{Bmatrix} \mathcal{P}_{k\alpha}(a')\nabla_\alpha n_j(a) \\ \frac{3\gamma_c}{\Omega}\mathcal{P}_{k\alpha}(a')\nabla_\alpha \delta R(a) n_j(a) \\ -\frac{\gamma_c R_0}{\Omega}\nabla_{s,j}\delta R(a)\mathcal{P}_{k\alpha}(a')\nabla_\alpha \\ \frac{\gamma_c R_0}{\Omega} n_j(a)\left[n_k^{(0)}(a')\nabla_{s,\alpha}\delta R(a') + n_\alpha^{(0)}(a')\nabla_{s,k}\delta R(a')\right]\nabla_\alpha \\ \frac{2\gamma_c}{\Omega} n_j(a)\mathcal{P}_{k\alpha}(a')\nabla_\alpha c(a',t)\delta R(a') \end{Bmatrix} c(a',t) \tag{S2.13}$$

As the contribution from normal force (30), $u_{i,2}^{(0)}$, $u_{i,2}^{(3)}$, and $u_{i,2}^{(4)}$ are readily calculated as

$$\begin{Bmatrix} u_{i,2}^{(0)} \\ u_{i,2}^{(3)} \\ u_{i,2}^{(4)} \end{Bmatrix} = \frac{\gamma_c R_0^2}{5\Omega\eta}\int da' \begin{Bmatrix} \mathcal{P}_{i\alpha}(a')\nabla_\alpha \\ \left[\frac{4}{3}n_i^{(0)}(a')\nabla_{s,\alpha}\delta R(a') + n_\alpha^{(0)}(a')\nabla_{s,i}\delta R(a')\right]\nabla_\alpha \\ \frac{2}{R_0}\mathcal{P}_{i\alpha}(a')\nabla_\alpha c(a',t)\delta R(a') \end{Bmatrix} c(a',t) \tag{S2.14}$$

Here the projection onto the normal vector of a spherical surface is

$$\mathcal{P}_{jk}(a') = \delta_{jk} - n_j(a')n_k(a'), \tag{S2.15}$$

which satisfies $n_j(a')\mathcal{P}_{jk}(a') = n_k(a')\mathcal{P}_{jk}(a') = 0$.

Each term is further expanded following (41) as (S2.1):

$$u_{i,2}^{(\alpha)} = \sum_{\alpha=0}^{4}\sum_{\beta=0}^{3} u_{i,2}^{(\alpha,\beta)}. \tag{S2.16}$$

Similar to $u_{i,1}$, we obtain

$$u_{i,2}^{(0,1)} = \frac{2\gamma_c A}{5D^2\eta} u_i \frac{\partial Q_2^{(0)}}{\partial s} + \frac{\gamma_c A}{5D^2\eta} u_j S_{ij}\left[-a_{2,0}^{(2)}\frac{\partial \bar{Q}_{2,2}^{(1)}}{\partial s} + R_0 a_{1,1}^{(2)}\frac{\partial \bar{Q}_{2,2}^{(1)}}{\partial s} + R_0 a_{0,2}^{(2)}\bar{Q}_{2,2}^{(1)}\right], \tag{S2.17}$$

$$u_{i,2}^{(0,2)} = -\frac{2\gamma_c A}{5D^3\eta}\dot{u}_i \frac{\partial Q_3^{(0)}}{\partial s} - \frac{\gamma_c A}{5D^2\eta}\dot{u}_j S_{ij}\left[-a_{2,0}^{(2)}\frac{\partial \bar{Q}_{2,2}^{(1)}}{\partial s} + R_0 a_{1,1}^{(2)}\frac{\partial \bar{Q}_{2,2}^{(1)}}{\partial s} + R_0 a_{0,2}^{(2)}\bar{Q}_{2,2}^{(1)}\right], \tag{S2.18}$$

$$u_{i,2}^{(0,3)} = \frac{2\gamma_c A}{5D^4\eta} u^2 u_i\left[-\frac{6}{5R_0^2}\frac{\partial Q_4^{(0)}}{\partial s} + \frac{6}{5R_0}\frac{\partial^2 Q_4^{(0)}}{\partial s^2} + \frac{3}{5}\frac{\partial^3 Q_4^{(0)}}{\partial s^3}\right] \tag{S2.19}$$

$$u_{i,2}^{(1,1)} = \frac{6A\gamma_c}{5D^2 R_0\eta} a_{2,0}^{(2)}\frac{\partial Q_2^{(0)}}{\partial s} u_j S_{ij} \tag{S2.20}$$

$$u_{i,2}^{(2,1)} = -\frac{7\gamma_c A}{15\eta D} u_j S_{ij} a_{1,1}^{(2)}\frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.21}$$

$$u_{i,2}^{(3,1)} = \frac{\gamma_c R_0 A}{15\eta D^2} a_{1,1}^{(2)} u_j S_{ij}\left(\frac{4}{R_0}\frac{\partial Q_2^{(0)}}{\partial s} + 3\frac{\partial^2 Q_2^{(0)}}{\partial s^2}\right), \tag{S2.22}$$

$$u_{i,2}^{(4,1)} = -\frac{2\gamma_c A}{5\eta D^2} a_{2,0}^{(2)} u_j S_{ij}\frac{\partial Q_2^{(0)}}{\partial s}. \tag{S2.23}$$

## B. second mode

We expand the right-hand side of (51) with the following terms; the contribution from normal force is

$$\begin{Bmatrix} M_{ij}^{(1,1)} \\ M_{ij}^{(1,2)} \\ M_{ij}^{(1,3)} \\ M_{ij}^{(1,4)} \end{Bmatrix} = \frac{\gamma_c}{\Omega}\int da \int da' \overline{n_i(a)n_j(a)}\mathsf{T}_{\alpha\beta}(\mathbf{r}(a),\mathbf{r}(a'))\begin{Bmatrix} R_0 n_\alpha(a) n_\beta(a')\kappa(a') \\ 2\nabla_{s,\alpha}\delta R(a) n_\beta(a') \\ -R_0\nabla_{s,\beta}\delta R(a') n_\alpha(a)\kappa(a') \\ 4\delta R(a') n_\alpha^{(0)}(a) n_\beta^{(0)}(a') \end{Bmatrix} c(a') \tag{S2.24}$$

and the contribution from tangential force is

$$\begin{Bmatrix} M_{ij}^{(2,1)} \\ M_{ij}^{(2,2)} \\ M_{ij}^{(2,3)} \\ M_{ij}^{(2,4)} \end{Bmatrix} = \frac{\gamma_c R_0}{\Omega}\int da \int da' \overline{n_i(a)n_j(a)}\mathsf{T}_{\alpha\beta}(\mathbf{r}(a),\mathbf{r}(a'))\begin{Bmatrix} n_\alpha(a)\mathcal{P}_{\beta\gamma}(a') \\ -\nabla_{s,\alpha}\delta R(a)\mathcal{P}_{\beta\gamma}(a') \\ n_\alpha(a)\left[\nabla_{s,\beta}\delta R(a') n_\gamma(a') + \nabla_{s,\gamma}\delta R(a') n_\beta(a')\right] \\ 2n_\alpha(a)\mathcal{P}_{\beta\gamma}(a')\frac{\delta R(a')}{R_0} \end{Bmatrix} (\nabla_\gamma c(a'))_I.$$

These are simplified as

$$\begin{Bmatrix} M_{ij}^{(1,1)} \\ M_{ij}^{(1,3)} \\ M_{ij}^{(1,4)} \end{Bmatrix} = \frac{4\gamma_c R_0^2}{35\Omega\eta}\int da' \begin{Bmatrix} \overline{n_i(a)n_j(a)}\kappa(a') \\ \frac{35\eta}{2R_0^2}\mathcal{B}_2^{(3)}\left(\nabla_{s,i}\delta R(a') n_j(a') + \nabla_{s,j}\delta R(a') n_i(a')\right) \\ -\frac{4}{R_0^2}\overline{n_i(a')n_j(a')}\delta R(a') \end{Bmatrix} c(a'). \tag{S2.25}$$

$$\begin{Bmatrix} M_{ij}^{(2,1)} \\ M_{ij}^{(2,3)} \\ M_{ij}^{(2,4)} \end{Bmatrix} = \frac{\gamma_c R_0}{\Omega}\int da' \begin{Bmatrix} \mathcal{B}_2^{(3)}\left(\delta_{i\gamma}n_j(a') + \delta_{j\gamma}n_i(a') - 2n_i(a')n_j(a')n_\gamma(a')\right) \\ \left[\mathcal{B}_2^{(3)}\left(n_j(a')\nabla_{s,i}\delta R(a') + n_i(a')\nabla_{s,j}\delta R(a')\right)n_\gamma(a') + \frac{4R_0}{35\eta}\overline{n_i(a')n_j(a')}\nabla_{s,\gamma}\delta R(a')\right] \\ \frac{2}{R_0}\mathcal{B}_2^{(3)}\left(\delta_{i\gamma}n_j(a') + \delta_{j\gamma}n_i(a') - 2n_i(a')n_j(a')n_\gamma(a')\right)\delta R(a') \end{Bmatrix} (\nabla_\gamma c(a'))_I$$

These are further expanded according to (41) for $\alpha = 1,2,3,4$ and $k = 1,2$ as

$$M_{ij}^{(k,\alpha)} = M_{ij}^{(k,\alpha,0)} + M_{ij}^{(k,\alpha,1)} + M_{ij}^{(k,\alpha,2)} + M_{ij}^{(k,\alpha,3)} + \cdots . \tag{S2.26}$$

For all $\alpha$ and $k = 1,2$

$$M_{ij}^{(k,\alpha,1)} = \mathcal{O}(u_k T_{ijk}), \tag{S2.27}$$

$$M_{ij}^{(k,\alpha,3)} = \mathcal{O}\left(u_i u_k u_l T_{jkl}\right) \tag{S2.28}$$

and for $\alpha \geq 2$, we obtain

$$M_{ij}^{(k,\alpha,2)} = \mathcal{O}(u_k u_l D_{ijkl}). \tag{S2.29}$$

The expressions of these terms are listed here.

$$M_{ij}^{(1,1,0)} = -\frac{4A\gamma_c R_0}{35D\eta} a_{2,0}^{(2)} \left( \frac{4}{R_0^2} Q_1^{(0)}(s) + \frac{2}{R_0} \bar{Q}_{1,2}^{(1)}(s) \right) S_{ij}. \tag{S2.30}$$

$$M_{ij}^{(1,1,2)} = -\frac{16\gamma_c A}{175D^3\eta} \overline{u_i u_j} \left( -\frac{1}{R_0} \frac{\partial Q_3^{(0)}}{\partial s} + \frac{\partial^2 Q_3^{(0)}}{\partial s^2} \right), \tag{S2.31}$$

$$M_{ij}^{(1,2,0)} = 0, \tag{S2.32}$$

$$M_{ij}^{(1,3,0)} = \frac{4A\gamma_c}{DR_0} \mathcal{B}_2^{(3)} a_{1,1}^{(2)} S_{ij} Q_1^{(0)}, \tag{S2.33}$$

$$M_{ij}^{(1,4,0)} = -\frac{16\gamma_c A}{35R_0\eta D} a_{2,0}^{(2)} Q_1^{(0)} S_{ij}, \tag{S2.34}$$

$$M_{ij}^{(2,1,0)} = \frac{2\gamma_c A}{D} \mathcal{B}_2 a_{1,1}^{(2)} \bar{Q}_{1,2} S_{ij}, \tag{S2.35}$$

$$M_{ij}^{(2,1,2)} = \frac{12\gamma_c A}{5R_0 D^3} \mathcal{B}_2^{(3)} \overline{u_i u_j} \left[ -\frac{1}{R_0} \frac{\partial Q_3^{(0)}}{\partial s} + \frac{\partial^2 Q_3^{(0)}}{\partial s^2} \right], \tag{S2.36}$$

$$M_{ij}^{(2,2,0)} = 0, \tag{S2.37}$$

$$M_{ij}^{(2,3,0)} = \frac{2\gamma_c A}{D} \mathcal{B}_2^{(3)} a_{1,1}^{(2)} \frac{\partial Q_1^{(0)}}{\partial s} S_{ij}, \tag{S2.38}$$

$$M_{ij}^{(2,4,0)} = 0. \tag{S2.39}$$

### C. third mode

Similar to the second mode, we expand the right-hand side of (56) with the following terms:

$$\begin{Bmatrix} N_{ijk}^{(1,1)} \\ N_{ijk}^{(1,2)} \\ N_{ijk}^{(1,3)} \\ N_{ijk}^{(1,4)} \end{Bmatrix} = \frac{\gamma_c R_0}{\Omega} \int da \int da' \overline{n_i(a) n_j(a) n_k(a)} \mathsf{T}_{\alpha\beta}(\mathbf{r}(a), \mathbf{r}(a')) \begin{Bmatrix} n_\alpha(a) n_\beta(a') \kappa(a') \\ \frac{2}{R_0} \nabla_{s,\alpha} \delta R(a) n_\beta(a') \\ -\nabla_{s,\beta} \delta R(a') n_\alpha(a) \kappa(a') \\ -\frac{4}{R_0} n_\alpha^{(0)}(a) n_\beta^{(0)}(a') \frac{\delta R(a')}{R_0} \end{Bmatrix} c(a') \tag{S2.40}$$

for the contribution from normal force, and

$$\left\{ N_{ijk}^{(2,\alpha)} \right\} = \frac{\gamma_c R_0}{\Omega} \int da \int da' \overline{n_i(a) n_j(a) n_k(a)} \mathsf{T}_{\alpha\beta}(\mathbf{r}(a), \mathbf{r}(a')) \begin{Bmatrix} n_\alpha(a) \mathcal{P}_{\beta\gamma}(a') \left(\nabla_\gamma c(a')\right)_I \\ -\nabla_{s,\alpha} \delta R(a) \mathcal{P}_{\beta\gamma}(a') \left(\nabla_\gamma c(a')\right)_I \\ n_\alpha(a) \left[\nabla_{s,\beta} \delta R(a') n_\gamma(a') + \nabla_{s,\gamma} \delta R(a') n_\beta(a')\right] \left(\nabla_\gamma c(a')\right)_I \\ 2n_\alpha^{(0)}(a) \mathcal{P}_{\beta\gamma}(a') \frac{\delta R(a')}{R_0} \left(\nabla_\gamma c(a')\right)_I \end{Bmatrix} \tag{S2.41}$$

for the contribution from tangential force. These are again simplified as

$$\begin{Bmatrix} N_{ijk}^{(1,1)} \\ N_{ijk}^{(1,3)} \\ N_{ijk}^{(1,4)} \end{Bmatrix} \simeq \frac{8\gamma_c R_0^2}{105\eta\Omega} \int da' \begin{Bmatrix} \overline{n_i(a')n_j(a')n_k(a')}\kappa(a') \\ \frac{1}{4R_0}\overline{\delta_{i\beta}n_j(a')n_k(a')}\nabla_{s,\beta}\delta R(a') \\ -\frac{4}{R_0^2}\overline{n_i(a')n_j(a')n_k(a')}\delta R(a) \end{Bmatrix} c(a') \tag{S2.42}$$

and

$$\begin{Bmatrix} N_{ijk}^{(2,1)} \\ N_{ijk}^{(2,3)} \\ N_{ijk}^{(2,4)} \end{Bmatrix} \simeq \frac{\gamma_c R_0^2}{105\eta\Omega} \int da' \begin{Bmatrix} \overline{\delta_{i\beta}n_j(a')n_k(a')}\mathcal{P}_{\beta\gamma}(a')\nabla_\gamma c(a') \\ \overline{\delta_{i\beta}n_j(a')n_k(a')}\nabla_{s,\beta}\delta R(a')n_\gamma(a')\nabla_\gamma c(a') + 8\int da'\overline{n_i(a')n_j(a')n_k(a')}\nabla_{s,\gamma}\delta R(a')\nabla_\gamma c(a') \\ \frac{2}{R_0}\overline{\delta_{i\beta}n_j(a')n_k(a')}\mathcal{P}_{\beta\gamma}(a')\delta R(a')\nabla_\gamma c(a') \end{Bmatrix} \tag{S2.43}$$

where

$$\overline{\delta_{i\beta}n_j(a')n_k(a')} = \delta_{i\beta}n_j(a')n_k(a') + \delta_{j\beta}n_i(a')n_k(a') + \delta_{k\beta}n_i(a')n_j(a') - \frac{1}{5}\left(\delta_{ij}\delta_{k\beta} + \delta_{ik}\delta_{j\beta} + \delta_{jk}\delta_{i\beta}\right). \tag{S2.44}$$

These are further expanded similar to (S2.26). The results are summarized here.

$$N_{ijk}^{(1,1,0)} \simeq -\frac{8A\gamma_c}{105\eta D} a_{3,0}^{(3)} \left[2\bar{Q}_{1,3}^{(1)} + \frac{10}{R_0}Q_1^{(0)}\right] T_{ijk}, \tag{S2.45}$$

$$N_{ijk}^{(1,1,1)} = -\frac{16\gamma_c A}{735\eta D^2}\left(\frac{\partial \bar{Q}_{2,2}^{(1)}}{\partial s} + \frac{2}{R_0}\frac{\partial Q_2^{(0)}}{\partial s} - \frac{2}{R_0}\bar{Q}_{2,2}^{(1)}\right)\overline{S_{ij}u_k}, \tag{S2.46}$$

$$N_{ijk}^{(1,1,3)} \simeq -\frac{32\gamma_c A}{35^2\eta D^4}\overline{u_i u_j u_k}\left(\frac{3}{R_0^2}\frac{\partial Q_4^{(0)}}{\partial s} - \frac{3}{R_0}\frac{\partial^2 Q_4^{(0)}}{\partial s^2} + \frac{\partial^3 Q_4^{(0)}}{\partial s^3}\right), \tag{S2.47}$$

$$N_{ijk}^{(1,2,0)} = 0, \tag{S2.48}$$

$$N_{ijk}^{(1,2,1)} = -\frac{4\gamma_c A}{35D^2\eta}\frac{1}{R_0}\frac{\partial Q_2^{(0)}}{\partial s}a_{2,0}^{(2)}\overline{u_k S_{ij}}, \tag{S2.49}$$

$$N_{ijk}^{(1,3,0)} \simeq \frac{2\gamma_c A}{35\eta D}a_{2,1}^{(3)}Q_1^{(0)}T_{ijk}, \tag{S2.50}$$

$$N_{ijk}^{(1,3,1)} = \frac{16\gamma_c A}{735D^2\eta}a_{2,0}^{(2)}\frac{1}{R_0}\frac{\partial Q_2^{(0)}}{\partial s}\overline{u_k S_{ij}}, \tag{S2.51}$$

$$N_{ijk}^{(1,4,0)} = -\frac{32\gamma_c A}{105\eta D R_0}a_{3,0}^{(3)}T_{ijk}Q_1^{(0)}, \tag{S2.52}$$

$$N_{ijk}^{(1,4,1)} = -\frac{32\gamma_c A}{105\eta D^2 R_0}u_\alpha\left[a_{4,0}^{(4)}D_{ijk\gamma} + \frac{1}{7}a_{2,0}^{(2)}\overline{S_{ij}\delta_{k\gamma}}\right]\frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.53}$$

$$N_{ijk}^{(2,1,0)} = \frac{\gamma_c A R_0}{35\eta D}a_{2,1}^{(3)}T_{ijk}\bar{Q}_{1,3}^{(1)}, \tag{S2.54}$$

$$N_{ijk}^{(2,1,1)} = \frac{4\gamma_c A}{735\eta D^2}a_{2,0}^{(2)}\left(3\frac{\partial \bar{Q}_{2,3}^{(1)}}{\partial s} - \frac{6}{R_0}\bar{Q}_{2,3}^{(1)}\right)\overline{u_k S_{ij}} \tag{S2.55}$$

$$N_{ijk}^{(2,1,3)} = \frac{24\gamma_c A}{35^2\eta D^4}\overline{u_i u_j u_k}\left(\frac{3}{R_0^2}\frac{\partial Q_4^{(0)}}{\partial s} - \frac{3}{R_0}\frac{\partial^2 Q_4^{(0)}}{\partial s^2} + \frac{\partial^3 Q_4^{(0)}}{\partial s^3}\right) \tag{S2.56}$$

$$N_{ijk}^{(2,2,0)} = 0, \tag{S2.57}$$

$$N_{ijk}^{(2,2,1)} = \frac{2\gamma_c A}{15\eta D^2} a_{2,0}^{(2)} \overline{S_{ij} u_k} \frac{1}{R_0} \frac{\partial Q_2^{(0)}}{\partial s}, \tag{S2.58}$$

$$N_{ijk}^{(2,3,0)} = \frac{\gamma_c R_0 A}{35\eta D} a_{2,1}^{(3)} \frac{\partial Q_1^{(0)}}{\partial s} T_{ijk}, \tag{S2.59}$$

$$N_{ijk}^{(2,3,1)} = \frac{8\gamma_c A}{735 D^2} a_{2,0}^{(2)} \overline{S_{jk} u_i} \left[ \frac{\partial^2 Q_2^{(0)}}{\partial s^2} - \frac{2}{R_0} \frac{\partial Q_2^{(0)}}{\partial s} \right], \tag{S2.60}$$

$$N_{ijk}^{(2,4,0)} = 0, \tag{S2.61}$$

$$N_{ijk}^{(2,4,1)} = \frac{8\gamma_c A}{735\eta D^2} a_{2,0}^{(2)} \overline{u_i S_{jk}} \frac{\partial Q_2^{(0)}}{\partial s}. \tag{S2.62}$$

## S3. SOME CALCULATIONS INCLUDING THE OSEEN TENSOR

In this section, we summarize useful results for calculation including Oseen tensor. First it has been obtained [1] that

$$\int da n_i(a') \mathsf{T}_{ij}(\mathbf{r}(a), \mathbf{r}(a')) n_j(a) Y_l^m(a) = E_l Y_l^m(a') \tag{S3.1}$$

with

$$E_l = \frac{R}{\eta} \frac{2l(l+1)}{(2l-1)(2l+1)(2l+3)}. \tag{S3.2}$$

We may also have [2]

$$\int da \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) n_i(a) n_k(a) = \frac{R_0}{5\eta} \delta_{ij} + \frac{R_0}{15\eta} n_i(a') n_j(a') \tag{S3.3}$$

$$\int da \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) n_i(a) n_k(a) \left[ \delta_{jl} - n_j(a') n_l(a') \right] = \frac{R_0}{5\eta} \left[ \delta_{il} - n_i(a') n_l(a') \right]. \tag{S3.4}$$

We may also use the following integral

$$\int \mathsf{T}_{ij}(\mathbf{r}(a), \mathbf{r}(a')) da = \frac{2R_0}{3\eta} \delta_{ij}. \tag{S3.5}$$

In the main text we encounter the integral $\int n_i(a) \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) \nabla_{s,j} Y_l^m(a) da$ . Since this integral is difficult to evaluate, we instead consider the following integral

$$\int \left[ n_i(a') \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) n_j(a) n_{\alpha_1}(a) n_{\alpha_2}(a) \cdots n_{\alpha_m} \right] da. \tag{S3.6}$$

Using integration by part, above integral with a gradient operator is replaced by (S3.6). Let us summarize the results for $m = 1, 2, 3$. For $m = 1$, (S3.6) is readily obtained using (S3.3)

$$\int \left[ n_i(a') \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) n_j(a) n_\alpha(a) \right] da = \frac{R_0}{15\eta} n_i(a') \left[ 3\delta_{k\alpha} + n_k(a') n_\alpha(a') \right]. \tag{S3.7}$$

For $m = 2$, we have

$$\begin{aligned} &\int \left[ n_\alpha(a) \mathsf{T}_{jk}(\mathbf{r}(a), \mathbf{r}(a')) n_j(a) n_\beta(a) \right] da \\ =& \frac{2R_0}{35\eta} n_\alpha(a') n_\beta(a') n_k(a') - \frac{4R_0}{105\eta} \delta_{\alpha\beta} n_k(a') + \frac{R_0}{35\eta} \left( \delta_{\alpha k} n_\beta(a') + \delta_{\beta k} n_\alpha(a') \right), \end{aligned} \tag{S3.8}$$

and for $m = 3$

$$\begin{aligned}
&\int \left[\mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)n_\alpha(a)n_\beta(a)n_\gamma(a)\right] da\\
=&\frac{R_0}{21\eta}n_\alpha(a')n_\beta(a')n_\gamma(a')n_k(a') + \frac{R_0}{105\eta}\left[\delta_{\alpha k}n_\beta(a')n_\gamma(a') + \delta_{\beta k}n_\alpha(a')n_\gamma(a') + \delta_{\gamma k}n_\alpha(a')n_\beta(a')\right]\\
&+\frac{4R_0}{105\eta}\left[\delta_{\alpha\beta}\delta_{\beta k} + \delta_{\alpha\gamma}\delta_{\beta k} + \delta_{\alpha k}\delta_{\beta\gamma}\right]. \qquad \text{(S3.9)}
\end{aligned}$$

In order to obtain the above results, we first calculate the integral

$$\int da n_i(a)\mathsf{T}_{ij}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)Y_l^m(a). \qquad \text{(S3.10)}$$

We follow the method in [1] in which the similar integral was calculated. This integral is rewritten using the Oseen tensor in Fourier space. Here we consider more general relation for using $\mathbf{n}^{(\mathrm{d})}$ rather than $\mathbf{n}$:

$$\begin{aligned}
&n_i^{(\mathrm{d})}(a)\mathsf{T}_{ij}(\mathbf{r}(a),\mathbf{r}(a'))n_j^{(\mathrm{d})}(a)Y_l^m(a)\\
=&\frac{1}{\eta R^2}\int da\int_{\mathbf{k}}\frac{1}{k^2}R_i(a)R_j(a)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)e^{ik_\alpha\left[R_\alpha(a)-R_\alpha(a')\right]}Y_l^m(a)\\
=&\frac{1}{\eta R^2}\int da\int_{\mathbf{k}}\frac{1}{k^2}R^2e^{ik_\alpha\left[R_\alpha(a)-R_\alpha(a')\right]}Y_l^m(a) - \frac{1}{\eta R^2}\int da\int_{\mathbf{k}}\frac{1}{k^4}\left(R_i(a)k_i\right)\left(R_j(a)k_j\right)e^{ik_\alpha\left[R_\alpha(a)-R_\alpha(a')\right]}Y_l^m(a) \qquad \text{(S3.11)}
\end{aligned}$$

The first term in (S3.11) becomes

$$\begin{aligned}
&\frac{1}{\eta}\int da\int_{\mathbf{k}}\frac{1}{k^2}e^{ik_\alpha\left[R_\alpha(a)-R_\alpha(a')\right]}Y_l^m(a)\\
=&\frac{1}{\eta}\int da\int_{\mathbf{k}}\frac{16\pi^2}{k^2}\sum_{l',m'}i^{l'}j_{l'}(kR(a))Y_{l'}^{m'}(\theta_k,\varphi_k)Y_{l'}^{m'*}(\theta,\varphi)\sum_{l'',m''}(-i)^{l''}j_{l''}(kR(a'))Y_{l''}^{m''*}(\theta_k,\varphi_k)Y_{l''}^{m''}(\theta',\varphi')Y_l^m(a)\\
=&\frac{1}{\eta}\int da\frac{2}{\pi}\sum_{l',m'}\int dk j_{l'}(kR(a))Y_{l'}^{m'*}(\theta,\varphi)j_{l'}(kR(a'))Y_{l'}^{m'}(\theta',\varphi')Y_l^m(a)\\
=&\frac{2}{\pi\eta}\sum_{l',m'}Y_{l'}^{m'}(a')\int da\mathcal{C}_{l'}(a,a')Y_{l'}^{m'*}(a)Y_l^m(a) \qquad \text{(S3.12)}
\end{aligned}$$

where we have used (37). Here we define

$$\begin{aligned}
\mathcal{C}_l(a,a') &= \int_0^\infty j_l(kR(a))j_l(kR(a'))dk\\
&\simeq \frac{\pi}{2(2l+1)R_0}\left[1 - \frac{1}{2R_0}\left(\delta R(a) + \delta R(a')\right)\right], \qquad \text{(S3.13)}
\end{aligned}$$

where the spherical Bessel function is expanded as

$$\begin{aligned}
j_l(kR) \simeq& j_l(kR_0) + k\delta Rj_l'(kR_0)\\
=& j_l(kR_0) + \frac{1}{2l+1}k\delta R\left[lj_{l-1}(kR_0) - (l+1)j_{l+1}(kR_0)\right]. \qquad \text{(S3.14)}
\end{aligned}$$

The second term in (S3.11) is calculated as

$$\begin{aligned}
&-i^2\frac{1}{\eta R^2}\int da\int_{\mathbf{k}}\frac{1}{k^4}\frac{\partial^2}{\partial b^2}e^{ik_\alpha\left[R_\alpha(a)b-R_\alpha(a')\right]}Y_l^m(a)\Big|_{b=1}\\
=&\frac{1}{\eta R^2}\int da\int_{\mathbf{k}}\frac{16\pi^2}{k^4}\frac{\partial^2}{\partial b^2}\sum_{l',m'}i^{l'}j_{l'}(bkR(a))Y_{l'}^{m'}(\theta_k,\varphi_k)Y_{l'}^{m'*}(\theta,\varphi)\sum_{l'',m''}(-i)^{l''}j_{l''}(kR(a'))Y_{l''}^{m''*}(\theta_k,\varphi_k)Y_{l''}^{m''}(\theta',\varphi')Y_l^m(a)\Big|_{b=1}\\
=&\frac{2}{\eta\pi}\sum_{l',m'}Y_{l'}^{m'}(\theta',\varphi')\int da\bar{\mathcal{C}}_{l'}(a,a')Y_{l'}^{m'*}(\theta,\varphi)Y_l^m(a) \qquad \text{(S3.15)}
\end{aligned}$$

where

$$\bar{C}_l(a,a') = \int_0^\infty j_l''(kR(a))j_l(kR(a'))dk$$
$$= -\frac{R(a')}{R(a)}\int_0^\infty j_l'(kR(a))j_l'(kR(a'))dk$$
$$= -\frac{2l(l+1)-1}{(2l-1)(2l+1)(2l+3)}\left(1+\frac{\delta R(a')}{R_0}-\frac{\delta R(a)}{R_0}\right)\frac{\pi}{2R_0}. \quad \text{(S3.16)}$$

Combining (S3.13) and (S3.16), we obtain for $\mathbf{n}^{(\mathrm{d})} \simeq \mathbf{n}$ in which no coupling between modes:

$$\int da n_i(a)\mathsf{T}_{ij}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)Y_l^m(a) = \frac{R_0}{\eta}\frac{2(l^2+l-1)}{(2l-1)(2l+1)(2l+3)}Y_l^m(a'). \quad \text{(S3.17)}$$

Now we outline the calculation of (S3.6). Let us consider for (S3.8)

$$\int \left[\mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)n_\alpha(a)n_\beta(a)\right] da. \quad \text{(S3.18)}$$

Respecting the symmetry of the integral, that is, invariance under $\alpha \leftrightarrow \beta$, the integral is decomposed into several third-rank tensors as

$$\int \left[n_\alpha(a)\mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)n_\beta(a)\right] da$$
$$= \mathcal{A}^{(3)}n_\alpha(a')n_\beta(a')n_k(a') + \mathcal{B}_1^{(3)}\delta_{\alpha\beta}n_k(a') + \mathcal{B}_2^{(3)}\left(\delta_{\alpha k}n_\beta(a')+\delta_{\beta k}n_\alpha(a')\right). \quad \text{(S3.19)}$$

First, by multiplying $n_\alpha(a')n_\beta(a')n_k(a')$, we have

$$\int \left[\left(n_\alpha(a)n_\alpha(a')\right)^2 \mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)n_k(a')\right] da = \mathcal{A}^{(3)} + \mathcal{B}_1^{(3)} + 2\mathcal{B}_2^{(3)}. \quad \text{(S3.20)}$$

The integral is calculated as

$$\mathcal{A}^{(3)} + \mathcal{B}_1^{(3)} + 2\mathcal{B}_2^{(3)} = \frac{R_0}{8\eta}\int_0^\pi \sin\theta\cos^2\theta\frac{\cos\theta-\sin^2\theta/2}{\sin\theta/2}d\theta = \frac{8R_0}{105\eta}. \quad \text{(S3.21)}$$

When $\alpha=\beta$, (S3.19) becomes

$$\int \left[\mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_j(a)\right] da = \mathcal{A}^{(3)}n_k(a') + 3\mathcal{B}_1^{(3)}n_k(a') + 2\mathcal{B}_2^{(3)}n_k(a'). \quad \text{(S3.22)}$$

The integral must vanish because uniform normal force never generates flow in an incompressible system, and thus $\mathcal{A}^{(3)} + 3\mathcal{B}_1^{(3)} + 2\mathcal{B}_2^{(3)} = 0$ . When $\alpha = k$, (S3.19) becomes

$$\int \left[n_j(a)\mathsf{T}_{jk}(\mathbf{r}(a),\mathbf{r}(a'))n_k(a)n_\beta(a)\right] da = \mathcal{A}^{(3)}n_\beta(a') + \mathcal{B}_1^{(3)}n_\beta(a') + 4\mathcal{B}_2^{(3)}n_\beta(a'). \quad \text{(S3.23)}$$

The integral is calculated with (S3.17) identifying $n_\beta(a')$ as $Y_1^m$, and we obtain $\mathcal{A}^{(3)} + \mathcal{B}_1^{(3)} + 4\mathcal{B}_2^{(3)} = \frac{2R_0}{15\eta}$ . Note that integration of (S3.8) with respect to $a'$ leads to zero for both sides of this equation. The same argument applies to more complicated integral of (S3.9). We may check the result (S3.9) by multiplying $R_0/\Omega$ and integrating over $a'$ for both sides using (D2), (D3), and (S3.5). Both sides result in $2R_0/15\eta$.

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[2] S. Yabunaka, T. Ohta, and N. Yoshinaga, J. Chem. Phys. **136**, 074904 (pages 8) (2012).