Persistence, Category Theory & Reeb Graphs

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Category Theory

What constitutes a category?

- Objects. $x, y, z, \ldots$
- Morphisms. $x \xrightarrow{f} y$
- A composition law. $x \xrightarrow{f} y \xrightarrow{g} z$ gives $x \xrightarrow{gf} z$
- The composition law is associative and has an identity at each object.

Examples of categories

- **Set** sets, functions
- **Top** topological spaces, continuous maps
- **Group** groups, homomorphisms
- **Ab** abelian groups, homomorphisms
- $\mathbf{Vect} = \mathbf{Vect}_{\mathbf{k}}$ vector spaces over $\mathbf{k}$, linear maps

Preordered sets

Let P be a set with a reflexive transitive relation $\leq$. Then

- objects = { elements of P }
- morphisms = { relations $x \leq y$ }

defines a category $\mathbf{P}$.

Directed graphs

A directed graph defines a category:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

or

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

or

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

(Identities and composites are implicit.)

Sublevelset persistent homology

Let $f : X \to \mathbf{R}$. Consider the category $\mathbf{n}$ defined by

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1,$$

and select $a_0 \leq a_1 \leq \cdots \leq a_{n-1}$. From

$$X^{a_0} \xrightarrow{\subseteq} X^{a_1} \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} X^{a_{n-1}},$$

construct

$$\mathrm{H}(X^{a_0}) \longrightarrow \mathrm{H}(X^{a_1}) \longrightarrow \cdots \longrightarrow \mathrm{H}(X^{a_{n-1}}).$$

Definitions

- $X^t := f^{-1}(-\infty, t]$ sublevelset
- $X_s := f^{-1}[s, +\infty)$ superlevelset
- $X_s^t := f^{-1}[s, t]$ interlevelset

Sublevelset persistent homology

The 'persistence module'

$$\mathrm{H}(X^{a_0}) \longrightarrow \mathrm{H}(X^{a_1}) \longrightarrow \cdots \longrightarrow \mathrm{H}(X^{a_{n-1}})$$

can be thought of as a **functor**

$$\mathbf{n} \xrightarrow{\mathrm{F}} \mathbf{Top} \xrightarrow{\mathrm{H}} \mathbf{Vect}.$$

This means:

- For each object of **n** we have a vector space.
- For each morphism of **n** we have a linear map.

Generalized persistence modules (Bubenik, Scott 2014)

A 'generalized persistence module' is simply a functor $\mathbb{V} : \mathbf{C} \to \mathbf{D}$.

- Usually $\mathbf{C}$ is a pre-ordered set, such as $\mathbf{n}, \mathbf{N}, \mathbf{Z}, \mathbf{R}$.
- Usually $\mathbf{D}$ is an abelian category, such as $\mathbf{Vect}, \mathbf{Ab}$.

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- Usually $\mathbf{D}$ is an ‘abelian’ category, such as $\mathbf{Vect}, \mathbf{Ab}$.

Categories of functors

The collection of functors $\mathbf{C} \to \mathbf{D}$ is itself a category, denoted $\mathbf{D}^{\mathbf{C}}$. The morphisms are **natural transformations** $\phi : \mathbb{V} \Rightarrow \mathbb{W}$, defined by the following data:

- For every $c \in C$ there is a map $\phi_c : V_c \to W_c$.
- For every map $f : c \to c'$ in C the diagram

$$\begin{tikzcd} V_c \arrow[r, "\phi_c"] \arrow[d, "{V[f]}"'] & W_c \arrow[d, "{W[f]}"] \\ V_{c'} \arrow[r, "\phi_{c'}"'] & W_{c'} \end{tikzcd}$$

is required to commute.

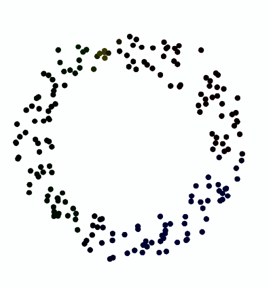
Category Theory

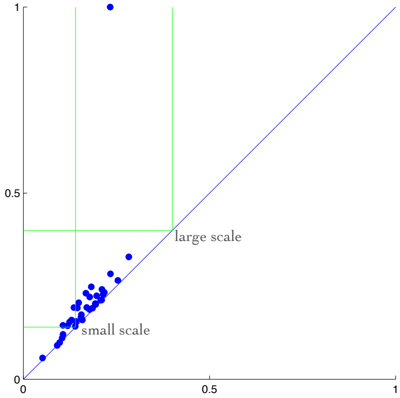
Why category theory?

- It is a convenient language for describing persistence modules.
- It gives clues to finding the ‘right’ definitions and concepts.
- It gives immediate access to deeper theorems.
- We are free to drop it when it doesn’t fit.

Story 1: Persistence diagrams

Persistent homology takes a filtered space $\mathbb{X} = \{X_t \mid t \in \mathbf{R}\}$ and returns a **barcode** of intervals $[p, q) \subset \mathbf{R}$ or a **persistence diagram** of points $(p, q) \in \mathbf{R}^2$.





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How is this defined?

Algorithmic approach (Edelsbrunner, Letscher, Zomorodian 2000).

- Discretize the t-variable.
- Present $\mathbb{X}$ as a finite list of cells, attached in sequence.
- Some cells σ generate new homology cycles.
- Other cells τ destroy cycles created by an earlier σ.
- There is an interval $[t_\sigma, t_\tau)$ for each such pair (σ, τ).
- There is an interval $[t_\sigma, +\infty)$ for each σ whose cycle is never destroyed.

Story 1: Persistence diagrams

Using commutative algebra (Zomorodian, Carlsson 2003).

- Discretize the t-variable to integers: $t = 0, 1, 2, \ldots$
- Present $\mathbb{X}$ as an increasing sequence:

$$\mathbb{X}: \quad X_0 \subset X_1 \subset X_2 \subset \ldots$$

- Apply a homology functor $\mathrm{H} = \mathrm{H}(-; \mathbf{k})$ to the sequence:

$$\mathrm{H}(\mathbb{X}): \quad \mathrm{H}(X_0) \to \mathrm{H}(X_1) \to \mathrm{H}(X_2) \to \ldots$$

- Observe that $\mathrm{H}(\mathbb{X})$ is a graded module over the polynomial ring $\mathbf{k}[z]$, where z acts by shifting to the right.
- Decompose this graded module as a direct sum of cyclic submodules.
- Summands $z^s\mathbf{k}[z]/(z^{t-s})$ are recorded as intervals $[s, t)$.
- Summands $z^s\mathbf{k}[z]$ are recorded as intervals $[s, +\infty)$.

Using quiver theory (Carlsson, dS 2010).

- Discretize the t-variable to integers: $t = 0, 1, \ldots, n-1$.
- Present $\mathbb{X}$ as a sequence of spaces with maps:

$$\mathbb{X}: \quad X_0 \to X_1 \to \cdots \to X_{n-1}$$

- Apply a homology functor $\mathrm{H} = \mathrm{H}(-; \mathbf{k})$ to the sequence:

$$\mathrm{H}(\mathbb{X}): \quad \mathrm{H}(X_0) \to \mathrm{H}(X_1) \to \cdots \to \mathrm{H}(X_{n-1})$$

- Observe that $\mathrm{H}(\mathbb{X})$ is a representation of the quiver $\bullet \to \bullet \to \ldots \to \bullet$.
- Decompose $\mathrm{H}(\mathbb{X})$ as a direct sum of indecomposable representations.
- According to Gabriel (1970), the indecomposables are precisely the **intervals**:

$$0 \to \cdots \to 0 \to \mathbf{k} \to \cdots \to \mathbf{k} \to 0 \to \cdots \to 0$$

The list of summands of $\mathrm{H}(\mathbb{X})$ gives the persistence intervals.

When the arrows have mixed orientations $\leftarrow, \rightarrow$, we get **zigzag persistence**.

What if we wish to work with a continuous parameter?

Interval decomposition

- Let $\mathbb{V}$ be a persistence module defined over the real numbers $\mathbf{R}$.
- Suppose

$$\mathbb{V} = \bigoplus_{k \in K} \mathbb{I}_{[a_k, b_k]}$$

where $\mathbb{I} = \mathbb{I}_{[a,b]}$ denotes the persistence module with

$$I_t = \begin{cases} \mathbf{k} & \text{if } t \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

and all maps i_s^t having full rank. (Open, half-open intervals allowed too.)

- Then we can define the persistence diagram to be

$$\mathrm{Dgm}(\mathbb{V}) = \{\!\{ (a_k, b_k) \mid k \in K \}\!\},$$

a multiset of points in the half-plane above the diagonal.

Story 1: Persistence diagrams

Problem

Not every $\mathbb{V}$ decomposes into intervals.

Theorem (Gabriel, Auslander, Ringel–Tachikawa, Webb, Crawley-Boevey)

Let $\mathbb{V}$ be a persistence module over $\mathbf{T} \subseteq \mathbf{R}$. In either of the following situations, $\mathbb{V}$ decomposes into interval modules:

- $\mathbf{T}$ is a finite set; or
- Every V_t is finite-dimensional.

On the other hand, there exists a persistence module over $\mathbf{Z}$ which does not admit an interval decomposition.

Solution (Chazal, dS, Glisse, Oudot 2016)

Define a measure which counts the number of persistence points in an arbitrary rectangle. Infer the existence of the persistence diagram. This works if the maps $V_s \to V_t$ are finite-rank whenever $s < t$.

Solution (Chazal, dS, Glisse, Oudot 2012)

Define a measure which counts the number of persistence points in an arbitrary rectangle. Infer the existence of the persistence diagram. This works if the maps $V_s \to V_t$ are finite-rank whenever $s < t$.

Definition 1 (non-functorial)

Let

$$\mu([a,b] \times [c,d]) = \mathrm{r}_b^c - \mathrm{r}_a^c - \mathrm{r}_b^d + \mathrm{r}_a^d$$

where $\mathrm{r}_s^t = \mathrm{rank}(V_s \to V_t)$.

Definition 2 (functorial)

Let

$$\mu([a,b] \times [c,d]) = \dim\left(M_{a,b,c,d}\mathbb{V}\right)$$

where

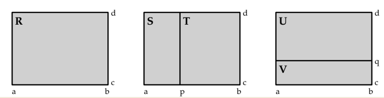
$$M_{a,b,c,d}\mathbb{V} = \left[\frac{\mathrm{Im}(v_b^c) \cap \mathrm{Ker}(v_c^d)}{\mathrm{Im}(v_a^c) \cap \mathrm{Ker}(v_c^d)}\right].$$

Note. Each $M_{a,b,c,d}$ extends to a functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Vect}$.

Story 1: Persistence diagrams

Solution step

It is necessary to show that μ is additive with respect to splitting a rectangle.



Proof 1 (for horizontal split)

$$\mathsf{r}_b^c - \mathsf{r}_a^c - \mathsf{r}_b^d + \mathsf{r}_a^d = (\mathsf{r}_p^c - \mathsf{r}_a^c - \mathsf{r}_p^d + \mathsf{r}_a^d) + (\mathsf{r}_b^c - \mathsf{r}_p^c - \mathsf{r}_b^d + \mathsf{r}_p^d)$$

Proof 2 (for horizontal split)

There is a short exact sequence

$$0 \to \left[\frac{\mathrm{Im}(v_p^c) \cap \mathrm{Ker}(v_c^d)}{\mathrm{Im}(v_a^c) \cap \mathrm{Ker}(v_c^d)}\right] \to \left[\frac{\mathrm{Im}(v_b^c) \cap \mathrm{Ker}(v_c^d)}{\mathrm{Im}(v_a^c) \cap \mathrm{Ker}(v_c^d)}\right] \to \left[\frac{\mathrm{Im}(v_b^c) \cap \mathrm{Ker}(v_c^d)}{\mathrm{Im}(v_p^c) \cap \mathrm{Ker}(v_c^d)}\right] \to 0$$

or, in other words, a short exact sequence of functors

$$0 \longrightarrow M_{a,p,c,d} \longrightarrow M_{a,b,c,d} \longrightarrow M_{p,b,c,d} \longrightarrow 0$$

Story 1: Persistence diagrams

Question (of Morozov)

Is the persistence diagram functorial?

Answer 1: Constructing a functorial persistence diagram

Let $\mathbb{V} : \mathbf{R} \to \mathbf{Vect}$ be a persistence module. Select

$$\cdots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \dots$$

The **functorial persistence diagram** with respect to (a_n) is the function

$$(m, n) \mapsto M_{a_m, a_{m+1}, a_n, a_{n+1}} \mathbb{V}$$

for integers $m < n$. At each point there is a vector space.

Pros and cons

- A map $\mathbb{V} \to \mathbb{W}$ between persistence modules induces a map between f.p.d.
- This method defines a persistence diagram in **any abelian category**.
- It is not so easy to change the discretization.
- **What is the right metric between these diagrams?**

Story 2: Interleaving

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

The map $\{$persistence modules$\} \to \{$diagrams$\}$ is 1-Lipschitz.

Relators

The metrics on the two spaces are defined in terms of 'relators'.

- Two persistence modules may be related by an **interleaving**.
- Two diagrams may be related by a **matching**.

Every relator, of each type, has a size associated with it. The metrics are defined by finding the infimum of the size of relators between a given pair of objects. (Compare the geodesic distance in a Riemannian manifold.)

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

If two persistence modules admit an ϵ-interleaving, then their persistence diagrams admit an ϵ-matching.

Definition

Let $\mathbb{V}, \mathbb{W}$ be persistence modules. An ϵ-interleaving between $\mathbb{V}, \mathbb{W}$ is a pair (Φ, Ψ) where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

$$\phi_t : V_t \to W_{t+\epsilon} \qquad\qquad \psi_t : W_t \to V_{t+\epsilon}$$

such that [various conditions].

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Glisse's Lemma (Chazal, Cohen-Steiner, Glisse, Guibas, Oudot 2009)

The proof of the stability theorem relies on the following fact. If $\mathbb{V}, \mathbb{W}$ are ϵ-interleaved, then there is a 1-parameter family

$$(\mathbb{V}_s \mid s \in [0, \epsilon])$$

with $\mathbb{V}_0 = \mathbb{V}$ and $\mathbb{V}_\epsilon = \mathbb{W}$, and where $\mathbb{V}_r, \mathbb{V}_s$ are $|r - s|$-interleaved for all r, s.

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$$\phi_t : V_t \to W_{t+\epsilon} \qquad\qquad \psi_t : W_t \to V_{t+\epsilon}$$

such that [various conditions].

The [various conditions] require the diagrams

$$\begin{tikzcd}
V_s \arrow[r, "v_t^s"] \arrow[dr, "\phi_s"] & V_t \arrow[rr, "v_{t+2\epsilon}^t"] \arrow[dr, "\phi_t"] & & V_{t+2\epsilon} \\
& W_{s+\epsilon} \arrow[r, "w_{t+\epsilon}^{s+\epsilon}"'] & W_{t+\epsilon} \arrow[ur, "\psi_{t+\epsilon}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& V_{s+\epsilon} \arrow[r, "v_{t+\epsilon}^{s+\epsilon}"] & V_{t+\epsilon} \arrow[dr, "\phi_{t+\epsilon}"] & \\
W_s \arrow[ur, "\psi_s"] \arrow[r, "w_t^s"'] & W_t \arrow[ur, "\psi_t"] \arrow[rr, "w_{t+2\epsilon}^t"'] & & W_{t+2\epsilon}
\end{tikzcd}$$

to commute for all $s < t$.

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$$\phi_t : V_t \to W_{t+\epsilon} \qquad \psi_t : W_t \to V_{t+\epsilon}$$

such that [various conditions].

The [various conditions] amount to the assertion that there is a unique way to get from any of the V_t, W_t to any other. All compositions of the $v_s^t, w_s^t, \phi_t, \psi_t$ with the same start and end point must agree.

Story 2: Interleaving

Definition

Let $\mathbb{V}, \mathbb{W}$ be persistence modules. An ϵ-interleaving between $\mathbb{V}, \mathbb{W}$ is a pair (Φ, Ψ) where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

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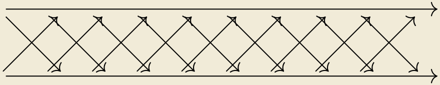
such that [various conditions].

Interleavor categories (Chazal, dS, Glisse, Oudot 2016)

An ϵ-interleaved pair of modules $(\mathbb{V}, \mathbb{W}, \Phi, \Psi)$ is 'the same thing' as a persistence module defined over the set $\mathbf{I} = \mathbf{R} \times \{0, \epsilon\}$ (two copies of the real line) with the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

$\mathbf{R} \times \{0, \epsilon\}$:



Interleavings for classical persistence modules

Two classical persistence modules $\mathbb{V}, \mathbb{W}$ are ϵ-interleaved iff the following functor extension problem has a solution:

$$\begin{tikzcd} & & \mathbf{Vect} & & \\ \mathbf{R} \arrow[urr, "\mathbb{V}"] \arrow[rr] & & \mathbf{R} \times \{0, \epsilon\} \arrow[u, dotted] & & \mathbf{R} \arrow[ull, "\mathbb{W}"'] \arrow[ll] \end{tikzcd}$$

Here $\mathbf{R} \times \{0, \epsilon\}$ has the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

Proof of Glisse's Lemma

Consider the set $\mathbf{J} = \mathbf{R} \times [0, \epsilon]$ with the partial order

$$(s, a) \leq (t, b) \Leftrightarrow s + |a - b| \leq t$$

This contains the interleavor category $\mathbf{I}$ as a sub-poset. An ϵ-interleaving between two persistence modules corresponds to a functor $\mathbf{I} \to \mathbf{Vect}$ which restricts to $\mathbb{V}, \mathbb{W}$ on the two respective copies of the real line.

An interpolation $(\mathbb{V}_t)$ is found constructing an extension of the functor to $\mathbf{J}$:

$$\begin{tikzcd} & \mathbf{Vect} \\ \mathbf{I} \arrow[ur] \arrow[r] & \mathbf{J} \arrow[u, dotted] \end{tikzcd}$$

Since $\mathbf{I}$ is a full subcategory of $\mathbf{J}$, and $\mathbf{Vect}$ contains all limits and colimits, the problem is solved by taking a left or right Kan extension.

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Story 2: Interleaving

Question (of Morozov)

Is the persistence diagram functorial?

Answer 2

The persistence diagram is a map

$$\{\text{persistence modules}\} \to \{\text{diagrams in the upper half-plane}\}$$

What are the morphisms that make these into categories?

- A morphism $\mathbb{V}_1 \to \mathbb{V}_2$ could be an interleaving pair (ϕ, ψ).
- A morphism $\mathrm{Dgm}_1 \to \mathrm{Dgm}_2$ could be a matching between points.

For both notions there is an associative composition law with identities.

Question (of Morozov, reworded)

Does an ϵ-interleaving between two persistence modules specify a ϵ-matching between their diagrams, in a way that respects composition?

Answer 2^+ (Bauer, Lesnick 2015)

Almost. See recent work of Ulrich Bauer and Michael Lesnick.

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$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

Interleavings for generalized persistence modules over a poset

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ are Ω-interleaved iff the following functor extension problem has a solution:

$$\begin{tikzcd} & \mathbf{C} & \\ \mathbf{P} \arrow[ur, "\mathbb{V}"] \arrow[r] & \mathbf{P} \cup_\Omega \mathbf{P} \arrow[u, dotted] & \mathbf{P} \arrow[ul, "\mathbb{W}"'] \arrow[l] \end{tikzcd}$$

Here $\mathbf{P} \cup_\Omega \mathbf{P}$ has the partial order

$$(s, a) \le (t, b) \Leftrightarrow \begin{cases} s \le t & \text{if } a = b \\ \Omega s \le t & \text{if } a \ne b \end{cases}$$

where $\Omega : \mathbf{P} \to \mathbf{P}$ is a translation.

Translations (Bubenik, dS, Scott 2015)

$\mathbf{Trans_P}$ is the poset of functions $\Omega : \mathbf{P} \to \mathbf{P}$ that are order-preserving and satisfy $x \leq \Omega x$ for all $x \in \mathbf{P}$.

Superlinear Families

A **superlinear family** is a 1-parameter family of translations of $\mathbf{P}$

$$(\Omega_\epsilon \mid \epsilon \in [0, \infty))$$

such that

$$\Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1 + \epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Sublinear Projections

A **sublinear projection** is a map $\pi : \mathbf{Trans_P} \to [0, \infty]$ such that

$$\pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)$$

for all $\Omega_1, \Omega_2 \in \mathbf{Trans_P}$.

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Examples of superlinear famlies

- $\mathbf{P} = \mathbf{R}$,
  $\Omega_\epsilon(t) = t + \epsilon$.
- $\mathbf{P} = \{\text{compact intervals in the real line}\}$,
  $\Omega_\epsilon([a, b]) = [a - \epsilon, b + \epsilon]$.
- $\mathbf{P} = \{\text{closed subsets of a metric space } X\}$,
  $\Omega_\epsilon(V) = V^\epsilon = \{x \in X \text{ such that } \mathrm{d}(x, V) \leq \epsilon\}$.

Superlinear Families

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such that

$$\Omega_{\epsilon_1}\Omega_{\epsilon_2} \leq \Omega_{\epsilon_1+\epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family (Ω_ϵ) of translations of **P**, we define the interleaving distance

$$\mathrm{d_i}(\mathbb{V}, \mathbb{W}) = \inf\,(\epsilon \mid \mathbb{V}, \mathbb{W} \text{ are } \Omega_\epsilon\text{-interleaved})$$

between generalized persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$.

Sublinear Projections

A **sublinear projection** is a map $\pi : \mathbf{Trans_P} \to [0, \infty]$ such that

$$\pi(\Omega_1\Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)$$

for all $\Omega_1, \Omega_2 \in \mathbf{Trans_P}$.

Interleaving distance (Bubenik, dS, Scott 2015)

Given a sublinear projection family $\pi : \mathbf{Trans_P} \to [0, \infty]$, we define the interleaving distance

$$\mathsf{d_i}(\mathbb{V}, \mathbb{W}) = \inf\left(\pi(\Omega) \mid \mathbb{V}, \mathbb{W} \text{ are } \Omega\text{-interleaved}\right)$$

between generalized persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$.

Functoriality

Suppose $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ and $H : \mathbf{C} \to \mathbf{D}$ are functors. Then

$$\mathrm{d_i}(H\mathbb{V}, H\mathbb{W}) \leq \mathrm{d_i}(\mathbb{V}, \mathbb{W})$$

for any superlinear family or sublinear projection.

Proof.

An Ω-interleaving of $\mathbb{V}, \mathbb{W}$ gives an Ω-interleaving of $H\mathbb{V}, H\mathbb{W}$:

$$\begin{tikzcd}
\mathbf{P} \arrow[d] \arrow[dr, "\mathbb{V}"] & & \\
\mathbf{P} \cup_\Omega \mathbf{P} \arrow[r, dotted] & \mathbf{C} \arrow[r, "H"] & \mathbf{D} \\
\mathbf{P} \arrow[u] \arrow[ur, "\mathbb{W}"'] & &
\end{tikzcd}$$

$\square$

Functoriality

Suppose $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ and $H : \mathbf{C} \to \mathbf{D}$ are functors. Then

$$\mathrm{d_i}(H\mathbb{V}, H\mathbb{W}) \leq \mathrm{d_i}(\mathbb{V}, \mathbb{W})$$

for any superlinear family or sublinear projection.

Example: sublevelset persistence

Let X be a topological space and $f, g : X \to \mathbf{R}$ be functions with $\|f - g\|_\infty \leq \epsilon$.

- The persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{R} \to \mathbf{Top}$ defined

  $$\mathbb{V}(t) = f^{-1}(-\infty, t], \qquad \mathbb{W}(t) = g^{-1}(-\infty, t],$$

  are ϵ-interleaved.
  (There are natural inclusions $\mathbb{V}(t) \subseteq \mathbb{W}(t + \epsilon)$ and $\mathbb{W}(t) \subseteq \mathbb{V}(t + \epsilon)$.)
- For any homology functor $H : \mathbf{Top} \to \mathbf{Vect}$, the persistence modules $H\mathbb{V}, H\mathbb{W} : \mathbf{R} \to \mathbf{Vect}$ are ϵ-interleaved.

Interleavings for generalized persistence modules over a poset

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{C}$ are Ω-interleaved iff the following functor extension problem has a solution:

$$\begin{tikzcd} & \mathbf{C} & \\ \mathbf{P} \arrow[ur, "\mathbb{V}"] \arrow[r] & \mathbf{P} \cup_\Omega \mathbf{P} \arrow[u, dotted] & \mathbf{P} \arrow[ul, "\mathbb{W}"'] \arrow[l] \end{tikzcd}$$

Here $\mathbf{P} \cup_\Omega \mathbf{P}$ has the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : \mathbf{P} \to \mathbf{P}$ is a translation.

Interleavings for generalized persistence modules over an arbitrary category

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{D} \to \mathbf{C}$ are Δ-interleaved iff the following functor extension problem has a solution:

$$\begin{tikzcd}
 & \mathbf{C} & \\
\mathbf{D} \arrow[ur, "\mathbb{V}"] \arrow[r, "I_1"'] & \Delta \arrow[u, dotted] & \mathbf{D} \arrow[ul, "\mathbb{W}"'] \arrow[l, "I_2"]
\end{tikzcd}$$

Here Δ is a **cospan**: the two functors I_1, I_2 are full-and-faithful and every object of Δ is of the form $I_1(d)$ or $I_2(d)$.

Example: dynamical system interleavings

Let **D** be the category defined by the directed graph

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

Thus **D** has one object and morphisms $\{0, 1, 2, 3, \dots\}$.

- Functors $\mathbf{D} \to \mathbf{Top}$ are **discrete dynamical systems**.

Let Δ_n be the category with two objects $\bullet_1$ and $\bullet_2$ and morphisms

$$\mathrm{Mor}(\bullet_1, \bullet_1) = \mathrm{Mor}(\bullet_1, \bullet_1) = \{0, 1, 2, 3, \dots\}$$
$$\mathrm{Mor}(\bullet_1, \bullet_2) = \mathrm{Mor}(\bullet_2, \bullet_1) = \{n, n+1, n+2, n+3, \dots\}$$

with addition as composition.

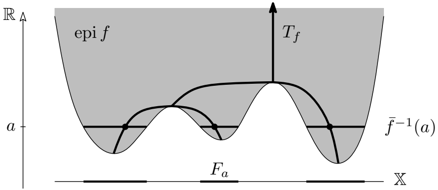
- Δ_n-interleavings are **shift-equivalences**.

Merge trees (Cagliari, Ferri, Pozzi 2001, & Morozov, Beketayev, Weber 2013)

- A functor $\mathbf{T} : \mathbf{R} \to \mathbf{Set}$ can be thought of as a **merge tree**.
- Let X be a topological space and $f : X \to \mathbf{R}$ a function. Then

$$\mathbf{T}(t) = \pi_0 f^{-1}(-\infty, t]$$
$$\mathbf{T}[s \leq t] = \pi_0 \left[f^{-1}(-\infty, t] \subseteq f^{-1}(-\infty, t] \right]$$

defines the **sublevelset merge tree** of (X, f).



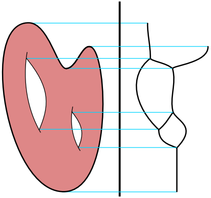
- If $f, g : X \to \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $\mathrm{d_i}(\mathbf{T}, \mathbf{U}) \leq \epsilon$.

Reeb graphs (dS, Munch, Patel 2016)

- A functor $\mathbf{F} : \mathbf{Int} \to \mathbf{Set}$ can be thought of as a **graph over the real line**. (Technically we require **F** to satisfy a **cosheaf condition**.)
- Let X be a topological space and $f : X \to \mathbf{R}$ a function. Then

$$\mathbf{F}_f(I) = \pi_0 f^{-1}(I)$$
$$\mathbf{F}_f[I \subseteq J] = \pi_0 \left[f^{-1}(I) \subseteq f^{-1}(J) \right]$$

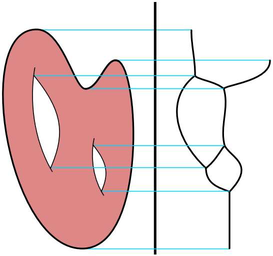
defines the **Reeb graph** of (X, f).



- If $f, g : X \to \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $\mathrm{d_i}(\mathbf{F}, \mathbf{G}) \leq \epsilon$.

Reeb graphs

- An **R-space** (X, f) is a topological space X with a map $f : X \to \mathbf{R}$.
- An **R**-space is a **Reeb graph** if X is a graph and each $f^{-1}(t)$ is finite.



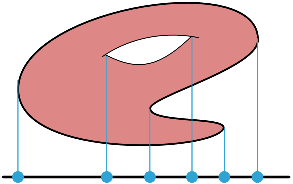
Reeb functor

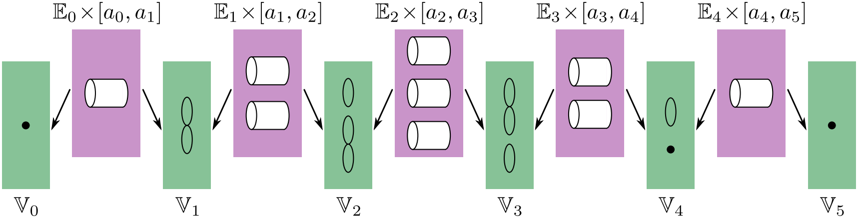
- The **Reeb functor** converts a (constructible) **R**-space into a Reeb graph:

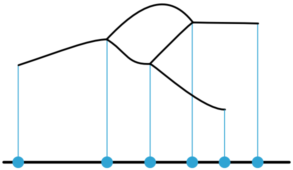
$$(X, f) \longmapsto ((X/\sim), \bar{f})$$

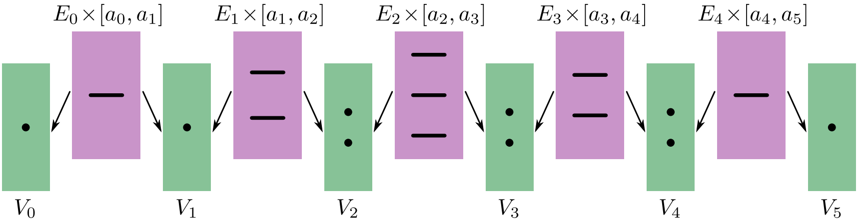
where $x \sim y$ iff x, y are in the same component of the same levelset of f.

Story 5: Reeb Graphs & Reeb Cosheaves



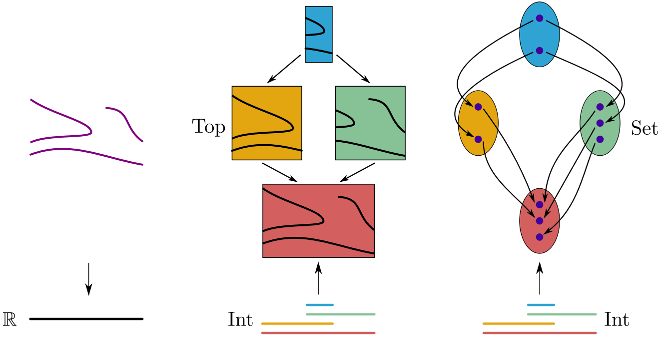





$E_0 \times [a_0, a_1]$
$E_1 \times [a_1, a_2]$
$E_2 \times [a_2, a_3]$
$E_3 \times [a_3, a_4]$
$E_4 \times [a_4, a_5]$
V_0
V_1
V_2
V_3
V_4
V_5

Reeb cosheaves (dS, Munch, Patel 2016)

- Let **Int** denote the poset of open intervals, with respect to inclusion.
- A Reeb graph gives rise to a functor $\mathbf{F} : \mathbf{Int} \to \mathbf{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.



$$\mathbf{F}(I = \bigcup I_\alpha) = \operatorname{colim}\left[\coprod_{\alpha,\beta} \mathbf{F}(I_\alpha \cap I_\beta) \rightrightarrows \coprod_\alpha \mathbf{F}(I_\alpha)\right]$$

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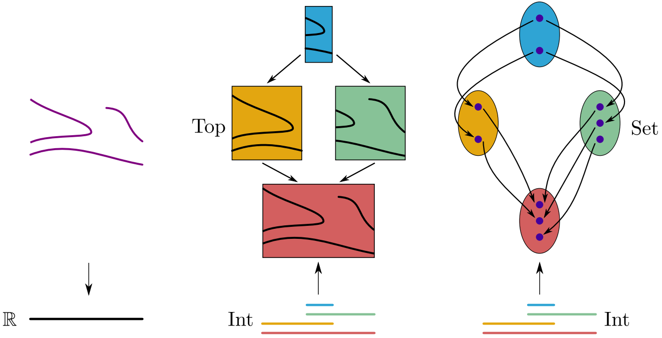

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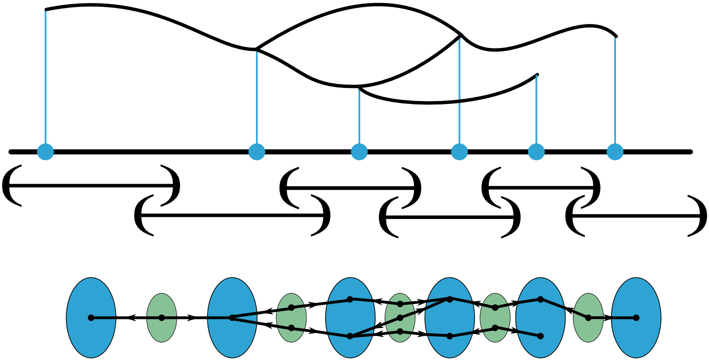
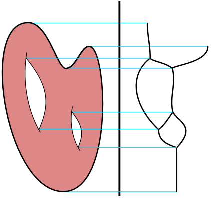


Reeb cosheaves (dS, Munch, Patel 2016)

- Let **Int** denote the poset of open intervals, with respect to inclusion.
- A Reeb graph corresponds to a functor $\mathbf{F} : \mathbf{Int} \to \mathbf{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.

$$
\begin{tikzcd}
 & & \mathrm{Pre} & \text{pre-cosheaves on } \mathbb{R} \\
\mathbb{R}\text{-spaces} & \mathbb{R}\text{-Top} \arrow[r, "\mathcal{C}"] & \mathrm{Csh} \arrow[u] & \text{cosheaves on } \mathbb{R} \\
\text{constructible } \mathbb{R}\text{-spaces} & \mathbb{R}\text{-Top}^{\mathrm{c}} \arrow[u] \arrow[r, "\mathcal{C}'"] \arrow[d, shift left, "\mathcal{R}"] & \mathrm{Csh}^{\mathrm{c}} \arrow[u] \arrow[d, equal] & \\
\text{Reeb graphs} & \mathrm{Reeb} \arrow[u, shift left, "\mathcal{I}"] \arrow[r, shift left, "\mathcal{C}''"] & \mathrm{Csh}^{\mathrm{c}} \arrow[l, shift left, "\mathcal{D}"] & \text{constructible cosheaves on } \mathbb{R}
\end{tikzcd}
$$

Story 5: Reeb Graphs & Reeb Cosheaves



Reeb functor (two versions)

- The **Reeb functor** converts a (constructible) **R**-space into a Reeb graph:

$$(X, f) \longmapsto ((X/\sim), \overline{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f.

or

- The **Reeb functor** converts a constructible **R**-space into a Reeb cosheaf:

$$\mathbf{F}(I) = \pi_0 f^{-1}(I)$$

$$\mathbf{G}[I \subseteq J] = \pi_0 \left[f^{-1}(I) \subseteq f^{-1}(J) \right]$$

Translation operators on **Int**

We define a 1-parameter semigroup (Ω_ϵ) of functors $\mathbf{Int} \to \mathbf{Int}$ by setting

$$\Omega_\epsilon(I) = I^\epsilon = \text{“}\epsilon\text{-neighbourhood of } I\text{”}$$

Reeb interleaving

An ϵ-**interleaving** between $\mathbf{F}, \mathbf{G}$ is given by two families of maps

$$\phi_I : \mathbf{F}(I) \to \mathbf{G}(I^\epsilon), \quad \psi_I : \mathbf{G}(I) \to \mathbf{F}(I^\epsilon)$$

which are natural with respect to inclusions $I \subseteq J$ and such that

$$\psi_{I^\epsilon} \circ \phi_I = \mathbf{F}[I \subseteq I^{2\epsilon}], \quad \phi_{I^\epsilon} \circ \psi_I = \mathbf{G}[I \subseteq I^{2\epsilon}]$$

for all I.

Stability Theorem

If $f, g : X \to \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $\mathrm{d_i}(\mathbf{F}, \mathbf{G}) \leq \epsilon$.

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Cosheaf Smoothing Theorem

If $\mathbf{F} : \mathbf{Int} \to \mathbf{Set}$ is a (constructible) cosheaf, then so is $\mathbf{F}\Omega_\epsilon : \mathbf{Int} \to \mathbf{Set}$.

Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.

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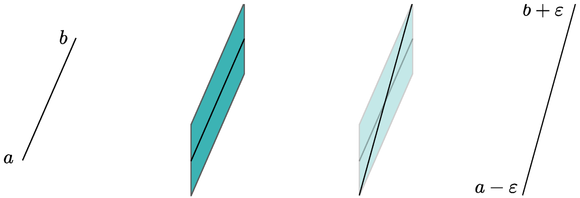

Story 5: Reeb Graphs & Reeb Cosheaves

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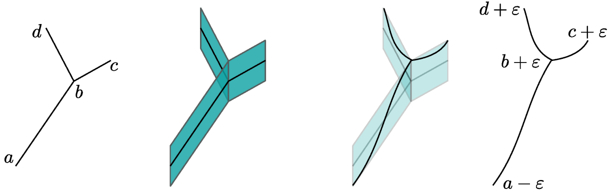

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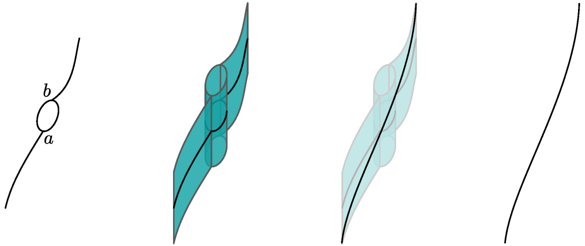

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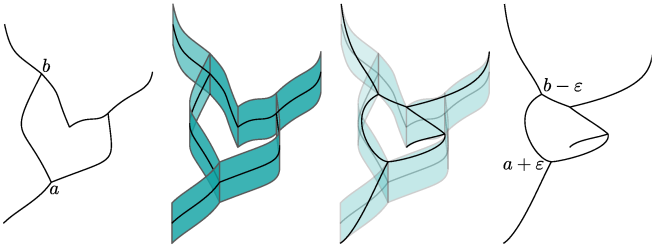

Progressive smoothing algorithm by Dmitriy Smirnov & Song Yu:

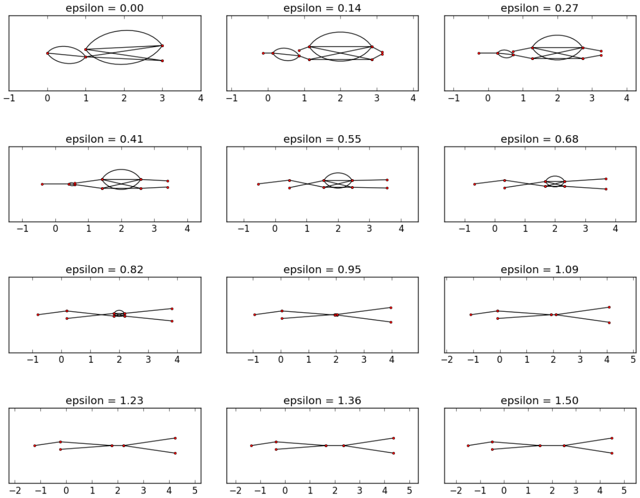

Image persistence (Cohen-Steiner, Edelsbrunner, Harer, Morozov 2009)

Let $\mathbb{V}, \mathbb{W} : \mathbf{P} \to \mathbf{Vect}$ be persistence modules and let $\Phi : \mathbb{V} \Rightarrow \mathbb{W}$. Then we can define a persistence module $\mathrm{Im}(\Phi)$ with

- $[\mathrm{Im}(\Phi)](t) = \mathrm{Im}(V_t \xrightarrow{\phi_t} W_t)$ for all t.
- $[\mathrm{Im}(\Phi)](s \leq t) =$ the map induced by the horizontal maps in:

$$\begin{tikzcd} V_s \arrow[r] \arrow[d, "\phi_s"'] & V_t \arrow[d, "\phi_t"] \\ W_s \arrow[r] & W_t \end{tikzcd}$$

We can similarly define $\mathrm{Ker}(\Phi)$ and $\mathrm{Coker}(\Phi)$.

Example

Suppose $p : X \to Y$ is a map of spaces, $f : X \to \mathbf{R}$, and $g : Y \to \mathbf{R}$. If $f \leq gp$, then p carries the t-sublevelset of f into the t-sublevelset of g, for all t, and the persistence module $\mathrm{Im}(\mathrm{H}(p))$ is defined.

Three ways of thinking of a map between persistence modules (over $\mathbf{N}$, say)

A functor $\mathbf{2} \to \mathbf{Vect}^{\mathbf{N}}$:

$$\begin{array}{c} F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \\ \Downarrow \\ G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \end{array}$$

A functor $\mathbf{N} \times \mathbf{2} \to \mathbf{Vect}$:

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & \cdots \\ G_0 \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & \cdots \end{tikzcd}$$

A functor $\mathbf{N} \to \mathbf{Vect}^{\mathbf{2}}$:

$$\begin{array}{ccccccc} F_0 & & F_1 & & F_2 & & \cdots \\ \downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow & \Rightarrow & \\ G_0 & & G_1 & & G_2 & & \cdots \end{array}$$

Story 6: Generalised Factors

The exponential law

The following categories of functors

$$(\mathbf{D}^{\mathbf{P}})^{\mathbf{W}} = \mathbf{D}^{\mathbf{P}\times\mathbf{W}} = (\mathbf{D}^{\mathbf{W}})^{\mathbf{P}}$$

are equal for any three categories $\mathbf{D}, \mathbf{P}, \mathbf{W}$.

Image, Kernel, Cokernel functors

The operations Im, Ker and Coker can be thought of as functors $\mathbf{Vect}^2 \to \mathbf{Vect}$.

- Each operation converts any $(V \xrightarrow{\alpha} W)$ into a vector space.
- Given a commutative square, there are induced maps between images, kernels, cokernels.

Proposition (Bubenik, dS, Scott)

The image persistence of $\Phi : \mathbb{V} \Rightarrow \mathbb{W}$ is equal to the composite

$$\mathbf{P} \xrightarrow{\quad \hat{\Phi} \quad} \mathbf{Vect}^2 \xrightarrow{\ \mathrm{Im}\ } \mathbf{Vect}$$

where $\hat{\Phi}$ is the interpretation of Φ as a functor $\mathbf{P} \to \mathbf{Vect}^2$.

Story 6: Generalised Factors

Generalized factor persistence (Bubenik, dS, Scott)

Given

- a category of persistence modules $\mathbf{D}^{\mathbf{P}}$;
- a category $\mathbf{W}$, which we call the *auxiliary category*;
- a functor $\mathbf{D}^{\mathbf{W}} \xrightarrow{N} \mathbf{E}$, which we call the *generalized factor*.

Then any functor $F : \mathbf{W} \to \mathbf{D}^{\mathbf{P}}$ determines a persistence module in $\mathbf{E}^{\mathbf{P}}$, by

$$\begin{array}{ccccc} (\mathbf{D}^{\mathbf{P}})^{\mathbf{W}} = \mathbf{D}^{\mathbf{W}\times\mathbf{P}} = (\mathbf{D}^{\mathbf{W}})^{\mathbf{P}} & & \longrightarrow & & \mathbf{E}^{\mathbf{P}} \\ F \longmapsto \hat{F} & & \longmapsto & & N\hat{F} \end{array}$$

Reductions of 2-dimensional persistence

Let $\mathbb{V} = (V(s,t)) \in \mathbf{Vect}^{\mathbf{R}\times\mathbf{R}}$ be a two-dimensional persistence module. Think of this as a family $(\mathbb{W}_t)$ of 1-dimensional persistence modules. We will define various generalized factors $N : \mathbf{Vect}^{\mathbf{R}} \to \mathbf{Vect}$.

- Fix a and define $N(\mathbb{W}) = \mathbb{W}(a)$.
- Fix $a < b$ and define $N(\mathbb{W}) = \mathrm{Im}(\mathbb{W}(a) \to \mathbb{W}(b))$.
- Fix $a < b \leq c < d$ and define

$$N(\mathbb{W}) = \left[\frac{\mathrm{Im}\left(\mathbb{W}(b) \to \mathbb{W}(c)\right) \cap \mathrm{Ker}\left(\mathbb{W}(c) \to \mathbb{W}(d)\right)}{\mathrm{Im}\left(\mathbb{W}(a) \to \mathbb{W}(c)\right) \cap \mathrm{Ker}\left(\mathbb{W}(c) \to \mathbb{W}(d)\right)}\right]$$

Then there is a 1-parameter persistence module associated to each of these functors.

Zigzag factors

Suppose $\mathbf{Z}$ is the category defined by:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

An element of $\mathbf{Vect}^{\mathbf{Z}}$ is a diagram

$$\mathbb{W}: \qquad W_1 \xrightarrow{f} W_2 \xleftarrow{g} W_3 \xrightarrow{h} W_4$$

Then, for example, the functor $\mathbf{Vect}^{\mathbf{Z}} \to \mathbf{Vect}$ defined by

$$N(\mathbb{W}) = \left[\frac{g(h^{-1}(0))}{f(W_1)}\right]$$

picks out the part of $\mathbb{W}$ supported over W_2, W_3.

Therefore, given a zigzag of persistence modules

$$\mathbb{V}_1 \xrightarrow{f} \mathbb{V}_2 \xleftarrow{g} \mathbb{V}_3 \xrightarrow{h} \mathbb{V}_4$$

we can constrict a single persistence module which extracts the $[2, 3]$ part.

Story 7: The observable category

Tame persistence modules

Let $\mathbb{V} : \mathbf{R} \to \mathbf{Vect}$ be a persistence module. If the maps $V_s \to V_t$ have finite rank whenever $s < t$, then $\mathbb{V}$ has a persistence diagram. If $\mathbb{V}$ has an interval decomposition, then the summands are identified exactly by the points in the diagram. However, it is not guaranteed that $\mathbb{V}$ has an interval decomposition.

Ephemeral modules (Chazal, Crawley-Boevey, dS 2016)

A persistence module $\mathbb{V}$ is *ephemeral* if $v_s^t = 0$ whenever $s < t$.
Then:

- The ephemeral modules comprise a **Serre subcategory** of the category of persistence modules.
- We can form the Serre quotient category by formally inverting all maps whose kernels and cokernels are ephemeral.
- In this category, every q-tame persistence module admits an interval decomposition.

Perhaps this is the 'correct' category for real-parameter persistence?

Story 7: The observable category

Definition

A **Serre subcategory** is a full subcategory $\mathbf{C}$ of an Abelian category such that for any short exact sequence

$$0 \longrightarrow \mathbb{U} \longrightarrow \mathbb{V} \longrightarrow \mathbb{W} \longrightarrow 0$$

we have

$$\mathbb{V} \in \mathbf{C} \quad \Leftrightarrow \quad \mathbb{U} \in \mathbf{C} \text{ and } \mathbb{W} \in \mathbf{C}.$$

Equivalently, the subcategory $\mathbf{C}$ is closed under subobjects, quotient objects, and extensions.

Noise systems (Scolamiero et al., 2016)

Noise in topological data analysis can be studied by considering a nested family $(\mathbf{C}_\epsilon \mid \epsilon \in [0, \infty))$ satisfying an enriched version of the Serre conditions:

$$\mathbb{V} \in \mathbf{C}_\epsilon \quad \Rightarrow \quad \mathbb{U} \in \mathbf{C}_\epsilon \text{ and } \mathbb{W} \in \mathbf{C}_\epsilon$$
$$\mathbb{V} \in \mathbf{C}_{\epsilon_1+\epsilon_2} \quad \Leftarrow \quad \mathbb{U} \in \mathbf{C}_{\epsilon_1} \text{ and } \mathbb{W} \in \mathbf{C}_{\epsilon_2}.$$

for any short exact sequence.

Acknowledgements

Collaborators

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