【CL1-3】

Information topology and probabilistic graphical models

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Baudot et Bennequin [2] have introduced a cohomology adapted to information theory. To this end, they use the constructions of topos theory [1]: an "information topos" is a ringed topos on a suitable site whose objects are σ-algebras. One can define a family of sheaves F_q, with $q > 0$, such that Shannon's entropy generates $H^1(F_1)$ and Tsallis' entropy S_q generates $H^1(F_q)$ when $q \neq 1$. Other information functions appear also as cocycles and the theory can be extended to the quantum case.

In this context, we identify the entropy as the solution of a problem of extension of algebras. We also present a combinatorial analogue of Shannon's axioms [5] for strings of size N with fixed type (the proportion of appearances of each symbol) that give the cocyle equation in the limit $N \to \infty$.

Each probabilistic graphical model [4] gives an information topos; accordingly, Bayesian networks, Markov fields and other interesting models find a place in our theory. The site is such that the usual Bethe-Kikuchi approximation is obtained by Möbius inversion on it, followed by a truncation [3]. We present a simplicial representation for factor graphs; the obstruction to find a global state (probability law) with prescribed marginals is strongly related to the topology of this representation. Some explicit examples will be discussed.

References

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